A Hitchhiker’s Guide to Γ-Convergence

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Why Do We Need Γ-Convergence?

- Γ-convergence, introduced by Ennio de Giorgi in 1975, is a notion of convergence that is appropriate for variational problems.

- Suppose that you are given a sequence of functionals \((F_n)_{n \in \mathbb{N}}\) on some space \(X\). Suppose that \(x_n\) minimizes \(F_n\). Does \(\lim_{n \to \infty} x_n\), if it exists at all, minimize anything?

- In what sense does \(F_n\) have to converge to some other functional \(F\) to ensure that minimizers of \(F_n\) converge to minimizers of \(F\)? Neither pointwise nor uniform convergence does the job, as will be shown.

- Γ-convergence is the answer!
A Cautionary Tale: Microstructure

Define a functional $F : H^1_0([0, 1]; \mathbb{R}) \to [0, +\infty]$ by

$$F(u) := \int_0^1 (u'(x)^2 - 1)^2 \, dx.$$ 

Define $F_n$ by restricting $F$ to the “polygonal functions” of mesh size $\frac{1}{2n}$:

$$F_n(u) := \begin{cases} F(u), & \text{if } u' \text{ is constant on } \left(\frac{i}{2n}, \frac{i+1}{2n}\right) \text{ for } i = 1, \ldots, 2n - 1, \\ +\infty, & \text{otherwise.} \end{cases}$$

$u_1$ minimizes $F_1$
A Cautionary Tale: Microstructure

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+\infty, & \text{otherwise.}
\end{cases}$$
A (non-unique) minimizer for $F_n$ is given by $u_n$, a succession of $n$ “tent functions” of height $1/2n$ and width $1/n$:

$$F_n(u_n) = 0 \text{ for each } n \in \mathbb{N}.$$

- $u_n \to u$ uniformly, where $u(x) = 0$ for all $x \in [0, 1]$.
- But $u$ is not a minimizer of $F$!

$$F(u) = 1 > 0 = F(u_1).$$

What has gone wrong? In one sense, the problem is that $F$ is not lower semicontinuous. A deeper reason is that $F_n$ does not $\Gamma$-converge to $F$. 
The Setting

- $\mathcal{X}$ will be a normed / metric / topological space.\(^1\) Most of the examples later on will take $\mathcal{X} = \mathbb{R}$; in most applications, $\mathcal{X}$ is a space of functions, e.g. $H^1_0(\Omega; \mathbb{R}^3)$.
- $\overline{\mathbb{R}}$ denotes the **extended real number line**, $\mathbb{R} \cup \{-\infty, +\infty\}$.
- We will consider functions $F: \mathcal{X} \to \overline{\mathbb{R}}$. For historical reasons, and because $\mathcal{X}$ is often a space of functions, $F$ is often called a **functional**.

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\(^1\)To make things easy, $\mathcal{X}$ is assumed to be a **sequential space**, so everything can be phrased in terms of convergence of sequences. In general, we need to use nets (a.k.a. Moore–Smith sequences) and be more careful with terminology like “limit point” vs. “cluster point”.

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**Definition (Limit Points)**

A **limit point** of a sequence \((x_n)_{n \in \mathbb{N}}\) in \(\mathcal{X}\) is a point \(x \in \mathcal{X}\) such that some subsequence \((x_{n_k})_{k \in \mathbb{N}}\) has \(\lim_{k \to \infty} x_{n_k} = x\).

**Definition (Lower and Upper Limits)**

The **lower limit** (or **limit inferior**) of a real sequence \((x_n)_{n \in \mathbb{N}}\) is the infimum of all its limit points:

\[
\liminf_{n \to \infty} x_n := \inf \left\{ x \in \overline{\mathbb{R}} \mid x \text{ is a limit point of } (x_n)_{n \in \mathbb{N}} \right\}.
\]

The **upper limit** (or **limit superior**) is the supremum of all its limit points:

\[
\limsup_{n \to \infty} x_n := \sup \left\{ x \in \overline{\mathbb{R}} \mid x \text{ is a limit point of } (x_n)_{n \in \mathbb{N}} \right\}.
\]
Lower and Upper Limits

Lemma (Basic Properties of Lower and Upper Limits)

- For any real sequence \((x_n)_{n \in \mathbb{N}}\), the lower and upper limits are well-defined in \(\mathbb{R}\) and satisfy

\[
-\infty \leq \liminf_{n \to \infty} x_n \leq \limsup_{n \to \infty} x_n \leq +\infty.
\]

