SEMINAR: ELLIPTISCHE FUNKTIONEN UND MODULFORMEN (HS13)

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Organisation

Participants should attend all talks and are strongly encouraged to ask questions. Each participant will be assigned a topic to present in the seminar (see below, for a brief summary of each topic). Each seminar will last 90 minutes and you should prepare written notes for your presentation. The written notes for your talk should be submitted to me two weeks before the date of your presentation for checking. We will then meet to discuss any necessary changes.

Seminar time: Wednesdays, 13-15hr (Y27 H28).

Summary of topics

(1) Meromorphe Funktionen (Seiten 152-158 in [1]).
We give $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ a topology and define a notion of meromorphic functions on open subsets $D \subset \bar{\mathbb{C}}$; we show they form a field $\mathfrak{M}(D)$. We define the Laurent expansion at ∞ and give notions of ∞ being a removable singularity, pole or essential singularity. We prove every meromorphic function on $\bar{\mathbb{C}}$ has a non-essential singularity at ∞ and that every analytic function $f : \bar{\mathbb{C}} \to \mathbb{C}$ is constant. We view S^2 as a model for $\bar{\mathbb{C}}$ and call it the Riemann sphere. A Möbius tranformation is a type of bijective mapping of the Riemann sphere to itself; we prove these form a group isomorphic to $PGL(2, \mathbb{C})$ (the quotient of $GL(2, \mathbb{C})$ by the subgroup of scalar matrices $\{tI : t \neq 0\}$) which is also isomorphic to the set of (meromorphic) automorphisms of $\bar{\mathbb{C}}$.

(2) Satz von Mittag-Leffler (Seiten 166-171 in [2]).
Given a discrete subset of points P in an open set $U \subset \mathbb{C}$ and $h_a(z) \in \mathbb{C}[1/(z-a)]$ for each $a \in P$ (so h_a looks like the principal part of a Laurent expansion of a meromorphic function at a pole a), the Mittag-Leffler theorem states there is a meromorphic function on U whose poles are precisely the set P and furthermore the principal part of the Laurent expansion of f at a is h_a. Thus the Mittag-Leffler theorem gives us a meromorphic function with prescribed poles.

(3) Weierstrasscher Produktsatz (Seiten 172-177 in [2]).
Given a discrete subset of points $N \subset \mathbb{C}$ and orders of vanishing $n_a \geq 1$ for each $a \in N$, we ask if there is an entire function (that is, holomorphic on the complex plane) whose zero set is precisely N and the order of each zero $a \in N$ is n_a. If f is a solution, then we see that $e^h f$ is also a solution for any other entire function h. For finite zero sets N, one can immediately construct a polynomial with the prescribed zeros. When N is infinite, if we order the zeros $a_i \in N$ so that $0 = |a_0| < |a_1| \leq |a_2| \leq \dots$, then the Weierstrass factorization theorem gives a solution as an infinite product of 'elementary factors' $u_i(z)$ indexed by the zeros $a_i \in N$.

(4) Liouvillesche Sätze (Seiten 251-260 in [1]).
An elliptic function is a meromorphic function on the complex plane such that $f(z + w) = f(z)$ for all points w in a lattice $L = \mathbb{Z}w_1 + \mathbb{Z}w_2 \subset \mathbb{C}$ where w_i are $\mathbb{R}$-linearly independent. By the first Liouville theorem, every elliptic function without poles is constant. The second and third theorems concern the residues of an elliptic function at its poles and the 'order' of an elliptic function. It then follows that an elliptic function has an equal number of poles and zeros (counting modulo the lattice) and there are no order 1 elliptic functions. The quotient $\mathbb{C}/L$ can be geometrically realised as a torus (i.e.

a compact Riemann surface of genus 1) and we can view elliptic functions as functions on $\mathbb{C}/L$.

(5) Weierstrassche $\wp$-Funktion (Seiten 262-271 in [1]).

