Algebra III (Algebraic Geometry II) WS15-16
Fachbereich Mathematik und Informatik
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Algebraic Geometry II

Exercise Sheet 8

Hand-in date: 10am, Monday 7th December.

Exercise 1. Let $\mathbb{G}_m$ act on $\mathbb{P}^3$ linearly by the representation

$$t \mapsto \operatorname{diag}(t, t, t^{-1}, t^{-1}).$$

For this action, calculate the $\mathbb{G}_m$-invariant subring in the homogeneous coordinate ring $R(\mathbb{P}^n) = k[x_0, x_1, x_2, x_3]$ and determine the semistable and stable sets for the action. Then explicitly write down the GIT quotient morphism.

Exercise 2. Prove that the notion of a good (respectively geometric) quotient is local with respect to the target: that is, if an affine algebraic group G acts on a scheme X and we have a G-invariant morphism $\varphi : X \to Y$ with an open cover U_i of Y such that $\varphi|_i : \varphi^{-1}(U_i) \to U_i$ are good (respectively geometric) quotients, then φ is a good (respectively geometric) quotient.

Exercise 3. Let G be a smooth affine algebraic group and let $H \subset G$ be a smooth closed subgroup. Throughout this exercise, we assume that the characteristic of k is zero for simplicity (although the results also hold in positive characteristic).

a) Show that there exists a finite dimensional representation $G \to \mathrm{GL}(V)$ and a line $L \subset V$ such that the stabiliser G_L of L (i.e. the stabiliser of $[L] \in \mathbb{P}(V)$) is H.

b) Show that there is a smooth quasi-projective variety G/H whose k-points are the $H(k)$-cosets in $G(k)$ and there is a quotient morphism $\pi : G \to G/H$ which sends $g \mapsto gH$. Furthermore, show that π is smooth and open. One can also show that π is a geometric quotient; we will assume this fact for the rest of the sheet.

c) If H is reductive, prove that $G \mapsto G/\!/H$ is a geometric quotient. Deduce that G/H is affine in this case.

d) Let $\mathbb{G}_a \subset \mathrm{SL}_2$ be embedded in the upper triangular unipotent subgroup. Prove that $\mathcal{O}(\mathrm{SL}_2)^{\mathbb{G}_a}$ is a polynomial algebra in two variables and $\mathrm{SL}_2/\mathbb{G}_a$ is quasi-affine variety equal to the image of the morphism $\mathrm{SL}_2 \to \operatorname{Spec} \mathcal{O}(\mathrm{SL}_2)^{\mathbb{G}_a}$.

Some hints for Exercise 3 are given on the following page.

Hints for Exercise 3.

a) Consider G acting on itself by left multiplication. Take a finite dimensional H-invariant subspace $W \subset I(H)$ containing generators for this ideal, and prove for $g \in G(k)$ that $g \cdot W \subset W$ if and only if $g \in H(k)$; then using the assumption that we are in characteristic zero, conclude that the stabiliser of W (i.e. the subgroup of g in G such that $g \cdot W \subset W$) is equal to H. Next take a finite dimensional G-invariant subspace W_G containing W and choose V to be a G-representation constructed as an exterior power of W_G.

b) Use part a) and construct G/H as an orbit of an action.