Freie Universität Berlin

The Lipschitz Distance on Outer Space

DANIEL LÜTGEHETMANN

Bachelor's Thesis

Berlin, December 2012

Supervisor: Prof. Dr. Holger Reich

Selbstständigkeitserklärung

Hiermit versichere ich, dass ich die vorliegende Bachelorarbeit selbstständig und nur unter Zuhilfenahme der angegebenen Quellen erstellt habe.

____

Daniel Lütgehetmann

Contents

Introduction

It is a commonly used technique to examine a complicated group by considering its action on geometric objects. The *Outer space* is a topological space introduced by Culler and Vogtmann in [CV86] with this purpose for the group of automorphisms of the free group. More exactly, there is a right-action of the *outer* automorphisms on Outer space, which are equivalence classes in the automorphism group $\mathrm{Aut}(F_n)$. The points in Outer space are equivalence classes of triples consisting of a finite graph, a homotopy equivalence to the n-rose (which is the graph with one vertex and n edges) and a metric. The outer automorphisms act on such an equivalence class by changing the homotopy equivalence, but the concern of this thesis isn't this action. Instead, the goal is to give a short introduction to defining a distance on this space. We will define a distance function, show that it behaves properly under the right-action and fulfills all axioms of a metric except symmetry. Furthermore, we will see that this asymmetry isn't even bounded by any constant factor, i.e. the difference between the distances in each direction can be arbitrarily big. At last, we will show that the computation of the distance between two elements of Outer space can be broken down to finding loops of two specific forms in one of the graphs and find the maximum ratio of lengths of these loops in the two graphs. For the proof of this theorem (going back to Tad White), we will introduce the concept of *train track structures* on graphs, which makes the proof very elegant.

The thesis is based on Bestvina's script [Bes12] and the article [FM11].

Chapter 1

Outer Space

In the first chapter we want to give some basic definitions which we will need throughout this thesis. Moreover, *Outer space* will be defined and we shortly review the right-action of the *outer automorphisms*.

1.1 Graphs

Definition 1.1. A finite one-dimensional CW complex Γ is called a *finite graph*. The set $\mathrm{V}(\Gamma)$ is the set of 0-cells, its elements are called *vertices* of Γ. The open 1-cells of Γ are called *edges* and denoted $\mathrm{E}(\Gamma)$. A *closed edge* is the topological closure of an edge. A topologically closed union of edges and vertices of a graph is called a *subgraph*.

To limit the repetitions of the word finite, we agree that in the sequel every graph is finite if not stated otherwise.

Definition 1.2. The *valence* of a vertex $v \in \Gamma$ is the maximal number of connected components of $U - \{v\}$ for arbitrarily small connected open neighbourhoods U of v in Γ.

We will consider continuous maps between graphs with their ordinary topology as a CW complex. In the definition of Outer space, a graph with one vertex and $n \geq 2$ edges will play an important role and as all such graphs are homotopy equivalent, we choose a representative and give it a name.

Definition 1.3. The graph with one vertex and $n \geq 2$ edges is called the *n-rose* $\mathcal{R}_n$.

Definition 1.4. A tree (in a graph) is a contractible (sub-)graph. A *maximal tree* in a graph Γ is a tree that contains all vertices of Γ.

Remark. In a connected graph, every tree is contained in a maximal tree. A maximal tree is not contained in any bigger tree in Γ.

Definition 1.5. The *rank* of a graph Γ is the number of edges in $\Gamma - T$ for a maximal tree T in Γ.

Remark. The rank is independent of the choice of a maximal tree, homotopy equivalent graphs have the same rank.

Definition 1.6. A *path* in a topological space X is a continuous map $\omega\colon [a,b] \to X$, $a < b$, it is called a *loop* if $\omega(a) = \omega(b)$ and then may be regarded as a continuous map $\omega\colon S^1 \to X$. A path is called *immersed* if it is locally injective, i.e. if for all $t \in [a,b]$ there exists an open neighbourhood $U \subset [a,b]$ of t, so that $\omega|_U$ is injective. By ω^{-1} we don't mean the actual inverse of the map but the path that is defined as $\omega^{-1}(s) = \omega(a+b-s)$ for all $s \in [a,b]$.

If not stated otherwise, we assume for all paths and loops $a = 0$ and $b = 1$. We now want to introduce a combinatorial way of describing paths and maps between graphs.

Let Γ be a graph, then every closed edge e is the image of an interval $[0,1]_e$ under the continuous surjective projection

$$p\colon \coprod_{e \in \mathrm{E}(\Gamma)} [0,1]_e \amalg V(\Gamma) \to \Gamma$$

from the definition of Γ as CW complex. A closed edge e can be considered as immersed path

$$\begin{aligned} e\colon [0,1] &\to \Gamma \\ t &\mapsto p(t_e), \end{aligned}$$

where t_e is t considered as element of $[0,1]_e$. We write $E\colon [0,1] \to \Gamma$ for the inverse path defined as $E(t) = e(1-t)$ for all $t \in [0,1]$. For a vertex v in Γ we get the constant path $v\colon [0,1] \to \Gamma$ that maps everything to v. For two paths $\omega, \nu\colon [0,1] \to \Gamma$, we want to consider the formal product $\omega\nu$ as the concatenation of those paths, i.e. as the (not necessarily continuous) map

$$\begin{aligned} \omega\nu\colon [0,1] &\to \Gamma \\ t &\mapsto \begin{cases} \nu(2t) & \text{for } 0 \le t < 1/2 \\ \omega(2t-1) & \text{for } 1/2 \le t \le 1. \end{cases} \end{aligned}$$

Note that we write concatenations of paths from right to left. The map $\omega\nu$ is again a path if and only if $\omega(0) = \nu(1)$, so a formal product $\omega_1\omega_2\cdots\omega_k$ defines a path if and only if $\omega_i(0) = \omega_{i+1}(1)$ for all $1 \le i < k$. If there is a one-step backtrack, i.e. two consecutive paths that are inverse to each other, erasing this backtrack from the product yields a path that is homotopic relative endpoints to the original path. Up to parametrisation, this multiplication of paths is associative. If a formal product of vertices, edges and their formal inverses defines a path in Γ, we call this product a *combinatorial path*. Given a map $f\colon \Gamma \to \Gamma'$ and a path $\omega_1\omega_2\cdots\omega_k$ in Γ it is clear that we have $f(\omega_1\omega_2\cdots\omega_k) = f(\omega_1)f(\omega_2)\cdots f(\omega_k)$, where we write $f(\omega)$ for the composition $f \circ \omega\colon [0,1] \to \Gamma'$.

Now let Γ, Γ' be two graphs where we assigned a path in Γ' to every closed edge of Γ. This clearly defines a continuous map $\phi\colon \coprod_{e\in \mathrm{E}(\Gamma)}[0,1]_e \to \Gamma'$, which induces a continuous map from Γ to Γ' if ϕ is compatible with the projection into the quotient. This means that the images under ϕ of any two endpoints of edges that get identified have to be the same. If Γ is connected, every vertex gets identified with the end of some edge and therefore, the resulting map is uniquely determined.

Hence, we may describe a continuous map between connected graphs Γ and Γ' by assigning a combinatorial path in Γ' to every edge of Γ in such a manner that these paths are compatible at the vertices of Γ. Obviously, not every map between graphs can be described in this fashion.

We now identify every graph's fundamental group with the free group of rank n–denoted F_n–in a certain manner.

1.1.1 Marked Graphs

Definition 1.7. A *marking* of a graph Γ is a homotopy equivalence $f\colon \mathcal{R}_n \to \Gamma$. The pair (Γ, f) is called a *marked graph.*

If we identify $\pi_1(\mathcal{R}_n)$ with F_n, the marking induces an identification of the fundamental group of Γ with F_n. This is an important fact that we will use frequently. The first observation about marked graphs is rather obvious.

Remark. Marked graphs are path-connected and therefore connected.

Proof. The graph $\mathcal{R}_n$ is path-connected and as it is homotopy equivalent to Γ, the latter also has to be path-connected. □

This means that most of our graphs will be connected. Since the identification of the fundamental group with the free group depends only on the homotopy class of the marking, there exist different marked graphs that should be considered equivalent. Therefore, we will look at equivalence classes of marked graphs under the following relation.

Definition 1.8. Two marked graphs (Γ, f) and (Γ', f') are called *equivalent* if there exists a homeomorphism $\phi\colon \Gamma \to \Gamma'$, so that $\phi f \simeq f'$.

Since ϕ is a homeomorphism, this clearly defines an equivalence relation. Instead of defining the homotopy equivalence f directly, it's often more convenient to specify the inverse marking, i.e. the homotopy inverse of f. To do this, we make the identification of the free group with the fundamental group of the rose explicit by orienting the edges of $\mathcal{R}_n$ for an appropriate $n \in \mathbb{N}$ and labeling them with a basis $a_1, \ldots, a_n$ of F_n. Then, we choose a maximal tree T in the (connected!) graph Γ, orient all edges in $\Gamma - T$ and label them with a basis of F_n expressed in words in the a_i. This labeling induces an inverse marking by collapsing the tree to the single vertex in $\mathcal{R}_n$ and sending each edge to the loop in $\mathcal{R}_n$ associated to the labeling. It is a homotopy equivalence because it sends the loops associated to one basis of F_n to loops associated to another basis of F_n.

