MODULI PROBLEMS AND GEOMETRIC INVARIANT THEORY

VICTORIA HOSKINS

Abstract

In this course, we study moduli problems in algebraic geometry and the construction of moduli spaces using geometric invariant theory. We start by giving the definitions of coarse and fine moduli spaces, with an emphasis on examples. We then explain how to construct group quotients in algebraic geometry via geometric invariant theory. Finally, we apply these techniques to construct moduli spaces of projective hypersurfaces and moduli spaces of semistable vector bundles on a smooth projective curve.

Contents

1. Introduction

In this course, we study moduli problems in algebraic geometry and constructions of moduli spaces using geometric invariant theory. A moduli problem is essentially a classification problem: we want to classify certain geometric objects up to some notion of equivalence (key examples are vector bundles on a fixed variety up to isomorphism or hypersurfaces in $\mathbb{P}^n$ up to projective transformations). We are also interested in understanding how these objects deform in families and this information is encoded in a moduli functor. An ideal solution to a moduli problem is a (fine) moduli space, which is a scheme that represents this functor. However, there are many simple moduli problems which do not admit such a solution. Often we must restrict our attention to well-behaved objects to construct a moduli space. Typically the construction of moduli spaces is given by taking a group quotient of a parameter space, where the orbits correspond to the equivalence classes of objects.

Geometric invariant theory (GIT) is a method for constructing group quotients in algebraic geometry and it is frequently used to construct moduli spaces. The core of this course is the construction of GIT quotients. Eventually we return to our original motivation of moduli problems and construct moduli spaces using GIT. We complete the course by constructing moduli spaces of projective hypersurfaces and moduli spaces of (semistable) vector bundles over a smooth complex projective curve.

Let us recall the quotient construction in topology: given a group G acting on a topological space X, we can give the orbit space $X/G := \{G \cdot x : x \in X\}$ the quotient topology, so that the quotient map $\pi : X \to X/G$ is continuous. In particular, π gives a quotient in the category of topological spaces. More generally, we can suppose G is a Lie group and X has the structure of a smooth manifold. In this case, the quotient X/G will not always have the structure of a smooth manifold (for example, the presence of non-closed orbits, usually gives a non-Hausdorff quotient). However, if G acts properly and freely, then X/G has a smooth manifold structure, such that π is a smooth submersion.

In this course, we are interested in actions of an affine algebraic group G (that is, an affine scheme with a group structure such that multiplication and inversion are algebraic morphisms). More precisely, we're interested in algebraic G-actions on an algebraic variety (or scheme of finite type) X over an algebraically closed field k. As most affine groups are non-compact, their actions typically have some non-closed orbits. Consequently, the topological quotient X/G will not be Hausdorff. However one could also ask whether we should relax the idea of having an orbit space, in order to get a quotient with better geometrical properties. More precisely, we ask for a categorical quotient in the category of finite type k-schemes; that is, a G-invariant morphism $\pi : X \to Y$ which is universal (i.e., every other G-invariant morphism $X \to Z$ factors uniquely through π). With this definition, it is not necessary for Y to be an orbit space and so we can allow π to identify some orbits in order to get an algebraic quotient.

Geometric invariant theory, as developed by Mumford in [25], shows that for a reductive group G acting on a quasi-projective scheme X (with respect to an ample linearisation) one can construct an open subvariety $U \subset X$ and a categorical quotient $U//G$ of the G-action on U which is a quasi-projective scheme. In general, the quotient will not be an orbit space but it contains an open subscheme V/G which is the orbit space for an open subset $V \subset U$. If X is an affine scheme, we have that $U = X$ and the categorical quotient is also an affine scheme and if X is a projective scheme, the categorical quotient is also projective. We briefly summarise the main techniques involved in GIT.

Let $X = \operatorname{Spec} A$ be an affine scheme of finite type over an algebraically closed field k; then $A = \mathcal{O}(X) := \mathcal{O}_X(X)$ is a finitely generated k-algebra. An algebraic G-action on X induces G-action on the ring $\mathcal{O}(X)$ of regular functions on X. For any G-invariant morphism $f : X \to Z$ of schemes, the image of the associated homomorphism $f^* : \mathcal{O}(Z) \to \mathcal{O}(X)$ is contained in the subalgebra $\mathcal{O}(X)^G$ of G-invariant functions. In particular, if $\mathcal{O}(X)^G$ is finitely generated as a k-algebra, then the associated affine scheme $\operatorname{Spec} \mathcal{O}(X)^G$ is also of finite type over k and the inclusion $\mathcal{O}(X)^G \hookrightarrow \mathcal{O}(X)$ induces a morphism $X \to X//G := \operatorname{Spec} \mathcal{O}(X)^G$, which is categorical quotient of the G-action on X. The affine GIT quotient $X \to X//G$ identifies any orbits whose closures meet, but restricts to an orbit space on an open subscheme of so-called stable points.

