Algebra III (Algebraic Geometry II) WS15-16
Fachbereich Mathematik und Informatik
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Algebraic Geometry II

Exercise Sheet 7

Hand-in date: 10am, Monday 30th November.

Exercise 1. Let G be a linearly reductive group. For rational actions of G on k-vector spaces (not necessarily finite dimensional), show that taking G-invariants is an exact functor. Deduce that if G acts on an affine scheme Y and $X \subset Y$ is a closed G-invariant subscheme, then $X//G$ is a closed subscheme of $Y//G$.

Exercise 2. Consider $\mathbb{G}_m$ acting on the quasi-affine variety $X := \mathbb{A}^2 - \{0\}$ by

$$t \cdot (x, x) = (tx, t^{-1}y).$$

Prove that this action does not have a geometric quotient in the category of varieties, but it has a geometric quotient in the category of schemes.

[Hint: the notion of a geometric quotient is local in the target, so cover the target by two open affine sets Y_i such that the preimages X_i are also affine and that the morphisms $X_i \to Y_i$ is an affine GIT quotient.]

Exercise 3. Let k be a field of characteristic $p > 0$ and let $C_p := \mathbb{Z}/p\mathbb{Z}$ be the cyclic group of order p. Let η be a generator for this group and consider the action of C_p on $\mathbb{A}^2$ by

$$\eta \cdot (x, y) = (x + y, y).$$

Prove that C_p is not linearly reductive, but there is a invariant homogeneous polynomial which is non-vanishing at the fixed point $(1, 0)$.