Example 1.9. We want to show that the two marked graphs shown in Figure 1.1 are equivalent. The unlabeled edges form a maximal tree, intersections of lines are

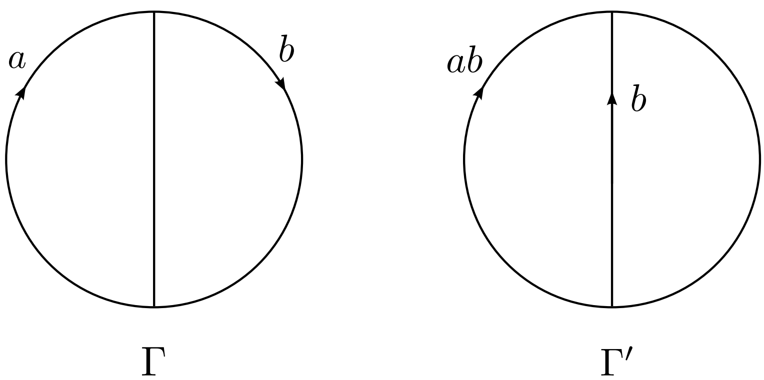

Figure 1.1: Two marked graphs with their labeling

vertices, the orientation of the edges is marked by the arrows and as always, loops are written from right to left. We named the edges of the rose a and b instead of a_1 and a_2 for easier reading. To show the equivalence, first name the edges of Γ according to Figure 1.2, the edges of Γ' are named c', d' and e' accordingly. First,

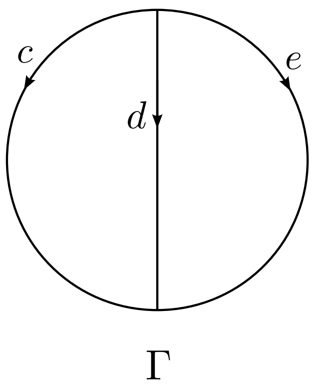

Figure 1.2: Naming of the edges of Γ

we reconstruct the marking from the given inverse marking, so we have to find the homotopy inverses of the following maps (x_0 is the vertex of $\mathcal{R}_2$):

$$\begin{aligned} f^{-1}\colon \Gamma &\to \mathcal{R}_2 \\ c &\mapsto A \\ d &\mapsto x_0 \\ e &\mapsto b \end{aligned} \qquad\qquad \begin{aligned} f'^{-1}\colon \Gamma' &\to \mathcal{R}_2 \\ c' &\mapsto BA \\ d' &\mapsto B \\ e' &\mapsto x_0. \end{aligned}$$

Recall that a capital letter denotes the path corresponding to the inversely oriented edge. The two maps are continuous because both vertices are mapped to x_0, no matter which edge we consider. The markings are then given by

$$\begin{aligned} f\colon \mathcal{R}_2 &\to \Gamma \\ a &\mapsto Cd \\ b &\mapsto De \end{aligned} \qquad\qquad \begin{aligned} f'\colon \mathcal{R}_2 &\to \Gamma' \\ a &\mapsto C'd' \\ b &\mapsto D'e', \end{aligned}$$

since then we have $f^{-1} \circ f \simeq \mathrm{id}_{\mathcal{R}_2}$, $f \circ f^{-1} \simeq \mathrm{id}_\Gamma$ and the same for f'. This is seen as follows.

For the maps with domain $\mathcal{R}_2$, i.e. $f^{-1} \circ f$ and $f'^{-1} \circ f'$, both edges start and end at the only vertex x_0. So any map given in combinatorial notation already maps this vertex to itself. Hence, we may show that such a map is homotopic to the identity by verifying that this is true on each edge individually with homotopies that fix the endvertices. One example for this is $f^{-1}(f(a)) = f^{-1}(Cd) = ax_0 \simeq a$ relative endvertices, the three remaining verifications are also easy.

For the maps $f \circ f^{-1}$ and $f' \circ f'^{-1}$ this doesn't hold as the lower vertex of Γ is not mapped to itself but to the upper vertex. To verify that the first map is homotopic to the identity, consider the continuous map $g\colon \Gamma \to \mathcal{R}_2$ that collapses the unlabeled maximal tree T to the vertex of $\mathcal{R}_2$. This is in fact a homotopy equivalence for which a homotopy inverse $g^{-1}\colon \mathcal{R}_2 \to \Gamma$ can be defined as follows: It sends the vertex of $\mathcal{R}_2$ to an arbitrary vertex v_0 in T. For every edge $\tilde{e}$ in $\mathcal{R}_2$ there exists exactly one edge e in $\Gamma - T$ with $g(e) = \tilde{e}$ and therefore a loop in Γ that starts at v_0, goes along T to the initial vertex of e, traverses e and goes back along T to v_0. The initial vertex of e naturally is $e(0)$ (considering e as path). Since all these loops start at v_0, assigning these loops to the edges really defines a continuous map g^{-1} on the whole rose. The algorithm for constructing g^{-1} is well-known from the theory of fundamental groups of graphs, so we won't proof that this is indeed a homotopy inverse of g.

More explicitly, we consider the continuous maps

$$\begin{aligned} g\colon \Gamma &\to \mathcal{R}_2 \\ c &\mapsto a \\ d &\mapsto x_0 \\ e &\mapsto b \end{aligned} \qquad\qquad \begin{aligned} g^{-1}\colon \mathcal{R}_2 &\to \Gamma \\ a &\mapsto Dc \\ b &\mapsto De. \end{aligned}$$

But then, any map $h\colon \Gamma \to \Gamma$ is homotopic to the identity on Γ if and only if $g \circ h \circ g^{-1}$ is homotopic to the identity on $\mathcal{R}_2$. So in the case of $f \circ f^{-1}$, we only have to examine

$g \circ f \circ f^{-1} \circ g^{-1}$, and this can be done combinatorially. One example for this is $g(f(f^{-1}(g^{-1}(a)))) = g(f(f^{-1}(Dc))) = g(f(x_0^{-1}A)) = g(\tilde{x}_0^{-1}Dc) = x_0^{-1}x_0^{-1}a \simeq a$ relative endvertices. Here, $\tilde{x}_0$ denotes the upper vertex of Γ. In the same way, we may prove this for the other edge of $\mathcal{R}_2$. With the described method, it is also easily seen that $f' \circ f'^{-1} \simeq \mathrm{id}_{\Gamma'}$ holds.

Now consider the homeomorphism

$$\begin{aligned} \phi\colon \Gamma &\to \Gamma' \\ c &\mapsto c' \\ d &\mapsto d' \\ e &\mapsto e' \end{aligned}$$

with its canonical continuous inverse. With this ϕ, we obviously get $\phi f \simeq f'$, so the two markings really are equivalent. To see that calling these two markings equivalent is plausible, we may look at a loop formed by going down the middle edge from the top and then going back over the left or right edge. In both markings, this loop represents a or $B \in F_n$, respectively (more precisely, it is mapped by both inverse markings to loops representing elements of the conjugacy class of a or B).

1.1.2 Metric Graphs

To define Outer space we need even more structure: we want not only a topology on the graph but also a metric. This leads to the following definition.

Definition 1.10. A *metric* on a finite graph Γ is a map $\ell\colon \mathrm{E}(\Gamma) \to (0, \infty)$. The value $\ell(e)$ is called the length of the edge $e \in \mathrm{E}(\Gamma)$, the volume of Γ is the sum over the lengths of all edges.

Given this map, we can view the graph Γ as metric space by treating the closed edges e as isometric images of closed intervals of length $\ell(e)$. Thus, we are able to assign a length to every immersed path that is contained in one single closed edge. Hence, we are able to assign a length to an immersed path in Γ by subdividing it into smaller paths such that each of them is contained in one closed edge, and summing over these lengths. The distance of two arbitrary points on the graph is then defined as the infimum over the lengths of all immersed paths from one point to the other. The topology induced by the metric is the same as the topology as CW complex.

Definition 1.11. Let $\omega\colon [a, b] \to X$ be a path in a metric space (X, d_X), then its *length* is defined as

$$||\omega|| := \sup_{a=t_0<t_1<\cdots<t_k=b} \sum_{l=1}^{k} d_X(\omega(t_{l-1}), \omega(t_l)),$$

where the supremum is taken over all finite partitions of $[a, b]$. A path is said to be *rectifiable* if its length is finite.

For two metric spaces $(G, d_G), (H, d_H)$ and a Lipschitz continuous map $f\colon G \to H$ we define $\sigma(f)$ as the smallest Lipschitz constant of f, i.e. as the smallest real number fulfilling

$$d_H(f(x), f(y)) \leq \sigma(f) d_G(x, y) \quad \text{for all } x, y \in G.$$

Lemma 1.12. *A Lipschitz continuous path $\omega\colon [a, b] \to X$ in a metric space (X, d_X) is always rectifiable and its length is bounded by $\sigma(\omega) \cdot (b - a)$.*

Proof. Let $\omega\colon [a, b] \to X$ be a Lipschitz continuous path, then we get

$$\begin{aligned} ||\omega|| &= \sup_{a=t_0<t_1<\cdots<t_k=b} \sum_{l=1}^{k} d_X(\omega(t_{l-1}), \omega(t_l)) \\ &\leq \sup_{a=t_0<t_1<\cdots<t_k=b} \sum_{l=1}^{k} \sigma(\omega) d_{[a,b]}(t_{l-1}, t_l) \\ &= \sigma(\omega)(b - a) < \infty. \end{aligned}$$

□

Remark. Every immersed path – and therefore every immersed loop – in a finite metric graph has finite length.