An important problem in GIT is determining when the ring of invariants $\mathcal{O}(X)^G$ is finitely generated; this is known as Hilbert's 14th problem. For $G = \mathrm{GL}_n$ over the complex numbers, Hilbert showed that the invariant ring is always finitely generated. However, for a group G built using copies of the additive group $\mathbb{G}_a$, Nagata gave a counterexample where $\mathcal{O}(X)^G$ is non-finitely generated. Furthermore, Nagata proved for any reductive group G, the ring of invariants $\mathcal{O}(X)^G$ is finitely generated. Consequently, (classical) GIT is concerned with the action of reductive groups; for developments on the theory for non-reductive groups, see [6].

The affine GIT quotient serves as a guide for the general approach: as every scheme is constructed by gluing affine schemes, the general theory is obtained by gluing affine GIT quotients. Ideally, we would to cover X by G-invariant open affine sets and glue the corresponding affine GIT quotients. The open G-affine sets are given by non-vanishing loci of invariant sections of a line bundle L on X, to which we have lifted the G-action. However, usually we cannot cover the whole of X with such open subsets, but rather only an open subset X^{ss} of X of so-called semistable points. In this case, we have a categorical quotient of X^{ss} which restricts to an orbit space on the stable locus X^s.

The definitions of (semi)stability are given in terms of the existence of invariant sections of a line bundle with certain properties. However, as calculating rings of invariants is difficult, one often instead makes use of a numerical criterion for semistability known as the Hilbert–Mumford criterion. More precisely, the Hilbert–Mumford criterion reduces the semistability of points in a projective scheme to the study of the weights of all 1-dimensional subtori $\mathbb{G}_m \subset G$.

The techniques of GIT have been used to construct many moduli spaces in algebraic geometry and finally we return to the construction of some important moduli spaces. The main examples that we cover in this course are the GIT constructions of moduli spaces of hypersurfaces and moduli spaces of (semistable) vector bundles on a smooth complex projective curve.

The main references for this course are the books of Newstead [31] and Mukai [24] on moduli problems and GIT, and the book of Mumford [25] on GIT.

Notation and conventions. Throughout we fix an algebraically closed field k; at certain points in the text we will assume that the characteristic of the field is zero in order to simplify the proofs. By a scheme, we always mean a finite type scheme over k. By a variety, we mean a reduced separated (finite type) scheme over k; in particular, we do not assume varieties are irreducible. We let $\mathcal{O}(X) := \mathcal{O}_X(X)$ denote the ring of regular functions on a scheme X. For a projective scheme X with ample line bundle L, we let $R(X, L)$ denote the homogeneous coordinate ring of X given by taking the direct sum of the spaces of sections of all non-negative powers of L.

Acknowledgements. First and foremost, I would like to thank Frances Kirwan for introducing me to geometric invariant theory. These lectures notes were written for a masters course at the Freie Universität Berlin in the winter semester of 2015 and I am grateful to have had some excellent students who actively participated in the course and engaged with the topic: thank you to Anna-Lena, Christoph, Claudius, Dominic, Emmi, Fei, Felix, Jennifer, Koki, Maciek, Maik, Maksymilian, Markus, and Vincent. Thanks also to Eva Martínez for all her hard work running the exercise sessions and to the students for persevering with the more challenging exercises. I am very grateful to Giangiacomo Sanna for many useful comments and suggestions on the script. Finally, I would especially like to thank Simon Pepin Lehalleur for numerous helpful discussions and contributions, and for his encyclopaedic knowledge of algebraic groups.

2. Moduli problems

2.1. **Functors of points.** In this section, we will make use of some of the language of *category theory*. We recall that a morphism of categories $\mathcal{C}$ and $\mathcal{D}$ is given by a *(covariant) functor* $F : \mathcal{C} \to \mathcal{D}$, which associates to every object $C \in \mathcal{C}$ an object $F(C) \in \mathcal{D}$ and to each morphism $f : C \to C'$ in $\mathcal{C}$ a morphism $F(f) : F(C) \to F(C')$ in D such that F preserves identity morphisms and composition. A *contravariant functor* $F : \mathcal{C} \to \mathcal{D}$ reverses arrows: so F sends $f : C \to C'$ to $F(f) : F(C') \to F(C)$.

The notion of a morphism of (covariant) functors $F, G : \mathcal{C} \to \mathcal{D}$ is given by a *natural transformation* $\eta : F \to G$ which associates to every object $C \in \mathcal{C}$ a morphism $\eta_C : F(C) \to G(C)$ in D which is compatible with morphisms $f : C \to C'$ in $\mathcal{C}$, i.e. we have a commutative square

$$\begin{array}{ccc} F(C) & \xrightarrow{\eta_C} & G(C) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(C') & \xrightarrow[\eta_{C'}]{} & G(C'). \end{array}$$

We note that if F and G were contravariant functors, the vertical arrows in this square would be reversed. If η_C is an isomorphism in $\mathcal{D}$ for all $C \in \mathcal{C}$, then we call η a *natural isomorphism* or simply an isomorphism of functors.