We now construct the most basic non-constant elliptic function, the Weierstrass $\wp$-Function, which has a single pole of order 2 at the lattice points (i.e. $\wp$ has pole order 2). The derivative of $\wp$ is computed and is shown to be an order 3 elliptic function; by Liouville's third theorem $\wp'$ has three zeros (modulo the lattice) which we compute explicitly. The Laurent expansion of $\wp$ at $z = 0$ is computed and the coefficients are so-called 'Eisenstein series' G_k. Finally, the field of elliptic functions $K(L)$ for a fixed lattice L is shown to be given by $K(L) = \mathbb{C}(\wp) + \mathbb{C}(\wp)\wp'$. In fact, after we have proved the Weierstrass $\wp$-Function satisfies a certain differential equation in the next talk, we shall see that we have an isomorphism $K(L) \cong \mathbb{C}(X)[Y]/ < Y^2 - 4X^3 + g_2X + g_3 >$ where $\wp \mapsto X$ and $\wp' \mapsto Y$.

(6) Tori und ebene elliptische Kurven (Seiten 272-279 in [1]).

We firstly prove that the Weierstrass $\wp$-Function satisfies a differential equation: $\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$ where $g_2 := g_2(L)$ and $g_3 := g_3(L)$ are constants determined by the lattice L. Complex projective space $\mathbb{P}^n_{\mathbb{C}}$ is the set of punctured lines through the origin in $\mathbb{C}^{n+1}$; we can also give it a topology and see it is as a complex manifold. We refer to $\mathbb{P}^1_{\mathbb{C}}$ as the Riemann sphere (or projective line) which as a set is equal to $\bar{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. Notions of affine plane curves $X \subset \mathbb{C}^2$ and projective plane curves $\tilde{X} \subset \mathbb{P}^2_{\mathbb{C}}$ are given. We define a bijective map from $\mathbb{C}/L - \{[0]\}$ to an affine plane curve $X(g_2, g_3) \subset \mathbb{C}^2$ and show that this can be extended to a bijection from $\mathbb{C}/L$ to a projective completion $\tilde{X}(g_2, g_3) \subset \mathbb{P}^2_{\mathbb{C}}$ of $X(g_2, g_3)$. We refer to $\tilde{X}(g_2, g_3)$ as the elliptic curve (associated to the lattice L).

(7) Additionstheorem (Seiten 281-285 in [1]).

Firstly, the addition theorem for the Weierstrass $\wp$-Function is stated as an equation relating $\wp(z + w)$ with $\wp(z)$ and $\wp(w)$. From this, we deduce a formula for $\wp(2z)$. As the torus $\mathbb{C}/L$ is an additive group, we can use the bijection $\mathbb{C}/L \to \tilde{X}(g_2, g_3)$ to give the elliptic curve a group structure. Then a geometric reformulation of the addition theorem states if $a, b, c \in \tilde{X}(g_2, g_3)$ satisfy $a + b + c = 0$, then they lie on a straight line $M \cong \mathbb{P}^1_{\mathbb{C}}$ in $\tilde{X}(g_2, g_3)$. The initial formulation of the addition theorem is then deduced from this geometric formulation and a proof of the geometric formulation is given using a theorem of Abel about the zeros and poles of elliptic functions (see Topic (9)).

(8) Elliptische Integrale und Perioden (Seiten 287-292 in [1]).

Given a degree 3 or 4 polynomial P without repeated roots, we consider the associated 'elliptic integral' of the first kind: $g(z) := \int_a^z 1/\sqrt{P(t)}dt$ (historically such integrals were used to compute arc lengths of ellipses). The main theorem states that the inverse function of such an elliptic integral is an elliptic function f. We reduce from degree four to degree three and then to the case when $P(t) = 4t^3 - g_2t - g_3$ where g_2 and g_3 are constants determined by a lattice L. Using the differential equation for the Weierstrass $\wp$-function, we deduce that $f = \wp$ is the inverse of the elliptic integral defined by the above P. The theory of elliptic functions (e.g. the addition formula) is applied to study elliptic integrals. Finally we geometrically justify the claim that given (g_2, g_3) satisfying $\Delta := g_2^3 - 27g_3^2 \neq 0$, there is a lattice L such that $(g_2, g_3) = (g_2(L), g_3(L))$.