Proof. Let $\omega\colon [0, 1] \to \Gamma$ be continuous, then it is uniformly continuous by compactness of $[0, 1]$ and we may choose $\delta > 0$, such that

$$d_{[0,1]}(s, t) < \delta \implies d_\Gamma(\omega(s), \omega(t)) < \varepsilon \quad \text{for all } s, t \in [0, 1],$$

where $\varepsilon > 0$ is the length of the shortest edge in the finite graph Γ. Once entered an edge, the path ω has to traverse it completely or stop, as 180° turns contradict its immersion at the turning point. By subdividing $[0, 1]$ into finitely many parts of length smaller than δ we therefore can achieve that ω maps each of them to at most two edges of Γ. But then, the finite sum over the lengths of all these edge pairs is an upper bound for the path's length. □

Definition 1.13. Let $\phi\colon \Gamma \to \Gamma'$ be a continuous map between metric graphs and ω a non-trivial path in Γ. Then the *slope of ϕ along ω* is defined as the quotient of the lengths of $\phi \circ \omega$ and ω, i.e.

$$\text{slope}_\omega(\phi) := \frac{||\phi \circ \omega||}{||\omega||}.$$

If e is a closed edge of Γ, we say that ϕ has *constant slope on e* if for all non-trivial paths $\omega\colon [0, 1] \to e$ the slope of ϕ along ω is the same.

1.2 Outer Space

Combining the different types of graphs of the previous section we will consider finite marked metric graphs with volume 1 where all vertices have at least valence three, written as (Γ, ℓ, f). Two such triples $(\Gamma, \ell, f), (\Gamma', \ell', f')$ are called *equivalent* if there exists an isometry (i.e. an isometric isomorphism) $\phi\colon \Gamma \to \Gamma'$, such that $\phi f \simeq f'$. Finally, we are ready to give the definition of the space we are interested in.

Definition 1.14. The *Outer space* is defined as

$$\mathcal{X}_n := \{(\Gamma, \ell, f) \,|\, \text{triples as described above, where } \pi_1(\Gamma) \cong F_n\} \,/\!\sim,$$

where the equivalence relation is the one defined above and $n \in \mathbb{N}$.

If there is no ambiguity, we may–to shorten notation–write $\Gamma = (\Gamma, \ell, f) \in \mathcal{X}_n$ instead of $[(\Gamma, \ell, f)] \in \mathcal{X}_n$. For easier reading we will also often switch between equivalence classes and representatives without making this explicit. Since conjugacy classes in F_n correspond to free homotopy classes of free loops in $\mathcal{R}_n$, we may examine them by sending these loops via the marking to our graph Γ. To consider the simplest representatives possible, the following proposition comes in handy.

Proposition 1.15. *Every free homotopy class of a free loop in a graph has a unique immersed Lipschitz continuous representative (up to parametrization of S^1).*

Proof. Let Γ be a graph and ω be a loop in Γ. Define a metric on Γ such that every edge has length 1. Consider the open cover $\mathcal{U}$ of Γ that consists of the open edges and the open sets

$$B_{1/2}(v) = \{x \in \Gamma \,|\, d(x, v) < 1/2\}$$

for all vertices v in Γ. Then, the preimages of the open sets in $\mathcal{U}$ under ω define an open cover of S^1. Lebesgue's number lemma guarantees the existence of a number $\varepsilon > 0$ such that for every point $x \in S^1$ the neighbourhood $B_\varepsilon(x)$ is fully contained in one of those open sets. So let $t_1, \ldots, t_k \in S^1$ be finitely many points such that for all $1 \le i < k$ the image of the closed arc between t_i and t_{i+1} (denoted $[t_i, t_{i+1}]$) under ω is fully contained in one of the open sets of $\mathcal{U}$. As all sets in $\mathcal{U}$ are contractible, the loop ω is homotopic relative $t_1, \ldots, t_k$ to a loop ω' that is linear on $[t_i, t_{i+1}]$ for all i.

But then, the preimage $\widetilde{V}$ of $\mathrm{V}(\Gamma)$ under ω' is finite and therefore, we may piecewise linearize ω' via a homotopy fixing $\widetilde{V}$. This is true since ω' restricted to some arc in S^1 disjoint to $\widetilde{V}$ maps to exactly one open edge, which obviously is contractible. By reparametrisation, we therefore get a loop ω'' homotopic to ω and finitely many points $s_1, \ldots, s_{l+1}$ such that every $\omega''(s_i)$ is a vertex of Γ and ω'' linearly maps $[s_i, s_{i+1}]$ onto exactly one closed edge e_i of Γ for all i.

The loop ω'' is immersed if and only if for all $1 \le i < l$ we have $e_i \neq e_{i+1}^{-1}$ and $e_1 \neq e_l^{-1}$. This is equivalent to $e_1 e_2 \cdots e_l$ being cyclically reduced as word with letters $\mathrm{E}(\Gamma)$ and its inverses. Since ω'' is a free loop, we may replace any part of it corresponding to a one-step backtrack $e_i = e_{i+1}^{-1}$ or $e_1 = e_l^{-1}$ by a constant path without changing the free homotopy class. By cyclically reducing the word and simultaneously changing and reparametrising ω'' accordingly, we therefore really get an immersed representative of ω in Γ. This may be done in such a manner that after every step, the resulting loop is piecewise linear and therefore Lipschitz continuous.

We now want to show that this representative is–up to reparametrisation of S^1–unique. Therefore, let ν be another immersed representative of ω. Fixing a maximal tree T in Γ, we may represent both ω'' and ν by reduced words in the edges of $\Gamma - T$ and their inverses. Since ν is freely homotopic to ω'', the words representing them considered as elements of the free group $\pi_1(\Gamma)$ have to lie in the same conjugacy class. But as both words are cyclically reduced, the word for ν really has to be a shift of the word for ω''. Hence, ν can be obtained from ω'' by reparametrising S^1. □

In this proof we used the fact that every conjugacy class of a free group contains a cyclically reduced representative that is unique up to shifts. To cyclically reduce a word, we first reduce it. Now we remove the first and the last letter if they are inverse to one another and repeat this until this is not the case anymore. The result is a cyclically reduced word that lies in the same conjugacy class as the reduction of the original word. It is now easily seen that among words whose reduction is in the same conjugacy class, the cyclic reduction is unique up to shifts. If α is a conjugacy class in F_n, we write $\alpha|\Gamma\colon S^1 \to \Gamma$ for the unique immersed Lipschitz continuous representative of the free homotopy class $f(\alpha)$. Using this representative we can define the length of conjugacy classes, which will be very important for the definition of the Lipschitz distance in the second chapter.

Definition 1.16. The length $\ell_\Gamma(\alpha)$ for a non-trivial conjugacy class α in F_n is defined as the length of $\alpha|\Gamma$ in Γ, which can be calculated by summing over the lengths of all traversed edges of $\alpha|\Gamma$.

At last, we want to quickly sketch the right-action of the outer automorphisms on Outer space.

1.3 Outer Automorphisms

Definition 1.17. The *inner automorphisms* of the free group are defined as

$$\operatorname{Inn}(F_n) := \left\{ f \in \operatorname{Aut}(F_n) \,\middle|\, \exists z \in F_n : f(x) = z^{-1}xz \ \forall x \in F_n \right\},$$

which is a normal subgroup of $\operatorname{Aut}(F_n)$. The *outer automorphisms* are defined as the quotient group

$$\operatorname{Out}(F_n) := \operatorname{Aut}(F_n)/\operatorname{Inn}(F_n).$$

Two representatives of elements in $\operatorname{Out}(F_n)$ are equivalent if the images of each word differ only by conjugation with one *fixed* element of F_n. Therefore, a word in F_n is mapped by equivalent automorphisms to representatives of the same conjugacy class. As the representatives are automorphisms, we can–by the identification $\pi_1(\mathcal{R}_n) \cong F_n$

as always–view an element $\Phi \in \operatorname{Out}(F_n)$ as a homotopy equivalence between $\mathcal{R}_n$ and itself with homotopy inverse Φ^{-1}. Hence, there is a right action of $\operatorname{Out}(F_n)$ on Outer space: For an element $\Phi \in \operatorname{Out}(F_n)$, the map $f\Phi := f \circ \Phi$ is a homotopy equivalence, so the following is well-defined:

$$(\Gamma, \ell, f) \cdot \Phi := (\Gamma, \ell, f\Phi) \quad \text{for } (\Gamma, \ell, f) \in \mathcal{X}_n.$$

As it turns out, this action has some good properties like properness. From these, we already can retrieve information about the outer automorphisms, but as said before this is not part of this thesis.

Chapter 2

The Lipschitz Distance on Outer Space

This chapter mainly concerns the definition of the Lipschitz distance and some facts about it. In the definition there is an infimum, but we will show that this is in fact a minimum and moreover that we can reduce the set in whose minimum we are interested in by a large amount. In the last part of this chapter we will prove that the Lipschitz distance is an asymmetric metric.

If we have two points in Outer space $(\Gamma, \ell, f), (\Gamma', \ell', f') \in \mathcal{X}_n$, we may look at continuous maps $\phi\colon \Gamma \to \Gamma'$ with $\phi f \simeq f'$, even if they aren't isometries. Such maps form the basis for the Lipschitz distance and therefore get their own name.

Definition 2.1. Given $(\Gamma, \ell, f), (\Gamma', \ell', f') \in \mathcal{X}_n$, a Lipschitz continuous map $\phi\colon \Gamma \to \Gamma'$ with $\phi f \simeq f'$ is called a *difference of markings map.*

2.1 The Lipschitz Distance

Definition 2.2. For $\Gamma, \Gamma' \in \mathcal{X}_n$, the *Lipschitz distance* is defined as

$$d(\Gamma, \Gamma') := \inf_{\phi} \log \sigma(\phi),$$

where the infimum is taken over all difference of markings maps $\phi\colon \Gamma \to \Gamma'$.

Note that there always exists a map ϕ with $\phi f \simeq f'$, for example $\phi := f' \circ f^{-1}$, where f^{-1} is a homotopy inverse of the marking f. Like in the proof of Proposition 1.15, we may use Lebesgue's number lemma to piecewise linearize ϕ, obtaining a Lipschitz continuous map. Therefore, there always exist difference of markings maps and the infimum is not taken over the empty set. We will see in Proposition 2.8 that the Lipschitz constant is at least one for all difference of markings maps, so the infimum really exists. First, we make the following observation to reduce the number of maps we have to consider.

