Berlin Mathematical School

Configuration Spaces in Algebraic Topology

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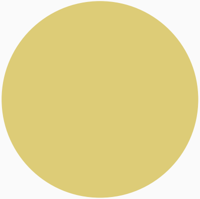

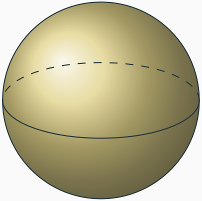

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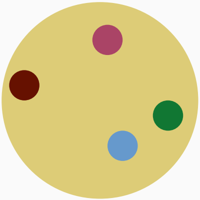

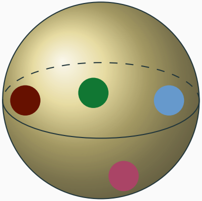

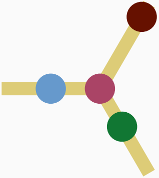

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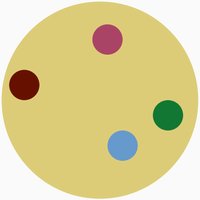
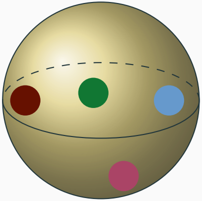
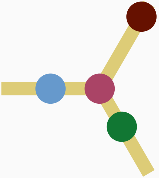
A configuration of n particles in X is given by choosing n distinct points in X, describing the positions of the particles. The set of all these configurations is called the n-th ordered configuration space of X.







What is a configuration space?

Definition

Let X be a topological space and $n \in \mathbb{N}$. The *n-th ordered configuration space of X* is the set

$$\mathrm{Conf}_n(X) := \left\{(x_1, \ldots, x_n) \,|\, x_i \neq x_j \text{ for } i \neq j\right\} \subset X^n$$

endowed with the subspace topology.

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The symmetric group Σ_n acts by changing labels, the *n-th unordered configuration space of X* is the quotient

$$\mathrm{UConf}_n(X) := \mathrm{Conf}_n(X)/\Sigma_n$$

with the quotient topology.

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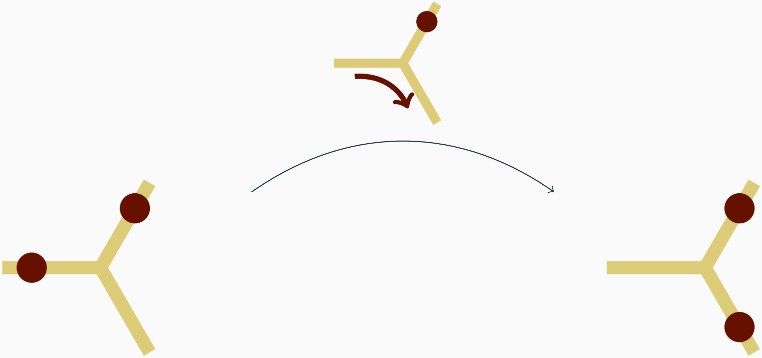

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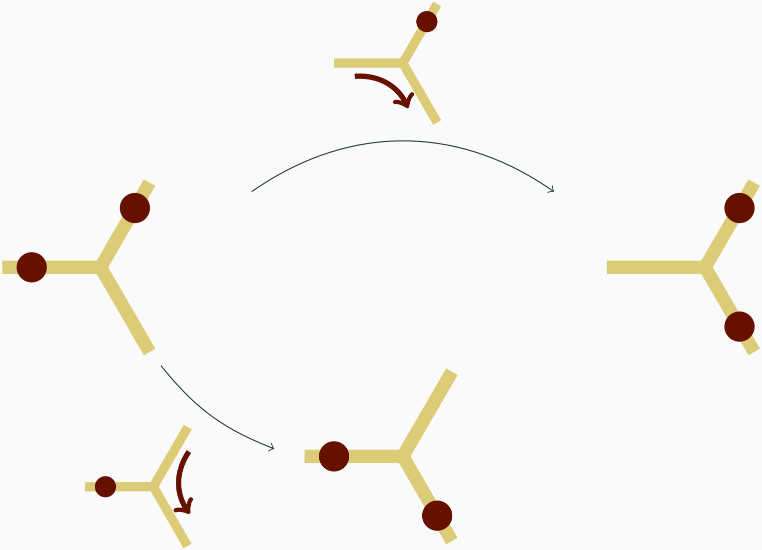