Remark 2.1. The focus of this course is moduli problems, rather than category theory and so we are doing naive category theory (in the sense that we allow the objects of a category to be a class). This is analogous to doing naive set theory without a consistent axiomatic approach. However, for those interested in category theory, this can all be handled in a consistent manner, where one pays more careful attention to the size of the set of objects. One approach to this more formal category theory can be found in the book of Kashiwara and Schapira [18]. Strictly speaking, in this case, one should work with the category of 'small' sets.

Let Set denote the category of sets and let Sch denote the category of schemes (of finite type over k).

Definition 2.2. The functor of points of a scheme X is a contravariant functor $h_X := \mathrm{Hom}(-, X) :$ Sch $\to$ Set from the category of schemes to the category of sets defined by

$$\begin{array}{lclccc} h_X(Y) & := & \mathrm{Hom}(Y, X) & & & \\ h_X(f : Y \to Z) & := & h_X(f) : & h_X(Z) & \to & h_X(Y) \\ & & & g & \mapsto & g \circ f. \end{array}$$

Furthermore, a morphism of schemes $f : X \to Y$ induces a natural transformation of functors $h_f : h_X \to h_Y$ given by

$$\begin{array}{rccc} h_{f,Z} : & h_X(Z) & \to & h_Y(Z) \\ & g & \mapsto & f \circ g. \end{array}$$

Contravariant functors from schemes to sets are called *presheaves on* Sch and form a category, with morphisms given by natural transformations; this category is denoted $\mathrm{Psh}(\mathrm{Sch}) := \mathrm{Fun}(\mathrm{Sch}^{\mathrm{op}}, \mathrm{Set})$, the category of presheaves on Sch. The above constructions can be phrased as follows: there is a functor $h : \mathrm{Sch} \to \mathrm{Psh}(\mathrm{Sch})$ given by

$$X \mapsto h_X \qquad (f : X \to Y) \mapsto h_f : h_X \to h_Y.$$

In fact, there is nothing special about the category of schemes here. So for any category $\mathcal{C}$, there is a functor $h : \mathcal{C} \to \mathrm{Psh}(\mathcal{C})$.

Example 2.3. For a scheme X, we have that $h_X(\operatorname{Spec} k) := \mathrm{Hom}(\operatorname{Spec} k, X)$ is the set of k-points of X and, for another scheme Y, we have that $h_X(Y)$ is the set of Y-valued points of X. Let $X = \mathbb{A}^1$ be the affine line; then the functor of points $h_{\mathbb{A}^1}$ associates to a scheme Y the set of functions on Y (i.e. morphisms $Y \to \mathbb{A}^1$). Similarly, for the scheme $\mathbb{G}_m = \mathbb{A}^1 - \{0\}$, the functor $h_{\mathbb{A}^1}$ associates to a scheme Y the set of invertible functions on Y.

Lemma 2.4 (The Yoneda Lemma)**.** *Let $\mathcal{C}$ be any category. Then for any $C \in \mathcal{C}$ and any presheaf $F \in \mathrm{Psh}(\mathcal{C})$, there is a bijection*

$$\{\text{natural transformsations } \eta : h_C \to F\} \longleftrightarrow F(C).$$

given by $\eta \mapsto \eta_C(Id_C)$.

Proof. Let us first check that this is surjective: for an object $s \in F(C)$, we define a natural transformation $\eta = \eta(s) : h_C \to F$ as follows. For $C' \in \mathcal{C}$, let $\eta_{C'} : h_C(C') \to F(C')$ be the morphism of sets which sends $f : C' \to C$ to $F(f)(s)$ (recall that $F(f) : F(C) \to F(C')$). This is compatible with morphisms and, by construction, $\eta_C(\mathrm{id}_C) = F(\mathrm{id}_C)(s) = s$.

For injectivity, suppose we have natural transformations $\eta, \eta' : h_C \to F$ such that $\eta_C(\mathrm{Id}_C) = \eta'_C(\mathrm{Id}_C)$. Then we claim $\eta = \eta'$; that is, for any C' in $\mathcal{C}$, we have $\eta_{C'} = \eta'_{C'} : h_C(C') \to F(C')$. Let $g : C' \to C$, then as η is a natural transformation, we have a commutative square

$$\begin{array}{ccc} h_C(C) & \xrightarrow{\eta_C} & F(C) \\ {\scriptstyle h_C(g)}\downarrow & & \downarrow{\scriptstyle F(g)} \\ h_C(C') & \xrightarrow[\eta_{C'}]{} & F(C'). \end{array}$$

It follows that

$$(F(g) \circ \eta_C)(\mathrm{id}_C) = (\eta_{C'} \circ h_C(g))(\mathrm{Id}_C) = \eta_{C'}(g)$$

and similarly, as η' is a natural transformation, that $(F(g) \circ \eta'_C)(\mathrm{id}_C) = \eta'_{C'}(g)$. Hence

$$\eta_{C'}(g) = F(g)(\eta_C(\mathrm{id}_C)) = F(g)(\eta'_C(\mathrm{id}_C)) = \eta'_{C'}(g)$$

as required. □

The functor $h : \mathcal{C} \to \mathrm{Psh}(\mathcal{C})$ is called the Yoneda embedding, due to the following corollary.