Proposition 2.3. *Let $\phi\colon \Gamma \to \Gamma'$ be a Lipschitz continuous map between $\Gamma, \Gamma' \in \mathcal{X}_n$. If $\tilde{\phi}\colon \Gamma \to \Gamma'$ has constant slope on all edges in Γ and $\phi \simeq \tilde{\phi}$ relative vertices, then $\sigma(\tilde{\phi}) \leq \sigma(\phi)$.*

Proof. Let the closed edge $e \in \mathrm{E}(\Gamma)$ be one of slope $\sigma(\tilde{\phi})$ under $\tilde{\phi}$. This has to exist because the Lipschitz constant equals the maximal slope and there are only finitely many edges. Consider the restrictions $\omega_1 := \tilde{\phi}|_e$ and $\omega_2 := \phi|_e$, which can be viewed as paths in Γ' between two points x and y. The first one is a linear immersed path, the latter one is homotopic relative endvertices and therefore has to traverse at least the same path as ω_1 (possibly plus some detours). This is seen as follows:

The universal cover $(\widetilde{\Gamma}', p)$ of the connected graph Γ' is a tree on which we fix some $\widetilde{x} \in p^{-1}(x)$. By the lifting property of this cover, the paths ω_1, ω_2 lift uniquely to paths $\widetilde{\omega}_1, \widetilde{\omega}_2$ in $\widetilde{\Gamma}'$ starting at $\widetilde{x}$, and as the homotopy relative endvertices between them also lifts uniquely, it follows $\widetilde{\omega}_1(1) = \widetilde{\omega}_2(1) =: \widetilde{y}$. As ω_1 is immersed, its lift $\widetilde{\omega}_1$ is an immersed path on the tree $\widetilde{\Gamma}'$ and therefore meets each point at most one time. Now assume that there exists some point $\widetilde{q} \neq \widetilde{x}, \widetilde{y}$ on this path which is not met by $\widetilde{\omega}_2$. Then, $\widetilde{\omega}_2$ connects $\widetilde{x}$ and $\widetilde{y}$ in $\widetilde{\Gamma}' - \widetilde{q}$. But as $\widetilde{\omega}_1$ passes $\widetilde{q}$ exactly once and $\widetilde{q}$ separates $\widetilde{\Gamma}'$ into two path components, $\widetilde{x}$ and $\widetilde{y}$ have to lie in different components of $\widetilde{\Gamma}' - \widetilde{q}$. ↯

Projecting back to Γ' leads to the stated fact.

As the paths are Lipschitz continuous, Lemma 1.12 implies that they are rectifiable and that the following holds:

$$\sigma(\tilde{\phi}) \cdot \ell(e) = ||\omega_1|| \leq ||\omega_2|| \leq \sigma(\phi) \cdot \ell(e).$$

Since e has positive length, the claim follows. □

This means that–in most cases–we can focus on maps with constant slope on the edges.

Definition 2.4. A difference of markings map ϕ is called *optimal* if it realizes the infimum of the Lipschitz distance and has constant slope on all edges.

2.2 Existence of Optimal Maps

For the existence of optimal maps we need some help from functional analysis, in particular the following standard theorem that is part of most textbooks in this area.

Theorem 2.5 (Arzelà-Ascoli, e.g. [MP78, pages 76-77])**.** *A subset $\mathcal{F} \subset C(X, Y)$ for a compact Hausdorff space X and a complete metric space Y has compact closure if and only if $\mathcal{F}$ is equicontinuous and the set $\{f(x) \mid f \in \mathcal{F}\}$ has compact closure in Y for all $x \in X$. The set $\mathcal{F}$ is called* equicontinuous *if for all $\varepsilon > 0$ and $x_0 \in X$ there exists a $\delta > 0$, such that*

$$d_X(x, x_0) < \delta \implies d_Y(f(x), f(x_0)) < \varepsilon \quad \textit{for all } f \in \mathcal{F}, x \in X.$$

The topology on $C(X,Y) = \{f\colon X \to Y \text{ continuous}\}$ we are talking about is induced by the metric

$$d(f,g) := \sup_{x\in X} d_Y(f(x),g(x)) \quad \text{for } f,g \in C(X,Y).$$

It is easy to see that a compact metric space has to be complete (if a Cauchy sequence has a convergent subsequence, then it is convergent itself). Since furthermore all metric spaces are Hausdorff and the closure of any subset of a compact set is compact, we may formulate a useful special case.

Corollary 2.6. *A subset $\mathcal{F} \subset C(X,Y)$ for two compact metric spaces X,Y has compact closure if and only if $\mathcal{F}$ is equicontinuous.*

This enables us to state the following proposition.

Proposition 2.7. *The infimum $\inf_\phi \log\sigma(\phi)$ over all $\phi\colon \Gamma \to \Gamma'$ difference of markings maps between (Γ,ℓ,f) and $(\Gamma',\ell',f') \in \mathcal{X}_n$ is realized.*

Proof. It suffices to show that $\inf_\phi \sigma(\phi)$ is realized, as we will see later that this infimum is always at least one (this will be proved without using this proposition). Define $c := \inf_\phi \sigma(\phi)$ and consider the set

$$F := \{\phi\colon \Gamma \to \Gamma' \mid \phi \text{ is } 2c\text{-Lipschitz continuous, } \phi f \simeq f'\} \subset C(\Gamma,\Gamma').$$

F is equicontinuous by construction and hence–by Corollary 2.6–the topological closure $\overline{F}$ is compact. We choose a sequence of difference of markings maps $\phi_k \in F$ with $\sigma(\phi_k) \xrightarrow{k\to\infty} c$ and get a convergent subsequence (ϕ_{k_l}) with $\phi_{k_l} \xrightarrow{l\to\infty} \phi \in \overline{F}$. Our next goal is to show the Lipschitz continuity of the limit ϕ.

Let $x,y \in \Gamma$ and $\varepsilon > 0$. Choose $l_0 \in \mathbb{N}$ such that we have $\sigma(\phi_{k_{l_0}}) \le c+\varepsilon$ and $d(\phi,\phi_{k_{l_0}}) \le \varepsilon \cdot d_\Gamma(x,y)$. With this l_0, it holds

$$\begin{aligned} d_{\Gamma'}(\phi(x),\phi(y)) &\le \underbrace{d_{\Gamma'}(\phi(x),\phi_{k_{l_0}}(x))}_{\le \varepsilon\cdot d_\Gamma(x,y)} + \underbrace{d_{\Gamma'}(\phi_{k_{l_0}}(x),\phi_{k_{l_0}}(y))}_{\le (c+\varepsilon) d_\Gamma(x,y)} + \underbrace{d_{\Gamma'}(\phi_{k_{l_0}}(y),\phi(y))}_{\le \varepsilon\cdot d_\Gamma(x,y)} \\ &\le (c+3\varepsilon)\cdot d_\Gamma(x,y). \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we get $\sigma(\phi) \le c$. To finish the proof, we have to show that $\phi f \simeq f'$ holds, since then $\phi \in F$ and because of that $\sigma(\phi) = c$.

Therefore, we choose a map $\tilde\phi$ in the sequence so that $d(\phi,\tilde\phi) < \varepsilon$, where $8\cdot\varepsilon > 0$ is smaller than the length of the shortest edge in Γ'. We now show $\phi \simeq \tilde\phi$, as this implies $\phi f \simeq \tilde\phi f \simeq f'$.

We define a homotopy H_1 from ϕ to ϕ_1 that moves the image of all vertices from their image under ϕ to that under $\tilde\phi$. Therefore, we choose a linear path ω_v between $\tilde\phi(v)$ and $\phi(v)$ for each vertex v in Γ and insert it at every edge starting at v such that the inserted part in Γ has sufficiently small length. Now that the length of each such path in Γ' is smaller than ε, we may do this in such a manner that the distance of ϕ and ϕ_1 is smaller than 3ε as seen in Appendix A.1. With this, we get $d(\tilde{\phi}, \phi_1) < 4\varepsilon =: \tilde{\varepsilon}$ since

$$d(\tilde{\phi}, \phi_1) \leq \underbrace{d(\tilde{\phi}, \phi)}_{<\varepsilon} + \underbrace{d(\phi, \phi_1)}_{\leq 3\varepsilon} < 4\varepsilon.$$

Due to the choice of ε, the upper bound for their distance $\tilde{\varepsilon}$ is smaller than half the length of the shortest edge in Γ'. Additionally, it holds $\phi_1(v) = \tilde{\phi}(v)$ for every vertex $v \in \mathrm{V}(\Gamma)$.

So we only have to consider the edges of Γ, i.e. we have to find a homotopy from ϕ_1 to $\tilde{\phi}$ relative vertices. Hence, let e be an edge of Γ, then we can view the two maps $\omega_1 := \tilde{\phi}|_{\overline{e}}$ and $\omega_2 := \phi_1|_{\overline{e}}$ as paths in Γ' with the same starting and ending point. Now consider the universal cover $p\colon \widetilde{\Gamma}' \to \Gamma'$ and lift the paths to $\widetilde{\omega}_1$ and $\widetilde{\omega}_2$ with the same starting point. If we are now able to show that their endpoints are the same, too, then they are homotopic relative endpoints because the universal cover is a tree. But then, this homotopy projects back to Γ' which leads edgewise to the homotopy from ϕ_1 to $\tilde{\phi}$, hence finishes the proof.