(9) Abelsches Theorem (Seiten 294-302 in [1]).

The theorem of Abel describes the permitted zeros and poles (modulo L) of an elliptic function. We start with a simpler case: consider the quotient of two polynomials as a function from the Riemann sphere to itself $f(z) = P(z)/Q(z) : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$, then f has the same number of poles as zeros (counted with multiplicity). Abel's theorem says there is an elliptic function with prescribed zeros a_i and poles b_i for $i = 1, \ldots, n$ if and only if $a_1 + \cdots + a_n \equiv b_1 + \cdots + b_n \bmod L$. The hardest direction of the proof - the construction of an elliptic function with prescribed poles satisfying the above congruence relation -

consumes most of the section and two proofs are given: one after Weierstrass and the other via theta functions.

(10) Gitter und elliptische Modulgruppe (Seiten 305-312 in [1]).

We define an equivalence relation on lattices by $L \sim L'$ if $L' = aL$ for $a \neq 0$; we see every L is equivalent to a lattice $L_\tau := \mathbb{Z} + \mathbb{Z}\tau$ where $\tau \in \mathbb{H} = \{z : \mathrm{Im} z > 0\}$. Every $M \in \mathrm{GL}(2, \mathbb{R})$ with positive determinant defines a conformal mapping $M : \mathbb{H} \to \mathbb{H}$. The modular group $\Gamma := \mathrm{SL}(2, \mathbb{Z})$ acts on the upper half plane $\mathbb{H}$ so that $\mathbb{Z} + \mathbb{Z}\tau \sim \mathbb{Z} + \mathbb{Z}\tau'$ where $\tau, \tau' \in \mathbb{H}$ if and only if $\tau' = M\tau$ for $M \in \Gamma$. We return to the problem of showing that given (g_2, g_3) satisfying $\Delta := g_2^3 - 27g_3^2 \neq 0$, there is a lattice L such that $(g_2, g_3) = (g_2(L), g_3(L))$. For $a \neq 0$, we relate $g_i(L)$ with $g_i(aL)$ for $i = 2, 3$ and construct an 'absolute j-invariant' for lattices as a rational function of the g_is so that $j(L) = j(aL)$. As j is constant on equivalence classes of lattices and every lattice is equivalent to some L_τ for $\tau \in \mathbb{H}$, we view it as a function $j : \mathbb{H} \to \mathbb{C}$.

(11) Eisensteinreihen und Modulfunktionen (Seiten 313-319 in [1]).

We recall that the Eisenstein series $G_k(L_\tau)$ appear as the coefficients in the Laurent expansion at 0 of the Weierstrass $\wp$-function associated to the lattice L_τ where $\tau \in \mathbb{H}$. The first result in this section is that the Eisenstein series G_k are analytic functions on $\mathbb{H}$. We study how they transform under the modular group Γ and their limit as $\tau \to \infty$. We prove the j-invariant is a modular function (that is, an analytic function on $\mathbb{H}$ which is Γ-invariant) - only the analytic part remains to be checked. We also prove that there is a 'Modulfigur' $\mathcal{F} \subset \mathbb{H}$ such that every point in $\mathbb{H}$ is mapped to a point in $\mathcal{F}$ by an element $M \in \Gamma$. Finally, the surjectivity of the j-invariant $j : \mathbb{H} \to \mathbb{C}$ is proved.

(12) Fundamentalbereich der Modulgruppe (Seiten 321-329 in [1]).