Corollary 2.5. *The functor $h : \mathcal{C} \to \mathrm{Psh}(\mathcal{C})$ is fully faithful.*

Proof. We recall that a functor is fully faithful if for every C, C' in $\mathcal{C}$, the morphism

$$\mathrm{Hom}_{\mathcal{C}}(C, C') \to \mathrm{Hom}_{\mathrm{Psh}(\mathcal{C})}(h_C, h_{C'})$$

is bijective. This follows immediately from the Yoneda Lemma if we take $F = h_{C'}$. □

Exercise 2.6. Show that if there is a natural isomorphism $h_C \to h'_C$, then there is a canonical isomorphism $C \to C'$.

The presheaves in the image of the Yoneda embedding are known as representable functors.

Definition 2.7. A presheaf $F \in \mathrm{Psh}(\mathcal{C})$ is called representable if there exists an object $C \in \mathcal{C}$ and a natural isomorphism $F \cong h_C$.

Question: Is every presheaf $F \in \mathrm{Psh}(\mathrm{Sch})$ representable by a scheme X?

The question has a negative answer, as we will soon see below. However, we are most interested in answering this question for special functors F, known as moduli functors, which classify certain geometric families. Before we introduce these moduli functors, we start with the naive notion of a moduli problem.

2.2. **Moduli problem.** A moduli problem is essentially a classification problem: we have a collection of objects and we want to classify these objects up to equivalence. In fact, we want more than this, we want a moduli space which encodes how these objects vary continuously in families; this information is encoded in a moduli functor.

Definition 2.8. A (naive) moduli problem (in algebraic geometry) is a collection $\mathcal{A}$ of objects (in algebraic geometry) and an equivalence relation $\sim$ on $\mathcal{A}$.

Example 2.9.

(1) Let $\mathcal{A}$ be the set of k-dimensional linear subspaces of an n-dimensional vector space and $\sim$ be equality.
(2) Let $\mathcal{A}$ be the set of n ordered distinct points on $\mathbb{P}^1$ and $\sim$ be the equivalence relation given by the natural action of the automorphism group PGL_2 of $\mathbb{P}^1$.
(3) Let $\mathcal{A}$ to be the set of hypersurfaces of degree d in $\mathbb{P}^n$ and $\sim$ can be chosen to be either equality or the relation given by projective change of coordinates (i.e. corresponding to the natural PGL_{n+1}-action).
(4) Let $\mathcal{A}$ be the collection of vector bundles on a fixed scheme X and $\sim$ be the relation given by isomorphisms of vector bundles.

Our aim is to find a scheme M whose k-points are in bijection with the set of equivalence classes $\mathcal{A}/\sim$. Furthermore, we want M to also encode how these objects vary continuously in 'families'. More precisely, we refer to $(\mathcal{A}, \sim)$ as a naive moduli problem, because there is often a natural notion of families of objects over a scheme S and an extension of $\sim$ to families over S, such that we can pullback families by morphisms $T \to S$.

Definition 2.10. Let $(\mathcal{A}, \sim)$ be a naive moduli problem. Then an extended moduli problem is given by

(1) sets $\mathcal{A}_S$ of families over S and an equivalence relation $\sim_S$ on $\mathcal{A}_S$, for all schemes S,
(2) pullback maps $f^* : \mathcal{A}_S \to \mathcal{A}_T$, for every morphism of schemes $T \to S$,

satisfying the following properties:

(i) $(\mathcal{A}_{\mathrm{Spec}\, k}, \sim_{\mathrm{Spec}\, k}) = (\mathcal{A}, \sim)$;
(ii) for the identity $\mathrm{Id} : S \to S$ and any family $\mathcal{F}$ over S, we have $\mathrm{Id}^*\mathcal{F} = \mathcal{F}$;
(iii) for a morphism $f : T \to S$ and equivalent families $\mathcal{F} \sim_S \mathcal{G}$ over S, we have $f^*\mathcal{F} \sim_T f^*\mathcal{G}$;
(iv) for morphisms $f : T \to S$ and $g : S \to R$, and a family $\mathcal{F}$ over R, we have an equivalence $(g \circ f)^*\mathcal{F} \sim_T f^*g^*\mathcal{F}$.

For a family $\mathcal{F}$ over S and a point $s : \mathrm{Spec}\, k \to S$, we write $\mathcal{F}_s := s^*\mathcal{F}$ to denote the corresponding family over $\mathrm{Spec}\, k$.

Lemma 2.11. *A moduli problem defines a functor* $\mathcal{M} \in \mathrm{Psh}(\mathrm{Sch})$ *given by*

$$\mathcal{M}(S) := \{\text{families over } S\}/\sim_S \qquad \mathcal{M}(f : T \to S) = f^* : \mathcal{M}(S) \to \mathcal{M}(T).$$

We will often refer to a moduli problem simply by its moduli functor. There can be several different extensions of a naive moduli problem.