- The lower and upper limits are equal if, and only if, \((x_n)_{n \in \mathbb{N}}\) is a convergent sequence, in which case

\[
\liminf_{n \to \infty} x_n = \lim_{n \to \infty} x_n = \limsup_{n \to \infty} x_n.
\]

Example

\[
\liminf_{n \to \infty} \sin n = \liminf_{n \to \infty} (-1)^n = -1 \quad \text{but} \quad \limsup_{n \to \infty} \sin n = \limsup_{n \to \infty} (-1)^n = +1
\]
**Definition (Epigraph)**

The **epigraph** of $F: \mathcal{X} \to \overline{\mathbb{R}}$ is the subset of $\mathcal{X} \times \overline{\mathbb{R}}$ that lies “on or above the graph” of $F$:

$$\text{epi}(F) := \left\{ (x, v) \in \mathcal{X} \times \overline{\mathbb{R}} \mid v \geq F(x) \right\}.$$
**Definition (Lower Semicontinuity)**

\( F : \mathcal{X} \to \overline{\mathbb{R}} \) is lower semicontinuous (or just lsc for short) if

\[
    x_n \to x \implies F(x) \leq \liminf_{n \to \infty} F(x_n);
\]

equivalently, \( \{ x \in \mathcal{X} \mid F(x) \leq t \} \) is closed in \( \mathcal{X} \) for each \( t \in \overline{\mathbb{R}} \); also equivalently, \( \text{epi}(F) \) is closed in \( \mathcal{X} \times \overline{\mathbb{R}} \).
Semicontinuity

Lower semicontinuity is very important in variational problems, because (on compacta) it ensures the existence of minimizers:

**Theorem**

Let $\mathcal{X}$ be compact and let $F : \mathcal{X} \to \mathbb{R}$ be lsc. Then there exists $x^* \in \mathcal{X}$ such that

$$F(x^*) = \inf_{x \in \mathcal{X}} F(x).$$

**Proof.**

Let $(x_n)_{n \in \mathbb{N}}$ be such that $\lim_{n \to \infty} F(x_n) = \inf_{x \in \mathcal{X}} F(x)$. Since $\mathcal{X}$ is compact, $(x_n)_{n \in \mathbb{N}}$ has a convergent subsequence $(x_{n_k})_{k \in \mathbb{N}}$ that converges to some $x^* \in \mathcal{X}$. Since $F$ is lsc,

$$F(x^*) \leq \lim_{k \to \infty} \inf_{x \in \mathcal{X}} F(x_{n_k}) = \inf_{x \in \mathcal{X}} F(x).$$
Definition (Lower Semicontinuous Envelope)

Given $F: \mathcal{X} \to \overline{\mathbb{R}}$, the lower semicontinuous envelope (or relaxation) of $F$ is the “greatest lsc function bounded above by $F$”:

$$F^{\text{lsc}}(x) := \sup \{ G(x) \mid G: \mathcal{X} \to \overline{\mathbb{R}} \text{ is lsc and } G \leq F \text{ on } \mathcal{X} \}$$

$$= \inf \left\{ \liminf_{n \to \infty} F(x_n) \mid (x_n)_{n \in \mathbb{N}} \subseteq \mathcal{X} \text{ and } x_n \to x \right\}.$$
### Definition (Pointwise Convergence)

Let $F_n : \mathcal{X} \to \overline{\mathbb{R}}$ for each $n \in \mathbb{N}$, and $F : \mathcal{X} \to \overline{\mathbb{R}}$. $F_n \to F$ pointwise if

$$\lim_{n \to \infty} |F_n(x) - F(x)| = 0.$$

### Definition (Uniform Convergence)

Let $F_n : \mathcal{X} \to \overline{\mathbb{R}}$ for each $n \in \mathbb{N}$, and $F : \mathcal{X} \to \overline{\mathbb{R}}$. $F_n \to F$ uniformly if

$$\lim_{n \to \infty} \sup_{x \in \mathcal{X}} |F_n(x) - F(x)| = 0.$$
**Γ-Convergence**

**Definition (Γ-Convergence)**

Let $F_n : \mathcal{X} \to \mathbb{R}$ for each $n \in \mathbb{N}$. We say that $(F_n)_{n \in \mathbb{N}}$ Γ-converges to $F : \mathcal{X} \to \mathbb{R}$, and write $\Gamma\text{-lim}_{n \to \infty} F_n = F$ or $F_n \Gamma \to F$, if