To accomplish our goal, we first define a metric on the universal cover in the obvious way and show that $d_{\widetilde{\Gamma}'}(\widetilde{\omega}_1, \widetilde{\omega}_2) < \tilde{\varepsilon}$. Assume not, then there exists a $t_0 \in (0, 1]$, such that $d_{\widetilde{\Gamma}'}(\widetilde{\omega}_1(t_0), \widetilde{\omega}_2(t_0)) = \tilde{\varepsilon}$ as both paths start at the same point. Now let ω be a path in Γ' between these points' projections $\omega_1(t_0)$ and $\omega_2(t_0)$ of length strictly smaller than $\tilde{\varepsilon}$, then this lifts to a path $\widetilde{\omega}$ in $\widetilde{\Gamma}'$ of the same length starting at $\widetilde{\omega}_1(t_0)$ and ending in a point P that maps to $\omega_2(t_0)$. In particular, the distance between P and $\widetilde{\omega}_1(t_0)$ is different from that between $\widetilde{\omega}_2(t_0)$ and $\widetilde{\omega}_1(t_0)$, so $P \neq \widetilde{\omega}_2(t_0)$, but it also holds

$$d_{\widetilde{\Gamma}'}(P, \widetilde{\omega}_2(t_0)) \leq \underbrace{d_{\widetilde{\Gamma}'}(P, \widetilde{\omega}_1(t_0))}_{<\tilde{\varepsilon}} + \underbrace{d_{\widetilde{\Gamma}'}(\widetilde{\omega}_1(t_0), \widetilde{\omega}_2(t_0))}_{=\tilde{\varepsilon}} < 2\tilde{\varepsilon}.$$

We therefore get two distinct points on the universal cover with distance strictly smaller than one edge length that are mapped to $\omega_2(t_0)$, which is not possible. ↯

In particular, we get $d_{\widetilde{\Gamma}'}(\widetilde{\omega}_1(1), \widetilde{\omega}_2(1)) < \tilde{\varepsilon}$, which implies because of the choice of $\tilde{\varepsilon}$ and the fact that $p(\widetilde{\omega}_1(1)) = p(\widetilde{\omega}_2(1))$ indeed $\widetilde{\omega}_1(1) = \widetilde{\omega}_2(1)$. □

Combined with Proposition 2.3 this shows that optimal maps always exist, which will make all following considerations much easier.

2.3 Properties of the Lipschitz Distance

The next proposition states that the Lipschitz distance is an asymmetric metric which behaves well under the right-action of the outer automorphisms.

Proposition 2.8. *(i)* $d(\Gamma_1, \Gamma_3) \leq d(\Gamma_1, \Gamma_2) + d(\Gamma_2, \Gamma_3)$ *for all* $\Gamma_i \in \mathcal{X}_n$

(ii) $d(\Gamma\Phi, \Gamma'\Phi) = d(\Gamma, \Gamma')$ *for all* $\Gamma, \Gamma' \in \mathcal{X}_n, \Phi \in F_n$

(iii) $d(\Gamma, \Gamma') \geq 0$ *for all* $\Gamma, \Gamma' \in \mathcal{X}_n$*, equality implies* $\Gamma = \Gamma'$.

To prove it, we need the following lemma.

Lemma 2.9. *Homotopy equivalences between finite connected graphs without vertices of valence* ≤ 1 *are always surjective.*

Proof. Let $\phi \colon \Gamma \to \Gamma'$ be a homotopy equivalence between finite connected graphs Γ, Γ' without vertices of valence ≤ 1. Assume $\phi(\Gamma) \subsetneq \Gamma'$, then let ϕ^{-1} be a homotopy inverse of ϕ and p a point in $\Gamma' - \phi(\Gamma)$. As the image of ϕ is compact and therefore closed in Γ', we even may assume p to lie on some open edge e. Now consider the subgraph Δ of Γ' which is defined as the union of all closed edges of Γ' except $\overline{e}$. We now show that $\Gamma' - \{p\}$ is connected, as this implies that Δ is connected, too.

Assume not, so there are exactly two disjoint connected components Γ'_1, Γ'_2 of $\Gamma' - \{p\}$. As $\phi(\Gamma)$ is connected, it is completely contained in one of the components, so assume that it is contained in Γ'_1. We have $\phi \circ \phi^{-1} \simeq \mathrm{id}_{\Gamma'}$, so for every free loop α in Γ', the loop $\phi \circ \phi^{-1} \circ \alpha$ is a representative of the free homotopy class of α that is fully contained in Γ'_1. Hence, Γ'_2 has to be contractible, which is not possible because there would be at least one vertex of Γ' with valence one. ↯

Therefore, Δ is connected and we may choose a maximal tree T in it. Since the valence of every vertex in Γ' is at least two, the subgraph Δ contains all vertices of Γ', so T is also a maximal tree in Γ'. Then, the fundamental group of Γ' is free on the edges in $\Gamma' - T$, so let β be a loop that is represented by the word e. The loop $\gamma := \phi \circ \phi^{-1} \circ \beta$ is freely homotopic to β, hence the word w representing this loop has to lie in the same conjugacy class as the word e. But as ϕ does not meet p, the path γ cannot traverse e completely and therefore, the letter e cannot be contained in w. This, however, is not possible since all conjugates of the word e contain the letter e at least once. ↯ □

Proof of Proposition 2.8. (i) In general, for Lipschitz continuous maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ between metric spaces X, Y, Z, the following is true for all $x, y \in X$:

$$d_Z\left(g \circ f(x), g \circ f(y)\right) \leq \sigma(g) d_Y(f(x), f(y)) \leq \sigma(g)\sigma(f) d_X(x, y).$$

Let $\Gamma_1 \xrightarrow{\phi_1} \Gamma_2$ and $\Gamma_2 \xrightarrow{\phi_2} \Gamma_3$ be optimal maps. Since $\phi_2 \circ \phi_1$ is a difference of markings map for Γ_1 and Γ_3, this leads to

$$\begin{aligned} d(\Gamma_1, \Gamma_3) &= \inf_{\phi} \log \sigma(\phi) \leq \log \sigma(\phi_2 \circ \phi_1) \leq \log(\sigma(\phi_2)\sigma(\phi_1)) \\ &= \log \sigma(\phi_2) + \log \sigma(\phi_1) = d(\Gamma_2, \Gamma_3) + d(\Gamma_1, \Gamma_2). \end{aligned}$$

(ii) Let f, f' be the markings of Γ, Γ' and $\nu \colon \Gamma \to \Gamma'$ be an optimal map, so that $\nu f \simeq f'$. But then also $\nu f \Phi \simeq f' \Phi$ holds and hence, ν is a difference of markings map for $\Gamma\Phi$ and $\Gamma'\Phi$. This implies $d(\Gamma\Phi, \Gamma'\Phi) \leq d(\Gamma, \Gamma')$ and the same argument for Φ^{-1} (Φ is an automorphism) leads to

$$d(\Gamma, \Gamma') = d(\Gamma\Phi\Phi^{-1}, \Gamma'\Phi\Phi^{-1}) \leq d(\Gamma\Phi, \Gamma'\Phi) \leq d(\Gamma, \Gamma'),$$

which means the inequalities are in fact equalities.

(iii) Let $\Gamma, \Gamma' \in \mathcal{X}_n$ and $\phi\colon \Gamma \to \Gamma'$ a difference of markings map. Assume that the maximal slope of ϕ is strictly smaller than one. Then it holds

$$\mathrm{vol}(\mathrm{im}(\phi)) \leq \sum_{e \in \mathrm{E}(\Gamma)} |\phi(e)| < \sum_{e \in \mathrm{E}(\Gamma)} 1 \cdot |e| = 1 = \mathrm{vol}(\Gamma'),$$

so ϕ is not surjective (the definition of volume is canonically extended to not only subgraphs but connected sub*sets* of graphs). Furthermore, the homotopy equivalence ϕf from the rose to Γ' is not surjective, but this is a contradiction to Lemma 2.9 (recall that in the definition of Outer space we only allowed graphs whose vertices have valence at least 3). $\lightning$ So the maximal slope of any difference of markings map is at least one and consequently, we have $d(\Gamma, \Gamma') \geq \log(1) = 0$.

Now, let $d(\Gamma, \Gamma') = 0$ and $\phi\colon \Gamma \to \Gamma'$ be an optimal map with $\sigma(\phi) = 1$. This implies in fact that all slopes are equal to one, since else $\mathrm{vol}(\mathrm{im}(\phi)) < 1$ like in the previous paragraph. The images of two distinct closed edges meet only in finitely many points as else the volume would decrease, too. This implies in fact that every vertex of Γ has to be mapped to a vertex in Γ' since the minimal valence in Γ is three. The intersection of the images of two distinct open edges cannot contain a point of an open edge of Γ' since then it would have non-zero volume. This is true as ϕ is locally injective (constant slope) and hence the intersection would contain some non-empty ε-ball. With similar argumentation, the map ϕ restricted to an edge can meet every point on an open edge of Γ' at most one time. Altogether, the graph Γ' results from Γ by identifying finitely many points through ϕ. We now want to show that identifying points of Γ increases the rank, contradicting the fact that the graphs are homotopy equivalent. As the number of identified points is finite, it suffices to show that the rank of Γ' is strictly greater than the rank of Γ in the case that ϕ identifies exactly two distinct points.

If $a \neq b \in \Gamma$ are those two points, we choose a maximal tree T in Γ with $a \in T$. In the case $b \notin T$, the point b is no vertex, the edge containing b is not in T and $\phi(T)$ is still contractible. The last fact is true because $\phi|_T$ is injective, T is compact and Γ' is Hausdorff, implying that T is homeomorphic to $\phi(T)$. The image of b under ϕ is a vertex and the edge e containing b splits under ϕ into two edges. This means that the image of e contains two edges of Γ', so $\Gamma' - \phi(T)$ contains one edge more than $\Gamma - T$. But as $\phi(T)$ still contains all vertices of Γ', it is even a maximal tree and therefore, the identification increased the rank of the graph.