Example 2.12. Let us consider the naive moduli problem given by vector bundles (i.e. locally free sheaves) on a fixed scheme X up to isomorphism. Then this can be extended in two different ways. The natural notion for a family over S is a locally free sheaf $\mathcal{F}$ over $X \times S$ flat over S, but there are two possible equivalence relations:

$$\begin{aligned} \mathcal{F} \sim'_S \mathcal{G} &\iff \mathcal{F} \cong \mathcal{G} \\ \mathcal{F} \sim_S \mathcal{G} &\iff \mathcal{F} \cong \mathcal{G} \otimes \pi_S^*\mathcal{L} \text{ for a line bundle } \mathcal{L} \to S \end{aligned}$$

where $\pi_S : X \times S \to S$. For the second equivalence relation, since $\mathcal{L} \to S$ is locally trivial, there is a cover S_i of S such that $\mathcal{F}|_{X\times S_i} \cong \mathcal{G}|_{X\times S_i}$. It turns out that the second notion of equivalence offers the extra flexibility we will need in order to construct moduli spaces.

Example 2.13. Let $\mathcal{A}$ consist of 4 ordered distinct points (p_1, p_2, p_3, p_4) on $\mathbb{P}^1$. We want to classify these quartuples up to the automorphisms of $\mathbb{P}^1$. We recall that the automorphism group of $\mathbb{P}^1$ is the projective linear group PGL_2, which acts as Möbius transformations. We define our equivalence relation by $(p_1, p_2, p_3, p_4) \sim (q_1, q_2, q_3, q_4)$ if there exists an automorphisms $f : \mathbb{P}^1 \to \mathbb{P}^1$ such that $f(p_i) = q_i$ for $i = 1, \dots, 4$. We recall that for any 3 distinct points (p_1, p_2, p_3) on $\mathbb{P}^1$, there exists a unique Möbius transformation $f \in \mathrm{PGL}_2$ which sends (p_1, p_2, p_3) to $(0, 1, \infty)$ and the cross-ratio of 4 distinct points (p_1, p_2, p_3, p_4) on $\mathbb{P}^1$ is given by $f(p_4) \in \mathbb{P}^1 - \{0, 1, \infty\}$, where f is the unique Möbius transformation that sends (p_1, p_2, p_3) to $(0, 1, \infty)$. Therefore, we see that the set $\mathcal{A}/\sim$ is in bijection with the set of k-points in the quasi-projective variety $\mathbb{P}^1 - \{0, 1, \infty\}$.

In fact, we can naturally speak about families of 4 distinct points on $\mathbb{P}^1$ over a scheme S: this is given by a proper flat morphism $\pi : \mathcal{X} \to S$ such that the fibres $\pi^{-1}(s) \cong \mathbb{P}^1$ are smooth rational curves and 4 disjoint sections $(\sigma_1, \dots, \sigma_4)$ of π. We say two families $(\pi : \mathcal{X} \to S, \sigma_1, \dots, \sigma_4)$ and $(\pi' : \mathcal{X}' \to S, \sigma'_1, \dots, \sigma'_4)$ are equivalent over S if there is an isomorphism $f : \mathcal{X} \to \mathcal{X}'$ over S (i.e. $\pi = \pi' \circ f$) such that $f \circ \sigma_i = \sigma'_i$.

There is a tautological family over the scheme $S = \mathbb{P}^1 - \{0, 1, \infty\}$: let $\pi : \mathbb{P}^1 - \{0, 1, \infty\} \times \mathbb{P}^1 \to S = \mathbb{P}^1 - \{0, 1, \infty\}$ be the projection map and choose sections $(\sigma_1(s) = 0, \sigma_2(s) = 1, \sigma_3(s) = \infty, \sigma_4(s) = s)$. It turns out that this family over $\mathbb{P}^1 - \{0, 1, \infty\}$ encodes all families parametrised by schemes S (in the language to come, $\mathcal{U}$ is a *universal family* and $\mathbb{P}^1 - \{0, 1, \infty\}$ is a *fine moduli space).*

Exercise 2.14. Define an analogous notion for families of n ordered distinct points on $\mathbb{P}^1$ and let the corresponding moduli functor be denoted $\mathcal{M}_{0,n}$ (this is the moduli functor of n ordered distinct points on the curve $\mathbb{P}^1$ of genus 0). For $n = 3$, show that $\mathcal{M}_{0,3}(\operatorname{Spec} k)$ is a single element set and so is in bijection with the set of k-points of $\operatorname{Spec} k$. Furthermore, show there is a tautological family over $\operatorname{Spec} k$.

2.3. **Fine moduli spaces.** The ideal situation is when there is a scheme that represents our given moduli functor.

Definition 2.15. Let $\mathcal{M} : \mathrm{Sch} \to \mathrm{Set}$ be a moduli functor; then a scheme M is a fine moduli space for $\mathcal{M}$ if it represents $\mathcal{M}$.