1. (asymptotic common lower bound) for every $x \in \mathcal{X}$ and every $(x_n)_{n \in \mathbb{N}}$ such that $x_n \to x$ in $\mathcal{X}$,
   \[ F(x) \leq \liminf_{n \to \infty} F_n(x_n); \]

2. (existence of recovery sequences) for every $x \in \mathcal{X}$, there exists some $(x_n)_{n \in \mathbb{N}}$ such that $x_n \to x$ in $\mathcal{X}$ and
   \[ F(x) \geq \limsup_{n \to \infty} F_n(x_n). \]
Remarks

- Note that $\Gamma$-convergence of functionals on $\mathcal{X}$ is phrased in terms of convergence of sequences of elements of $\mathcal{X}$. If we change the norm/metric/topology on $\mathcal{X}$, then we change the topology of $\Gamma$-convergence as well.

- One can even have “hybrid” notions of convergence, e.g. Mosco convergence, a.k.a. “weak $\Gamma$-lim inf and strong $\Gamma$-lim sup” convergence.

- Modulo a few technicalities, $\Gamma$-convergence on a topological vector space $\mathcal{X}$ can be equivalently expressed in terms of the minimum energies of the $F_n$, having been “tilted” by an arbitrary continuous load: $F_n \xrightarrow{\Gamma} F$ if, and only if, for every continuous linear map $\ell: \mathcal{X} \to \mathbb{R}$,

$$\lim_{n \to \infty} \inf_{x \in \mathcal{X}} (F_n(x) + \langle \ell, x \rangle) = \inf_{x \in \mathcal{X}} (F(x) + \langle \ell, x \rangle).$$
Convergence of Sets: Kuratowski Convergence

Consider a metric space \((\mathcal{X}, d)\). The distance from a point \(x \in \mathcal{X}\) to a set \(A \subseteq \mathcal{X}\) is defined by

\[
d(x, A) := \inf_{a \in A} d(x, a).
\]

**Definition (Kuratowski limits)**

Let \((A_n)_{n \in \mathbb{N}}\) be a sequence of subsets of \(\mathcal{X}\). Let

\[
\begin{align*}
\text{K-lim inf } A_n & := \left\{ x \in \mathcal{X} \left| \limsup_{n \to \infty} d(x, A_n) = 0 \right. \right\}, \\
\text{K-lim sup } A_n & := \left\{ x \in \mathcal{X} \left| \liminf_{n \to \infty} d(x, A_n) = 0 \right. \right\}.
\end{align*}
\]

If these two are equal, then their common value is called the **Kuratowski limit** \(\text{K-lim}_{n \to \infty} A_n\).
Examples of Kuratowski Convergence

Examples

In $\mathcal{X} = \mathbb{R}$ with its usual Euclidean distance,

$$\lim_{n \to \infty} \frac{1}{n} \mathbb{Z} =$$
Examples of Kuratowski Convergence

**Examples**

In $\mathcal{X} = \mathbb{R}$ with its usual Euclidean distance,

$$\lim_{n \to \infty} \frac{1}{n} \mathbb{Z} = \mathbb{R},$$
Examples of Kuratowski Convergence

Examples

In \( X = \mathbb{R} \) with its usual Euclidean distance,

\[
\lim_{n \to \infty} \frac{1}{n} \mathbb{Z} = \mathbb{R},
\]

\[
\lim_{n \to \infty} n \mathbb{Z} = \mathbb{R}.
\]
Examples of Kuratowski Convergence

In $X = \mathbb{R}$ with its usual Euclidean distance,

$$\text{K-lim } \lim_{n \to \infty} \frac{1}{n} \mathbb{Z} = \mathbb{R},$$

$$\text{K-lim } \lim_{n \to \infty} n \mathbb{Z} = \{0\}.$$
Examples of Kuratowski Convergence

Examples

In $\mathcal{X} = \mathbb{R}^2$ with its usual Euclidean distance, let

$$A_n := \{(x, y) \in \mathbb{R}^2 \mid |x|^n + |y|^n < 1\}.$$

Then

$$\lim_{n \to \infty} A_n =$$
Examples of Kuratowski Convergence

**Examples**

In $\mathcal{X} = \mathbb{R}^2$ with its usual Euclidean distance, let

$$A_n := \{(x, y) \in \mathbb{R}^2 \mid |x|^n + |y|^n < 1\}.$$

Then

$$\operatorname{K-lim}_{n \to \infty} A_n = \{(x, y) \mid |x| \leq 1 \text{ and } |y| \leq 1\},$$
Examples of Kuratowski Convergence