We recall that the modular group Γ acts on $\mathbb{H}$ and we write Γ_p for the stabiliser of a point $p \in \mathbb{H}$ under this action. We say p is an elliptic fixed point of Γ is this stabiliser contains an element other than $\pm I$. Recall that the upper half plane $\mathbb{H}$ is covered by translates $M\mathcal{F}$ of the 'Modulfigur' $\mathcal{F} \subset \mathbb{H}$ as M ranges over Γ; we next classify the elements $M \in \Gamma$, such that $\mathcal{F} \cap M\mathcal{F} \neq \emptyset$ and also describe this intersection. Finally we see that for $M \in \Gamma$ the following are equivalent: M has a fixed point in $\mathbb{H}$; M had finite order; and either $M = \pm I$ or M is an 'elliptic element' of M (that is, $|\mathrm{Tr}(M)| < 2$). Hence, the elliptic fixed points in $\mathbb{H}$ correspond precisely to the elliptic elements of Γ.

(13) Komplexe Fourierreihen und k/12-Formel (Seiten 147-150, 330-335 in [1]).

Generalising the notion of a modular function, we define a (weight k) modular form to be an analytic function $f : \mathbb{H} \to \mathbb{C}$ which satisfies some equation (depending on k) describing how f transforms under the modular group Γ and is 'holomorphic as $z \to i\infty$'. We can also speak about meromorphic modular forms which are meromorphic functions from $\mathbb{H}$ to the Riemann sphere $\bar{\mathbb{C}}$. Every (meromorphic) modular form has a Fourier expansion and associated Laurent expansion in a punctured disc around zero; we use this to extend the modular form to $\mathbb{H} \cup \{i\infty\}$ and require that $i\infty$ is a removable singularity or perhaps a pole (if the modular form is meromorphic). The $k/12$-formula can be seen as an analogue of Liouville's theorem (that an elliptic function has an equal number of poles and zeros): more precisely, we see that for a non-zero modular function of order k, a weighted sum over the orders of all poles and zeros in $\mathbb{H} \cup \{i\infty\}$ (modulo Γ) is $k/12$.

(14) Algebra der Modulformen (Seiten 335-342 in [1]).

Using the $k/12$-formula we prove: every modular form of negative weight is zero; every weight zero modular form is constant and every modular form of positive weight has at least one zero in $\mathbb{H} \cup \{i\infty\}$. The Eisenstein series G_{2k} are weight $2k$ modular forms for $k \geq 2$ and we prove that G_4 and G_6 have exactly one first order zero each. We also use the $k/12$-formula to show that the j-invariant $j : \mathbb{H}/\Gamma \to \mathbb{C}$ is bijective. We prove the field of modular functions (which are precisely the weight zero modular forms) is equal to $\mathbb{C}(j)$. Finally, we study the $\mathbb{C}$-algebra $\mathcal{A}(\Gamma)$ of all modular forms (graded by order). We use monomials in G_4 and G_6 to give a finite $\mathbb{C}$-basis of the space of weight k modular forms and in particular calculate the dimension of this space. The culmination of these results is that $\mathcal{A}(\Gamma)$ is a polynomial ring in two variables corresponding to the Eisenstein series G_4 and G_6.

(15) Moduli von eindimensionalen komplexen Tori.

The conclusion of our seminar is a description of the moduli space of 1-dimensional complex tori. A lattice $L \subset \mathbb{C}$ defines a torus $\mathbb{C}/L$ and two tori are biholomorphic if and only if their corresponding lattices are equivalent (i.e. related by a non-zero scalar). Therefore, we consider the set of equivalence classes of lattices and try to give this set a geometric description. We have already seen that every lattice is equivalent to a lattice $\mathbb{Z} \oplus \mathbb{Z}\tau$ for $\tau \in \mathbb{H}$ and, moreover, the modular group $\Gamma := \mathrm{SL}(2,\mathbb{Z})$ acts on $\mathbb{H}$ such that the orbits correspond to equivalence classes of lattices. Hence, we consider the 'moduli space' $\mathcal{M} := \mathbb{H}/\Gamma$ which is a 1-dimensional complex manifold (the Γ-action is not free, but is properly discontinuous). Finally, we see that the j invariant yields a biholomorphism $\mathcal{M} \to \mathbb{C}$.

References.

[1] D. Freitag, R. Busam *Funktionentheorie* (1993)
[2] W. Fischer, I. Lieb *Funktionentheorie* (1994)

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