Let's carefully unravel this definition: M is a fine moduli space for $\mathcal{M}$ if there is a natural isomorphism $\eta : \mathcal{M} \to h_M$. Hence, for every scheme S, we have a bijection

$$\eta_S : \mathcal{M}(S) := \{\text{families over } S\}/\sim_S \longleftrightarrow h_M(S) := \{\text{morphisms } S \to M\}.$$

In particular, if $S = \operatorname{Spec} k$, then the k-points of M are in bijection with the set $\mathcal{A}/\sim$. Furthermore, these bijections are compatible with morphisms $T \to S$, in the sense that we have a commutative diagram

$$\begin{array}{ccc} \mathcal{M}(S) & \xrightarrow{\eta_S} & h_M(S) \\ {\scriptstyle \mathcal{M}(f)}\downarrow & & \downarrow{\scriptstyle h_M(f)} \\ \mathcal{M}(T) & \xrightarrow[\eta_T]{} & h_M(T). \end{array}$$

The natural isomorphism $\eta : \mathcal{M} \to h_M$ determines an element $\mathcal{U} = \eta_M^{-1}(\mathrm{id}_M) \in \mathcal{M}(M)$; that is, $\mathcal{U}$ is a family over M (up to equivalence).

Definition 2.16. Let M be a fine moduli space for $\mathcal{M}$; then the family $\mathcal{U} \in \mathcal{M}(M)$ corresponding to the identity morphism on M is called the universal family.

This family is called the universal family, as any family $\mathcal{F}$ over a scheme S (up to equivalence) corresponds to a morphism $f : S \to M$ and, moreover, as the families $f^*\mathcal{U}$ and $\mathcal{F}$ correspond to the same morphism $\mathrm{id}_M \circ f = f$, we have

$$f^*\mathcal{U} \sim_S \mathcal{F};$$

that is, any family is equivalent to a family obtained by pulling back the universal family.

Remark 2.17. If a fine moduli space for $\mathcal{M}$ exists, it is unique up to unique isomorphism: that is, if (M, η) and (M', η') are two fine moduli spaces, then they are related by unique isomorphisms $\eta'_M((\eta_M)^{-1}(\mathrm{Id}_M)) : M \to M'$ and $\eta_{M'}((\eta'_{M'})^{-1}(\mathrm{Id}_{M'})) : M' \to M$.

We recall that a presheaf $F : \mathrm{Sch} \to \mathrm{Set}$ is said to be a sheaf in the Zariski topology if for every scheme S and Zariski cover $\{S_i\}$ of S, the natural map

$$\{f \in F(S)\} \longrightarrow \{(f_i \in F(S_i))_i : f_i|_{S_i \cap S_j} = f_j|_{S_j \cap S_i} \text{ for all } i, j\}$$

is a bijection. A presheaf is called a separated presheaf if these natural maps are injective.

Exercise 2.18.

(1) Show that the functor of points of a scheme is a sheaf in the Zariski topology. In particular, deduce that for a presheaf to be representable it must be a sheaf in the Zariski topology.
(2) Consider the moduli functor of vector bundles over a fixed scheme X, where we say two families $\mathcal{E}$ and $\mathcal{F}$ are equivalent if and only if they are isomorphic. Show that the corresponding moduli functor fails to be a separable presheaf (it may be useful to consider the second equivalence relation we introduced for families of vector bundles in Exercise 2.12).

Example 2.19. Let us consider the projective space $\mathbb{P}^n = \mathrm{Proj}\, k[x_0, \dots, x_n]$. This variety can be interpreted as a fine moduli space for the moduli problem of lines through the origin in $V := \mathbb{A}^{n+1}$. To define this moduli problem carefully, we need to define a notion of families and equivalences of families. A family of lines through the origin in V over a scheme S is a line bundle $\mathcal{L}$ over S which is a subbundle of the trivial vector bundle $V \times S$ over S (by subbundle we mean that the quotient is also a vector bundle). Then two families are equivalent if and only if they are equal.

Over $\mathbb{P}^n$, we have a tautological line bundle $\mathcal{O}_{\mathbb{P}^n}(-1) \subset V \times \mathbb{P}^n$, whose fibre over $p \in \mathbb{P}^n$ is the corresponding line in V. This provides a tautological family of lines over $\mathbb{P}^n$. The dual of the tautological line bundle is the line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$, known as the Serre twisting sheaf. The important fact we need about $\mathcal{O}_{\mathbb{P}^n}(1)$ is that it is generated by the global sections $x_0, \dots, x_n$.