If $b \in T$, the path along T from a to b projects under ϕ to a cycle. Since $\phi(T)$ still contains all vertices of Γ', we have to reduce it by at least one edge to get a maximal tree in Γ', so the rank increased, too.

Thus, no points are identified, ϕ is an isometry and therefore $\Gamma = \Gamma'$ in $\mathcal{X}_n$. $\square$

Chapter 3

Calculating the Distance of Two Graphs

This chapter concerns calculating the distance of two concrete points in Outer space. From the definition, we get an easy way to calculate an upper bound: we take an arbitrary difference of markings map, calculate the Lipschitz constant and take the logarithm of this number. The next proposition shows a way to determine a lower bound, and if the two bounds agree, we have calculated the distance.

Proposition 3.1. *If α is a non-trivial conjugacy class in F_n, then*

$$\log \frac{\ell_{\Gamma'}(\alpha)}{\ell_{\Gamma}(\alpha)} \leq d(\Gamma, \Gamma') \quad \textit{for } \Gamma, \Gamma' \in \mathcal{X}_n.$$

Proof. Let $\phi\colon \Gamma \to \Gamma'$ be an optimal map between $\Gamma, \Gamma' \in \mathcal{X}_n$ and α a non-trivial conjugacy class in F_n. Then, the formula is equivalent to $\ell_{\Gamma'}(\alpha) \leq \sigma(\phi) \cdot \ell_{\Gamma}(\alpha)$. The loop $\phi \circ \alpha|\Gamma$ is a Lipschitz continuous representative of α in Γ' and therefore rectifiable. Since it may not be immersed, this immediately leads to

$$\ell_{\Gamma'}(\alpha) \leq ||\phi(\alpha|\Gamma)|| \leq \sigma(\phi) \cdot ||\alpha|\Gamma|| = \sigma(\phi) \cdot \ell_{\Gamma}(\alpha).$$

□

Example 3.2. We want to compare the distances $d(G, H_\varepsilon)$ and $d(H_\varepsilon, G)$ for the graphs in Figure 3.1 which we will consider as equivalently marked, so the left edges of both marked graphs represent the same conjugacy class in F_n. Of course, the same is true for the right edges. Considering the canonical difference of markings map $\phi\colon G \to H_\varepsilon$ that sends the left edge of G with constant slope to the left edge of H and the same for the right edge, we get the slopes 2ε on the left and $2 - 2\varepsilon$ on the right edge, so we know

$$d(G, H_\varepsilon) \leq \log \max\{2\varepsilon, 2 - 2\varepsilon\}.$$

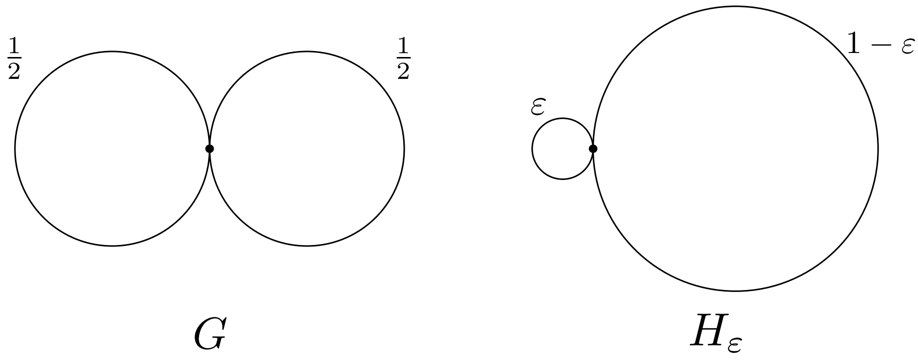

Figure 3.1: The graphs G and H_ε for $0 < \varepsilon < 1$

Now consider the conjugacy class α_l in F_2 which is represented both in G and H_ε by the left loop. With the previous proposition we get

$$d(G, H_\varepsilon) \geq \log \frac{\ell_{H_\varepsilon}(\alpha_l)}{\ell_G(\alpha_l)} = \log 2\varepsilon.$$

By considering the conjugacy class α_r which represents the right loop, we get $d(G, H_\varepsilon) \geq \log(2 - 2\varepsilon)$. In total, this leads to

$$d(G, H_\varepsilon) = \log \max\{2\varepsilon, 2 - 2\varepsilon\} < \log 2 \quad \text{for all } 0 < \varepsilon < 1.$$

For the distance $d(H_\varepsilon, G)$ we consider once again the conjugacy class α_l and get

$$d(H_\varepsilon, G) \geq \log \frac{\ell_G(\alpha)}{\ell_{H_\varepsilon}(\alpha)} = \log \frac{1}{2\varepsilon} \xrightarrow{\varepsilon \to 0} \infty.$$

This shows that the Lipschitz distance is no metric due to its lack of symmetry. Furthermore, we see that the asymmetry isn't even bounded by any constant factor, so the Lipschitz distance can be considered *very* asymmetric. By considering α_r and the canonical difference of markings map we even can calculate the distance explicitly, yielding

$$d(H_\varepsilon, G) = \log \max \left\{ \frac{1}{2\varepsilon}, \frac{1}{2 - 2\varepsilon} \right\}.$$

In general, it is very complicated to guess an optimal map so that it would be convenient if we could skip this step. This is indeed the case and with the next theorem we formulate the main result of this chapter: there is an easy way to effectively compute distances in Outer space.

Theorem 3.3 (White)**.** *Let* $\Gamma, \Gamma' \in \mathcal{X}_n$. *There always exists a conjugacy class* α *in* F_n *so that*

$$\log \frac{\ell_{\Gamma'}(\alpha)}{\ell_\Gamma(\alpha)} = d(\Gamma, \Gamma').$$

Such conjugacy classes are called *witnesses*. The proof of the theorem comes later in this chapter, it makes use of an additional structure on the graph Γ which shall be described in the next section.

3.1 Train Track Structure

Let G, H be metric graphs without any further restrictions, $x \in G$ an arbitrary point and $\phi\colon G \to H$ a continuous map with constant slope on the edges of G. We now define the set of directions at $x \in G$ named $T_x(G)$ as germs of isometries from $[0, \varepsilon]$ to G starting at x, so two such isometries $[0, \varepsilon_i] \to G$ represent the same germ if and only if they agree on an interval $[0, \varepsilon]$ with $0 < \varepsilon \leq \min\{\varepsilon_1, \varepsilon_2\}$. For a vertex $v \in G$, our map ϕ naturally induces a mapping

$$\phi_*\colon T_v(G) \to T_{\phi(v)}(H)$$

since for a direction $\nu \in T_v(G)$, $\phi \circ \nu$ is an isometry stretched by the slope of the corresponding edge (choose ε smaller than the length of the edge) and hence can be identified with a direction at $\phi(v)$ by reparametrisation. So we are able to define the following equivalence relation on $T_v(G)$:

$$d_1 \sim d_2 :\iff \phi_*(d_1) = \phi_*(d_2) \quad \text{for directions } d_1, d_2 \in T_v(G). \tag{3.1}$$

Informally two directions are *equivalent* if their images under ϕ point into the same direction. With this construction in mind, we define the following structure on G.

Definition 3.4. For all vertices v of the graph G, let $\sim_v$ be an equivalence relation on the germs of isometries $T_v(G)$. Then the collection of equivalence classes in $T_v(G)/{\sim_v}$ for all vertices v is called a *train track structure on* G. The equivalence classes are called *gates*.

So by construction, the map $\phi\colon G \to H$ considered earlier defines a train track structure on G by the equivalence relations (3.1) for every vertex v of G. If we visualize our graphs, it is convenient to draw the ends of edges that define equivalent directions as tangent to each other. We now want to consider an immersed path $\omega\colon [a, b] \to G$. Every $t_0 \in [a, b]$ that gets mapped to a vertex is called a *turn*. For every turn, we consider the two maps

$$\begin{aligned} i_\pm\colon [0, \varepsilon] &\to G \\ s &\mapsto \omega(t_0 \pm s) \end{aligned}$$

for an appropriate $\varepsilon > 0$. Since these maps are locally injective, they can be identified with isometries and therefore define a gate.

Definition 3.5. For every turn, the gate defined by i_- is called *entering gate*, the one defined by i_+ *exiting gate* of the turn. If the gates are distinct, the turn is called *legal*, otherwise it is called *illegal*. A path is called *legal* if every turn is legal.

3.2 The Existence of Witnesses

After this short section about train track structures, we are now able to give the proof of the theorem.

Definition 3.6. For an optimal map $\phi\colon \Gamma \to \Gamma'$ the *tension graph* Δ_ϕ is defined as the subgraph of Γ which contains exactly those edges where the slope of ϕ is maximal.

Since ϕ has constant slope on its edges, the tension graph is equipped with a train track structure as introduced in the previous section. The whole construction was only necessary for the following lemma, which obviously is very important for the proof of the theorem.