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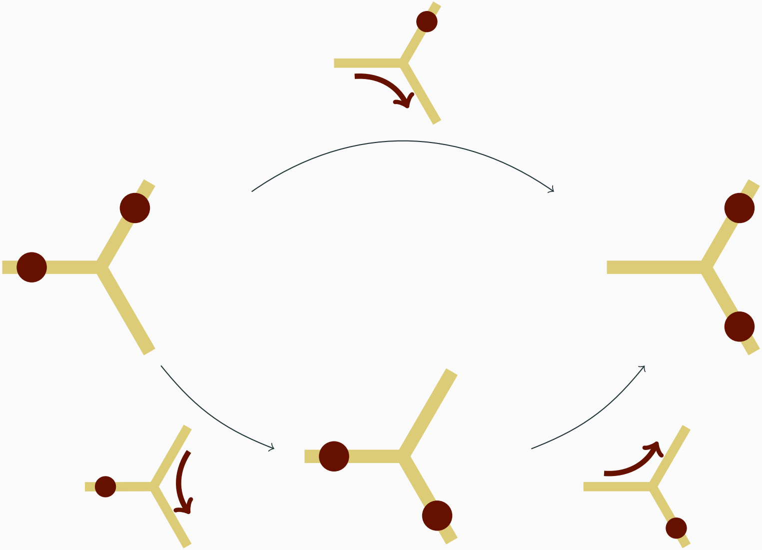

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Two paths are called homotopic if we can continuously deform one into the other. How many different homotopy classes of paths connecting two configurations are there?

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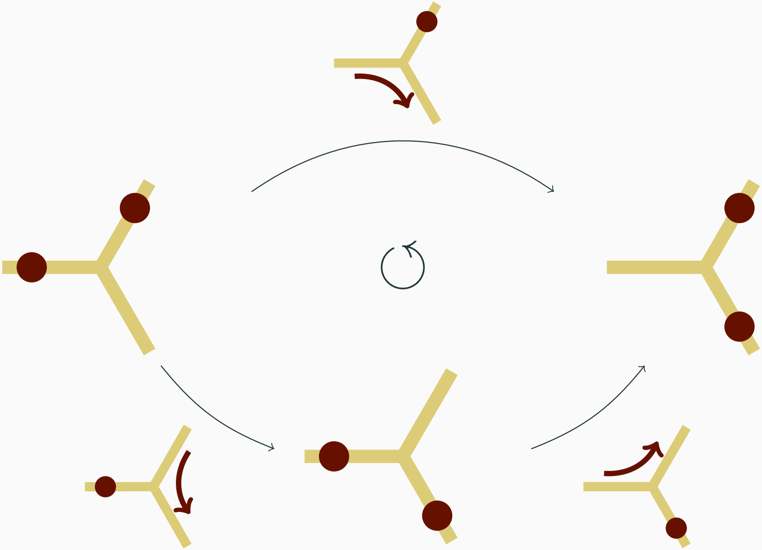

What can topology tell us about configuration spaces?

The difference of two such paths is a *closed loop* in the configuration space. We are therefore interested in the fundamental group $\pi_1(\mathsf{UConf}_n(X))$ of $\mathsf{UConf}_n(X)$, which is the set of homotopy classes of closed loops starting in a fixed configuration.

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One can show that $\pi_1(\mathrm{UConf}_2(Y)) \cong \mathbb{Z}$ is generated by the loop described above. This shows that every path as above (up to homotopy) can be written as a sum of those two paths.

What can topology tell us about configuration spaces?

An invariant that is easier to calculate and that still captures this fact is singular homology: the first homology group $H_1(\mathrm{UConf}_2(Y)) \cong \mathbb{Z}$ is generated by the homology class of the loop constructed above.

What can topology tell us about configuration spaces?

Meta question

What can we say about the singular homology groups $H_i(\mathrm{Conf}_n(X))$ and $H_i(\mathrm{UConf}_n(X))$ in dependence on the space X and the natural numbers i and n?

Configuration spaces of graphs

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However, for $n \to \infty$ the dimension of $H_1(\mathrm{Conf}_n(D^2))$ grows polynomially, whereas even for small graphs G (like star graphs) the dimension of $H_1(\mathrm{Conf}_n(G))$ grows factorially.

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For a triple $L \subset G \cap K$ of finite graphs denote by G_k the result of gluing k copies of K onto G along L. The space G_k admits an action of the symmetric group Σ_k, so $H_*(\mathrm{Conf}_n(G_k))$ is a Σ_k-representation.

Graph stabilization

Theorem (L)
Assume that either $i \leq 2$ or that G and K are trees. Then for each $n \in \mathbb{N}$ the sequence $k \mapsto H_i(\mathrm{Conf}_n(G_k))$ satisfies representation stability.

Read as
Assume that either $i \leq 2$ or that G and K are trees. Then for each n the homology groups $H_i(\mathrm{Conf}_n(G_k))$ for all $k \in \mathbb{N}$ can be computed by a finite computation. Additionally, the dimension of $H_i(\mathrm{Conf}_n(G_k))$ is eventually polynomial in k.

Particle stabilization

Theorem (L)

Let G be a finite 3-vertex connected graph with at least four essential vertices and without self-loops. Then the FI-module $n \mapsto H^1(Conf_n(G))$ is finitely generated in degree 2.

Read as

Let G be a graph with many paths between any pair of vertices. Then the homology groups $H_1(\mathrm{Conf}_n(G))$ for all $n \in \mathbb{N}$ can be determined by a finite computation. Additionally, the dimension of $H_1(\mathrm{Conf}_n(G))$ is eventually polynomial in n.

Thanks for listening!