Given any morphism of schemes $f : S \to \mathbb{P}^n$, the line bundle $f^*\mathcal{O}_{\mathbb{P}^n}(1)$ is generated by the global sections $f^*(x_0), \dots, f^*(x_n)$. Hence, we have a surjection $\mathcal{O}_S^{n+1} \to f^*\mathcal{O}_{\mathbb{P}^n}(1)$. For locally free sheaves, pull back commutes with dualising and so

$$f^*\mathcal{O}_{\mathbb{P}^n}(-1) \cong (f^*\mathcal{O}_{\mathbb{P}^n}(1))^\vee.$$

Dually the above surjection gives an inclusion $\mathcal{L} := f^*\mathcal{O}_{\mathbb{P}^n}(-1) \to \mathcal{O}_S^{n+1} = V \times S$ which determines a family of lines in V over S.

Conversely, let $\mathcal{L} \subset V \times S$ be a family of lines through the origin in V over S. Then, dual to this inclusion, we have a surjection $q : V^\vee \times S \to \mathcal{L}^\vee$. The vector bundle $V^\vee \times S$ is generated by the global sections $\sigma_0, \dots, \sigma_n$ corresponding to the dual basis for the standard basis on V. Since q is surjective, the dual line bundle $\mathcal{L}^\vee$ is generated by the global sections $q \circ \sigma_0, \dots, q \circ \sigma_n$. In particular, there is a unique morphism $f : S \to \mathbb{P}^n$ given by

$$s \mapsto [q \circ \sigma_0(s) : \cdots : q \circ \sigma_n(s)]$$

such that $f^*\mathcal{O}_{\mathbb{P}^n}(-1) = \mathcal{L} \subset V \times S$ (for details, see [14] II Theorem 7.1).

Hence, there is a bijective correspondence between morphisms $S \to \mathbb{P}^n$ and families of lines through the origin in V over S. In particular, $\mathbb{P}^n$ is a fine moduli space and the tautological family is a universal family. The keen reader may note that the above calculations suggests we should rather think of $\mathbb{P}^n$ as the space of 1-dimensional quotient spaces of a $n+1$-dimensional vector space (a convention that many algebraic geometers use).

Exercise 2.20. Consider the moduli problem of d-dimensional linear subspaces in a fixed vector space $V = \mathbb{A}^n$, where a family over S is a rank d vector subbundle $\mathcal{E}$ of $V \times S$ and the equivalence relation is given by equality. We denote the associated moduli functor by $\mathcal{G}r(d,n)$.

We recall that there is a projective variety $\mathrm{Gr}(d,n)$ whose k-points parametrise d-dimensional linear subspaces of k^n, called the *Grassmannian variety.* Let $\mathcal{T} \subset V \times \mathrm{Gr}(d,n)$ be the tautological family over $\mathrm{Gr}(d,n)$ whose fibre over a point in the Grassmannian is the corresponding linear subspace of V. In this exercise, we will show that the Grassmannian variety $\mathrm{Gr}(d,n)$ is a fine moduli space representing $\mathcal{G}r(d,n)$.

Let us determine the natural isomorphism $\eta : \mathcal{G}r(d,n) \to h_{\mathrm{Gr}(d,n)}$. Consider a family $\mathcal{E} \subset V \times S$ over S. As $\mathcal{E}$ is a rank d vector bundle, we can pick an open cover $\{U_\alpha\}$ of S on which $\mathcal{E}$ is trivial, i.e. $\mathcal{E}|_{U_\alpha} \cong U_\alpha \times \mathbb{A}^d$. Then, since we have $U_\alpha \times \mathbb{A}^d \cong \mathcal{E}|_{U_\alpha} \subset V \times S|_{U_\alpha} = \mathbb{A}^n \times U_\alpha$, we obtain a homomorphism $U_\alpha \times \mathbb{A}^d \hookrightarrow U_\alpha \times \mathbb{A}^n$ of trivial vector bundles over U_α. This determines a $n \times d$ matrix with coefficients in $\mathcal{O}(U_\alpha)$ of rank d; that is, a morphism $U_\alpha \to M^d_{n\times d}(k)$, to the variety of $n \times d$ matrices of rank d. By taking the wedge product of the d rows in this matrix, we obtain a morphism $f_\alpha : U_\alpha \to \mathbb{P}(\wedge^d(k^n))$ with image in the Grassmannian $\mathrm{Gr}(d,n)$. Using the fact that the transition functions of $\mathcal{E}$ are linear, verify that these morphisms glue to define a morphism $f = f_{\mathcal{E}} : S \to \mathbb{P}(\wedge^d(k^n))$ such that $f^*\mathcal{T} = \mathcal{E}$. In particular, this procedure determines the natural isomorphism: $\eta_S(\mathcal{E}) = f_{\mathcal{E}}$.

For a comprehensive coverage of the Grassmannian moduli functor and its representability, see [8] Section 8. The Grassmannian moduli functor has a natural generalisation to the moduli problem of classifying subsheaves of a fixed sheaf (or equivalently quotient sheaves with a natural notion of equivalence). This functor is representable by a *quot scheme* constructed by Grothendieck [9, 10] (for a survey of the construction, see [33]). Let us mention two special cases of this construction. Firstly, if we take our fixed sheaf to be the structure sheaf of a scheme X, then we are considering ideal sheaves and obtain a Hilbert scheme classifying subschemes of X. Secondly, if we take our fixed sheaf to be a locally free coherent sheaf $\mathcal{E}$ over X and consider quotient line bundles of $\mathcal{E}$, we obtain the projective space bundle $\mathbb{P}(\mathcal{E})$ over X (see [14] II §7).