Lemma 3.7. *Let $\Gamma, \Gamma' \in \mathcal{X}_n$ with $d(\Gamma, \Gamma') = \log \lambda$ and $\phi\colon \Gamma \to \Gamma'$ an optimal map. If for a non-trivial conjugacy class α in F_n the immersed representative $\alpha|\Gamma$ is contained in the tension graph Δ_ϕ and is legal, then*

$$\frac{\ell_{\Gamma'}(\alpha)}{\ell_\Gamma(\alpha)} = \lambda.$$

Proof. Since the slope on the tension graph is constantly λ, we just need to show that $\phi \circ \alpha|\Gamma$ is immersed in Γ'. This loop $\phi \circ \alpha|\Gamma =: \beta$ is a representative of α in Γ' that is locally injective at all points $t \in S^1$ that $\alpha|\Gamma$ doesn't map to a vertex. This is true since we can choose a neighbourhood U of t in which $\alpha|\Gamma$ is injective and doesn't pass a vertex. The restriction of ϕ to $\alpha|\Gamma(U)$ then has constant slope, so by shrinking U we may assume that $\phi|_{\alpha|\Gamma(U)}$ is injective. Restricted to this neighbourhood, β is injective, hence β is locally injective at t. But β also has to be locally injective at all $t \in S^1$ that get mapped to vertices of Γ due to legality of the turns, thus $\phi \circ \alpha|\Gamma$ is immersed. □

Inspired from this lemma, we want to prove the existence of legal loops with as few restrictions as possible. The visualisation of the following proof can be viewed in Figure 3.2.

Lemma 3.8. *If every vertex of a graph G with train structure has at least two gates, then G contains a legal loop in the form of either*

- *an embedded circle traversing each edge at most once*
- *the red loop in the fourth image of Figure 3.2, traversing the edges on the circles exactly once and those between the circles (there don't have to be any) in each direction exactly once.*

These types of loops are called candidates.

Proof. Construct an arbitrary legal path (by extending under consideration of the gates) until it intersects itself and consider only the cyclic part of this path, starting at the intersection. This is possible since there are at least two gates at every vertex. If the path is legal we are finished, so let's assume it is not. Then continue until the path intersects itself for the second time. If the path uses three distinct gates at the second intersection, a legal embedded circle is easily found. But even if not, we can construct a legal loop as follows. By following the path from the beginning, it is divided by the intersections into three parts, to which we will refer to as paths a, b, c. In the case that the two intersections coincide, let a,c be the two parts of the path and b the constant path at the intersection. If the second intersection is on the embedded circle, either bc (first image of Figure 3.2) or $a^{-1}c$ (second image) is legal, producing a legal circle. If not, then either c or $b^{-1}cba$ is legal (third and fourth image). □

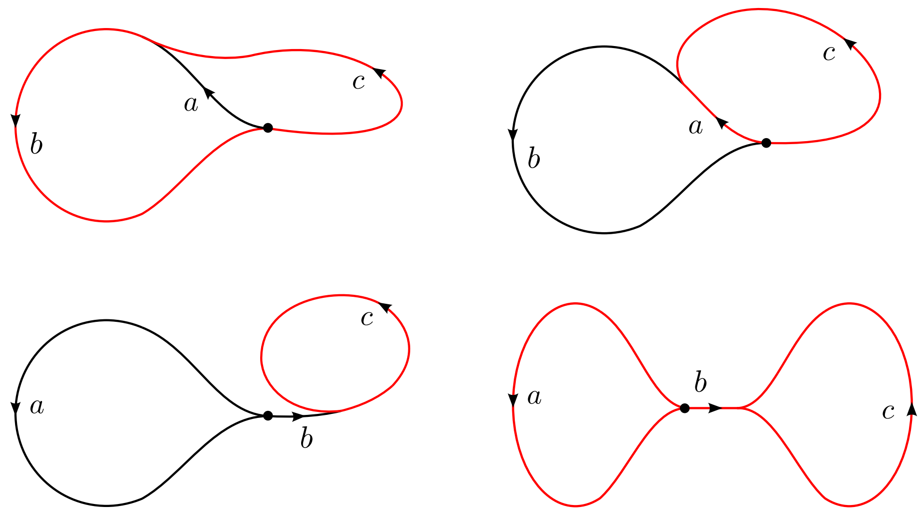

Figure 3.2: Possible legal loops in the proof of Lemma 3.8, the starting point is marked as black dot, the four cases are in the same order as in the proof

We are now ready to prove the theorem.

Proof of Theorem 3.3. With the two lemmata from above we only have to show that there exists an optimal map where every vertex of the tension graph has at least two gates. Let ϕ be an arbitrary optimal map with Lipschitz constant λ. If it has at least two gates at every vertex in the tension graph, we are finished, so assume that there exists a vertex $v \in \Delta_\phi$ with only one gate. Then, we try to manipulate ϕ in such a way that this vertex won't be in the tension graph of the thereby created difference of markings map $\tilde{\phi}$.

For that, we define a homotopy $H \colon \Gamma \times [0,1] \to \Gamma'$ from our ϕ to another difference of markings map $\tilde{\phi}$. This map H is stationary on all vertices but v, which is moved slightly in the direction $\phi_*(d_v)$ where $d_v \in T_v(\Delta_\phi)$ is one representative of the only gate. We do this linearly, i.e. for all $t \in [0,1]$ the map $H(-,t)$ is linear on all edges. Therefore, the slope on edges not incident to v is not changed. If an edge is incident to v and contained in the tension graph, the slope of ϕ on this edge decreased due to the movement of $\phi(v)$ and hence is strictly smaller than λ. Among the edges incident to v that aren't contained in Δ_ϕ, there may be some on which the slope of ϕ increased because of this perturbation, but as these are only finitely many we can make this adjustment small enough so that the slope of $\tilde{\phi}$ on those edges is still strictly less than λ.

All together, we now have a difference of markings map $\tilde{\phi}$ which has constant slope on the edges and where the maximal slope is $\leq \lambda$. Since λ already is the smallest possible Lipschitz constant, $\tilde{\phi}$ is even an optimal map. Furthermore, we have $\Delta_{\tilde{\phi}} \subsetneq \Delta_\phi$ since v and its incident edges aren't contained in the new tension graph.

As λ is the minimal Lipschitz constant for difference of markings maps between Γ and Γ', we cannot remove all the vertices from the tension graph of ϕ by this operation, and because there are only finitely many vertices, we eventually get an optimal map whose tension graph has at least two gates at every vertex. □

We hereby proved that witnesses always exist. Furthermore, we have seen that we can compute the distance $d(\Gamma, \Gamma')$ by finding all non-trivial candidates in Γ and taking the logarithm of the maximum ratio of lengths in Γ' and Γ for those finitely many loops, i.e.

$$d(\Gamma, \Gamma') = \log \max_{\substack{\alpha|\Gamma \\ \text{candidate}}} \frac{\ell_{\Gamma'}(\alpha)}{\ell_\Gamma(\alpha)}.$$

Since in every graph in Outer space the length of a conjugacy class $[w]$ is equal to the length of $[w^{-1}]$, we in fact only have to consider half the amount of candidates.

3.3 Exemplary Computation of the Lipschitz Distance

After all this theory, we want to actually compute the two distances between a pair of points in Outer space with this method. We will hereby see that even for small graphs we have to consider a relatively big number of candidates, as finite doesn't necessarily mean few.

Example 3.9. Let R_3 be the 3-rose with inverse marking a, b, c on the edges and edge length $\frac{1}{3}$ each. We now calculate the distance from this graph to the graph Γ described in Figure 3.3. First, we have to reconstruct the marking from the given inverse marking. We therefore name the edges of Γ so that the inverse marking is

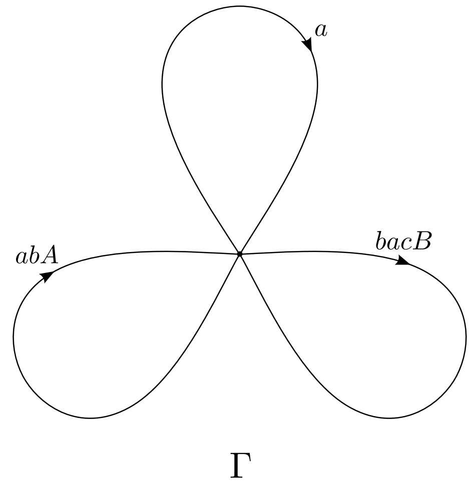

Figure 3.3: A graph with edge length $\frac{1}{3}$ each. The edge on top is called d, those at the bottom e (left) and f (right).

given by

$$\begin{aligned} f^{-1} \colon \Gamma &\to R_3 \\ d &\mapsto a \\ e &\mapsto abA \\ f &\mapsto bacB. \end{aligned}$$

We now claim that the marking is given by

$$\begin{aligned} f \colon R_3 &\to \Gamma \\ a &\mapsto d \\ b &\mapsto Ded \\ c &\mapsto D^2 EdfDed. \end{aligned}$$

We easily calculate

$$f(abA) = dDedD \simeq e \text{ and } f(bacB) = Ded^2 D^2 EdfDedDEd \simeq f$$

relative endvertices. With $f(a) = d$, this implies that $f \circ f^{-1} \simeq \mathrm{id}_\Gamma$, the other way round is also easy. The marking of R_3 is obvious, so we may skip that. Computing the distance $d(R_3, \Gamma)$ means finding all candidates in R_3 and calculating the corresponding ratios. All these candidates look like those in Figure 3.4, so we consider those for different combinations of edges. Hereby, we don't have to consider candidates that are inverses of candidates we already took into account. For each possible form, there are three different candidates in the graph. In the tables, they are listed in the same order as in Figure 3.4.

candidate in R_3	immersed loop in Γ	ℓ_Γ/ℓ_{R_3}	
a	d	$\frac{1}{3}/\frac{1}{3} = 1$	
b	e	$\frac{1}{3}/\frac{1}{3} = 1$	
c	$DEdfDe$	$\frac{6}{3}/\frac{1}{3} = 6$	$*$
ba	ed	$\frac{2}{3}/\frac{2}{3} = 1$	
ca	f	$\frac{1}{3}/\frac{2}{3} = \frac{1}{2}$	
cb	$DEdfDe^2$	$\frac{7}{3}/\frac{2}{3} = \frac{7}{2}$	
Ba	Ed	$\frac{2}{3}/\frac{2}{3} = 1$	
Ca	$EdFDed^2$	$\frac{7}{3}/\frac{2}{3} = \frac{7}{2}$	
Cb	$FDed^2$	$\frac{5}{3}/\frac{2}{3} = \frac{5}{2}$	