**Examples**

In $\mathcal{X} = \mathbb{R}^2$ with its usual Euclidean distance, let

$$A_n := \{(x, y) \in \mathbb{R}^2 \mid |x|^n + |y|^n < 1\}.$$  

Then

$$\lim_{n \to \infty} A_n = \{(x, y) \mid |x| \leq 1 \text{ and } |y| \leq 1\},$$

and, sending $n \to 0$, i.e. $1/n \to \infty$,

$$\lim_{n \to 0} A_n =$$
Examples of Kuratowski Convergence

Examples

In $\mathcal{X} = \mathbb{R}^2$ with its usual Euclidean distance, let

$$A_n := \{(x, y) \in \mathbb{R}^2 \mid |x|^n + |y|^n < 1\}.$$  

Then

$$K\text{-lim}_{n \to \infty} A_n = \{(x, y) \mid |x| \leq 1 \text{ and } |y| \leq 1\},$$

and, sending $n \to 0$, i.e. $1/n \to \infty$,

$$K\text{-lim}_{n \to 0} A_n = (\{0\} \times [-1, 1]) \cup (-1, 1] \times \{0\}).$$
The following theorem, which relates $\Gamma$-convergence and Kuratowski convergence, is very useful if you have some intuition about the epigraphs of the functionals under consideration:

**Theorem**

$F_n$ $\Gamma$-converges to $F$ on $\mathcal{X}$ if, and only if, the epigraphs of the $F_n$ $K$-converge to the epigraph of $F$ in $\mathcal{X} \times \overline{\mathbb{R}}$:

$$\Gamma\lim_{n \to \infty} F_n = F \iff K\lim_{n \to \infty} \text{epi}(F_n) = \text{epi}(F).$$
Properties of $\Gamma$-Limits

**Lemma**

*Kuratowski limits are always closed sets.*

**Theorem**

*$\Gamma$-limits are always lower semicontinuous.*

**Proof.**
Properties of Γ-Limits

Lemma

Kuratowski limits are always closed sets.

Theorem

Γ-limits are always lower semicontinuous.

Proof.

Suppose that $F_n \xrightarrow{\Gamma} F$ on $\mathcal{X}$. Then $\text{epi}(F_n) \xrightarrow{K} \text{epi}(F)$ in $\mathcal{X} \times \overline{\mathbb{R}}$. Since it is a Kuratowski limit, $\text{epi}(F)$ has to be closed. Hence, $F$ is lower semicontinuous.
Properties of Γ-Limits

**Theorem (Γ and Pointwise Convergence)**

If \( F_n \xrightarrow{\Gamma} F \) and \( F_n \to G \) pointwise, then \( F \leq G \).

**Theorem (Γ and Uniform Convergence)**

If \( F_n \xrightarrow{unif} F \), then \( F_n \xrightarrow{\Gamma} F^{lsc} \).

**Corollary**

The Γ-limit of a constant sequence \((F)_{n \in \mathbb{N}}\) is . . .
Properties of $\Gamma$-Limits

**Theorem ($\Gamma$ and Pointwise Convergence)**

If $F_n \xrightarrow{\Gamma} F$ and $F_n \rightarrow G$ pointwise, then $F \leq G$.

**Theorem ($\Gamma$ and Uniform Convergence)**

If $F_n \xrightarrow{unif} F$, then $F_n \xrightarrow{\Gamma} F^{lsc}$.

**Corollary**

The $\Gamma$-limit of a constant sequence $(F)_{n \in \mathbb{N}}$ is $F^{lsc}$. 
Properties of $\Gamma$-Limits

**Theorem (Well-Definedness of $\Gamma$-Limits)**

If $F_n \xrightarrow{\Gamma} F$, then every subsequence $(F_{n_k})_{k \in \mathbb{N}}$ of $(F_n)_{n \in \mathbb{N}}$ also $\Gamma$-converges to $F$.

**Theorem (Stability of $\Gamma$-Limits under Continuous Perturbations)**

If $F_n \xrightarrow{\Gamma} F$ and $G$ is continuous, then

$$F_n + G \xrightarrow{\Gamma} F + G.$$
Properties of $\Gamma$-Limits

**Theorem (Fundamental Theorem of $\Gamma$-Convergence)**

If $F_n \xrightarrow{\Gamma} F$ and $x_n$ minimizes $F_n$, then every limit point of the sequence $(x_n)_{n \in \mathbb{N}}$ is a minimizer of $F$.

- Compare this theorem with the microstructure example in the beginning: in that example, the functionals do not $\Gamma$-converge, since the minimizers $(u_n)_{n \in \mathbb{N}}$ of $(F_n)_{n \in \mathbb{N}}$ converge to a $u$ that does not minimize $F$.