Table 3.1: Ratios of candidates in R_3

candidate in Γ	immersed loop in R_3	ℓ_{R_3}/ℓ_Γ	
d	a	$\frac{1}{3}/\frac{1}{3} = 1$	
e	b	$\frac{1}{3}/\frac{1}{3} = 1$	
f	ac	$\frac{2}{3}/\frac{1}{3} = 2$	
ed	ab	$\frac{2}{3}/\frac{2}{3} = 1$	
fd	$bacBa$	$\frac{5}{3}/\frac{2}{3} = \frac{5}{2}$	
fe	$bacBabA$	$\frac{7}{3}/\frac{2}{3} = \frac{7}{2}$	$*$
Ed	aB	$\frac{2}{3}/\frac{2}{3} = 1$	
Fd	$bCABa$	$\frac{5}{3}/\frac{2}{3} = \frac{5}{2}$	
Fe	$bCABabA$	$\frac{7}{3}/\frac{2}{3} = \frac{7}{2}$	$*$

Table 3.2: Ratios of candidates in Γ

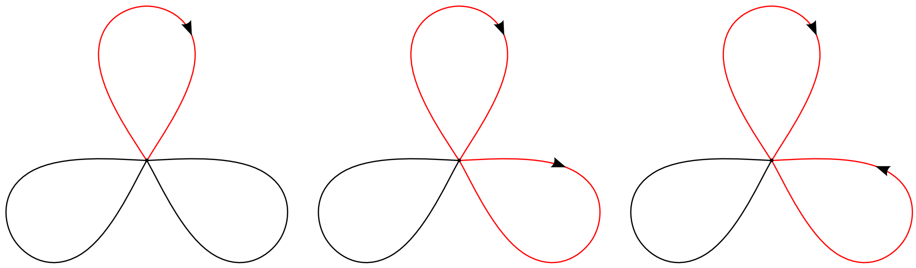

Figure 3.4: Possible candidates in R_3 (up to permutation of the edges)

Table 3.1 leads to the result that the maximal ratio is 6 and therefore we have

$$d(R_3, \Gamma) = \log 6.$$

For the other distance we have to consider candidates in Γ, which have the same possible forms as those in R_3. From Table 3.2 we can conclude that

$$d(\Gamma, R_3) = \log \frac{7}{2},$$

which shows once more the asymmetry of the Lipschitz distance. We also see that there are even two different candidates–namely fe and Fe–which are witnesses of this distance.

Conclusion

In this thesis we examined the metric structure of Outer space. After defining the distance function we have seen that there always exist optimal maps, and we therefore can discard all difference of markings maps that aren't linear on the edges. As there are still many of them, we wanted to make the computation even simpler and proved the fact that there always exist witnesses in certain forms. We therefore showed the validity of the alternative definition

$$d(\Gamma, \Gamma') = \max_{\substack{\alpha|\Gamma \\ \text{candidate}}} \log \frac{\ell_{\Gamma'}(\alpha)}{\ell_{\Gamma}(\alpha)} \qquad \text{for } \Gamma, \Gamma' \in \mathcal{X}_n.$$

The crucial fact is that the candidates in Γ are only finitely many.

With these facts in mind, we now have the basics to consider other properties of this distance. For instance, we could wonder if we can define a topology induced by this asymmetric metric and examine if it is equivalent to other topologies defined on Outer space (e.g. the simplicial topology). Interesting is also the question how we can symmetrize the Lipschitz distance to get a proper metric, because if we do this in a certain manner, the induced topology is indeed the simplicial topology. One could also think about completeness, which is different when considering the asymmetric or the symmetric metric. Another topic not covered in this thesis is the fact that there exists a deformation retract of the whole space on which the Lipschitz distance *is* quasi-symmetric, i.e. there exist universal constants that constrain the difference between $d(\Gamma, \Gamma')$ and $d(\Gamma', \Gamma)$.

Most of these and many other results are covered in [FM11], which provides a good starting point to getting useful information about outer automorphisms by examining the Lipschitz distance.

Acknowledgements

I would like to thank Professor Holger Reich for his expert guidance and his helpful answers to all questions that arised while writing this thesis.

Bibliography

[Bes12] Mladen Bestvina. PCMI Lectures on the geometry of Outer space. August 2012.

[CV86] Marc Culler and Karen Vogtmann. Moduli of graphs and automorphisms of free groups. *Invent. Math*, 84(1):91–119, 1986.

[FM11] Stefano Francaviglia and Armando Martino. Metric properties of Outer space. *Pub. Mat.*, 55:433–473, 2011.

[MP78] A. Mukherjea and K. Pothoven. *Real and Functional Analysis.* Plenum Press, New York and London, 1978.

Appendix

A.1 Explicit Homotopies

Here we give the explicit homotopy H_1 for the proof of Proposition 2.7, so all notation relates to that proof. Let $\varepsilon' := \min\{\frac{\varepsilon}{\sigma(\phi)}, \frac{1}{4}\}$ and choose for every vertex v of Γ a linear path $\omega_v\colon [0,\varepsilon'] \to \Gamma'$ from $\tilde{\phi}(v)$ to $\phi(v)$ of length at most ε. We don't define ϕ_1 but rather the restriction $\phi_1|_e$ for every closed edge e in Γ. Let v_1, v_2 be the endvertices of e. Identifying the edge with $[0,1]$, the map is then given by

$$\begin{aligned} \phi_1|_e\colon [0,1] &\to \Gamma' \\ s &\mapsto \begin{cases} \omega_{v_1}(s) & 0 \le s < \varepsilon' \\ \phi(g(s)) & \varepsilon' \le s \le 1-\varepsilon' \\ \omega_{v_2}(1-s) & 1-\varepsilon' < s \le 1, \end{cases} \end{aligned}$$

where $g\colon [0,1] \to [0,1]$ is the continuous reparametrisation $g(s) = \frac{s-\varepsilon'}{1-2\varepsilon'}$. Since $\omega_{v_1}(\varepsilon') = \phi(g(\varepsilon'))$ and $\omega_{v_2}(1-\varepsilon') = \phi(g(1-\varepsilon'))$, the map is continuous. Two edges sharing a vertex map it to the same element in Γ', so this indeed defines a continuous map ϕ_1 on the whole graph Γ.

We now show that this map is homotopic to ϕ by defining a homotopy on each closed edge. With the same notational conventions as before, define

$$\begin{aligned} H_1|_e\colon [0,1]\times[0,1] &\to \Gamma' \\ (s,t) &\mapsto \begin{cases} \omega_{v_1}(s+(1-t)\varepsilon') & 0 \le s < t\varepsilon' \\ \phi(g(s,t)) & t\varepsilon' \le s \le 1-t\varepsilon' \\ \omega_{v_2}(1-s+(1-t)\varepsilon') & 1-t\varepsilon' < s \le 1, \end{cases} \end{aligned}$$

with the continuous map $g\colon [0,1]^2 \to [0,1], (s,t) \mapsto \frac{s-t\varepsilon'}{1-2t\varepsilon'}$ for all $t \in [0,1]$. The map $H_1|_e$ is easily checked to be continuous. The movement of the endvertices' images does only depend on the parameter t and the vertex, so all these maps together indeed define a homotopy H_1 on the whole graph. Being $H_1(-,0) = \phi$ and $H_1(-,1) = \phi_1$

this leads to $\phi \simeq \phi_1$.

At last, we want to show that $d(\phi, \phi_1) \leq 3\varepsilon$ holds. So let e be a closed edge of Γ and $s \in [0,1]$ one point on this edge (with the obvious identification as before). If $s \in [0, \varepsilon']$, we get $d_\Gamma(s, 0) \leq \varepsilon' \cdot \ell(e) \leq \varepsilon'$ and therefore

$$d_{\Gamma'}(\phi(s), \phi_1(s)) \leq \underbrace{d_{\Gamma'}(\phi(s), \phi(0))}_{\leq \varepsilon' \cdot \sigma(\phi) \leq \varepsilon} + \underbrace{d_{\Gamma'}(\phi(0), \phi_1(0))}_{= d_{\Gamma'}(\phi(0), \tilde{\phi}(0)) < \varepsilon} + \underbrace{d_{\Gamma'}(\phi_1(0), \phi_1(s))}_{\leq ||\omega_{v_1}|| \leq \varepsilon} < 3\varepsilon.$$

A similar argument shows the same for $s \in [1 - \varepsilon', 1]$, so let $s \in (\varepsilon', 1 - \varepsilon')$. If we consider $s, t \in [0, 1]$ as elements of the edge e, we have $d_\Gamma(s, t) = d_{[0,1]}(s, t) \cdot \ell(e)$, which leads to

$$\begin{aligned}
d_{\Gamma'}(\phi(s), \phi_1(s)) &= d_{\Gamma'}(\phi(s), \phi(g(s))\\
&\leq \sigma(\phi) \cdot d_{[0,1]}(s, g(s)) \cdot \ell(e)\\
&\leq \sigma(\phi) \cdot |g(s) - s|\\
&= \sigma(\phi) \cdot \left| \frac{\varepsilon'(2s-1)}{1-2\varepsilon'} \right|\\
&\leq \frac{\varepsilon}{1-2\varepsilon'} \cdot \underbrace{|(2s-1)|}_{<1-2\varepsilon'}\\
&< \varepsilon.
\end{aligned}$$

Consequently, this implies $d(\phi, \phi_1) \leq 3\varepsilon$.