- Note, however, that this theorem may be vacuously true, e.g. if $F_n$ has no minimizer, if $(x_n)_{n \in \mathbb{N}}$ has no convergent subsequences, or the space $\mathcal{X}$ is non-compact.
Example

Define \( F_n : \mathbb{R} \to \mathbb{R} \) by \( F_n(x) := \cos(nx) \). What is the \( \Gamma \)-limit of this sequence?

\[
\Gamma \text{-} \lim_{n \to \infty} \cos(nx) =
\]
Example

Define $F_n : \mathbb{R} \to \mathbb{R}$ by $F_n(x) := \cos(nx)$. What is the $\Gamma$-limit of this sequence?

$$\Gamma \lim_{n \to \infty} \cos(nx) = -1 \text{ for all } x \in \mathbb{R}.$$
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Example

Define $F_n : \mathbb{R} \to \mathbb{R}$ by $F_n(x) := \frac{1}{2}x^2 + \sin(nx)$. What is the $\Gamma$-limit of this sequence?

$$\Gamma \lim_{n \to \infty} \left( \frac{1}{2}x^2 + \sin(nx) \right) = \ldots$$
Examples / Exercises

Example

Define $F_n : \mathbb{R} \rightarrow \mathbb{R}$ by $F_n(x) := \cos(nx)$. What is the $\Gamma$-limit of this sequence?

$$\Gamma\lim_{n \to \infty} \cos(nx) = -1 \text{ for all } x \in \mathbb{R}.$$ 

Example

Define $F_n : \mathbb{R} \rightarrow \mathbb{R}$ by $F_n(x) := \frac{1}{2}x^2 + \sin(nx)$. What is the $\Gamma$-limit of this sequence?

$$\Gamma\lim_{n \to \infty} \left(\frac{1}{2}x^2 + \sin(nx)\right) = \frac{1}{2}x^2 - 1.$$
Example (Regularization of a Step Function)

Define $F_n : \mathbb{R} \rightarrow \mathbb{R}$ by

$$F_n(x) := \begin{cases} 
0, & x \leq -1/n, \\
\frac{nx + 1}{2}, & -\frac{1}{n} \leq x \leq \frac{1}{n}, \\
1, & x \geq \frac{1}{n}.
\end{cases}$$

$$\lim_{n \to \infty} F_n(x) =$$

$$\Gamma\text{-lim}_{n \to \infty} F_n(x) =$$
Example (Regularization of a Step Function)

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1, & x \geq \frac{1}{n}.
\end{cases}$$

$$\lim_{n \to \infty} F_n(x) = \begin{cases} 
0, & x < 0, \\
\frac{1}{2}, & x = 0, \\
1, & x > 0.
\end{cases}$$

$$\Gamma\text{-lim}_{n \to \infty} F_n(x) = \begin{cases} 
0, & x \leq 0, \\
1, & x > 0.
\end{cases}$$
Example

Define $F_n : \mathbb{R} \to \mathbb{R}$ by

$$F_n(x) := nx \exp(nx).$$

$$\lim_{n \to \infty} F_n(x) =$$

$$\Gamma\text{-lim}_{n \to \infty} F_n(x) =$$
Define $F_n : \mathbb{R} \to \mathbb{R}$ by

$$F_n(x) := nx \exp(nx).$$

$$\lim_{n \to \infty} F_n(x) = \begin{cases} 
0, & x \leq 0, \\
+\infty, & x > 0.
\end{cases}$$

$$\Gamma\text{-lim}_{n \to \infty} F_n(x) = \begin{cases} 
0, & x < 0, \\
-e^{-1}, & x = 0, \\
+\infty, & x > 0.
\end{cases}$$
Example

Define $F_n : \mathbb{R} \to \mathbb{R}$ by

$$F_n(x) := nx \exp(-2n^2 x^2).$$

$$\lim_{n \to \infty} F_n(x) =$$

$$\Gamma\text{-lim}_{n \to \infty} F_n(x) =$$
Define $F_n : \mathbb{R} \to \mathbb{R}$ by

$$F_n(x) := nx \exp(-2n^2x^2).$$

$$\lim_{n \to \infty} F_n(x) = 0 \text{ for all } x \in \mathbb{R}.$$ 

$$\Gamma\text{-lim}_{n \to \infty} F_n(x) = \begin{cases} -\frac{1}{2}e^{-1/2}, & x = 0, \\ 0, & x \neq 0. \end{cases}$$
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