2.4. **Pathological behaviour.** Unfortunately, there are many natural moduli problems which do not admit a fine moduli space. In this section, we study some examples and highlight two particular pathologies which prevent a moduli problem from admitting a fine moduli space, namely:

(1) The jump phenomena: moduli may jump in families (in the sense that we can have a family $\mathcal{F}$ over $\mathbb{A}^1$ such that $\mathcal{F}_s \sim \mathcal{F}_{s'}$ for all $s, s' \in \mathbb{A}^1 - \{0\}$, but $\mathcal{F}_0 \nsim \mathcal{F}_s$ for $s \in \mathbb{A}^1 - \{0\}$).
(2) The moduli problem may be unbounded (in that there is no family $\mathcal{F}$ over a scheme S which parametrises all objects in the moduli problem).

Example 2.21. We consider the naive moduli problem of classifying endomorphisms of a n-dimensional k-vector space. More precisely $\mathcal{A}$ consists of pairs (V,T), where V is an n-dimensional k-vector space and T is an endomorphism of V. We say $(V,\phi) \sim (V',\phi')$ if there exists an isomorphism $h : V \to V'$ compatible with the endomorphisms i.e. $h \circ \phi = \phi' \circ h$. We extend this to a moduli problem by defining a family over S to be a rank n vector bundle $\mathcal{F}$ over S with an endomorphism $\phi : \mathcal{F} \to \mathcal{F}$. Then we say $(\mathcal{F},\phi) \sim_S (\mathcal{G},\phi')$ if there is an isomorphism $h : \mathcal{F} \to \mathcal{G}$ such that $h \circ \phi = \phi' \circ h$. Let $\mathcal{E}nd_n$ be the corresponding moduli functor.

For any $n \geq 2$, we can construct families which exhibit the jump phenomena. For concreteness, let $n = 2$. Then consider the family over $\mathbb{A}^1$ given by $(\mathcal{F} = \mathcal{O}_{\mathbb{A}^1}^{\oplus 2}, \phi)$ where for $s \in \mathbb{A}^1$

$$\phi_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}.$$

For $s, t \neq 0$, these matrices are similar and so $\phi_t \sim \phi_s$. However, $\phi_0 \nsim \phi_1$, as this matrices have distinct Jordan normal forms. Hence, we have produced a family with the jump phenomenon.

Example 2.22. Let us consider the moduli problem of vector bundles over $\mathbb{P}^1$ of rank 2 and degree 0.

We claim there is no family $\mathcal{F}$ over a scheme S with the property that for any rank 2 degree 0 vector bundle $\mathcal{E}$ on $\mathbb{P}^1$, there is a k-point $s \in S$ such that $\mathcal{F}|_s \cong \mathcal{E}$. Suppose such a family $\mathcal{F}$ over a scheme S exists. For each $n \in \mathbb{N}$, we have a rank 2 degree 0 vector bundle $\mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}(-n)$ (in fact, by Grothendieck's Theorem classifying vector bundles on $\mathbb{P}^1$, every rank 2 degree 0 vector bundle on $\mathbb{P}^1$ has this form). Furthermore, we have

$$\dim H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}(-n)) = \dim_k(k[x_0,x_1]_n \oplus k[x_0,x_1]_{-n}) = \begin{cases} 2 & \text{if } n = 0, \\ n+1 & \text{if } n \geq 1. \end{cases}$$

Consider the subschemes $S_n := \{s \in S : \dim H^0(\mathbb{P}^1, \mathcal{F}_s) \geq n\}$ of S, which are closed by the semi-continuity theorem (see [14] III Theorem 12.8). Then we obtain a decreasing chain of closed subschemes

$$S = S_2 \supsetneq S_3 \supsetneq S_4 \supsetneq \dots.$$

each of which is distinct as $\mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}(-n) \in S_{n+1} - S_{n+2}$. The existence of this chain contradicts the fact that S is Noetherian (recall that for us scheme means scheme of finite type over k). In particular, the moduli problem of vector bundles of rank 2 and degree 0 is unbounded.

In fact, we also see the jump phenomena: there is a family $\mathcal{F}$ of rank 2 degree 0 vector bundles over $\mathbb{A}^1 = \operatorname{Spec} k[s]$ such that

$$\mathcal{F}_s = \begin{cases} \mathcal{O}_{\mathbb{P}^1}^{\oplus 2} & s \neq 0 \\ \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) & s = 0. \end{cases}$$

To construct this family, we note that

$$\operatorname{Ext}^1(\mathcal{O}_{\mathbb{P}^1}(1), \mathcal{O}_{\mathbb{P}^1}(-1)) \cong H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})^* \cong k$$

by Serre duality. Therefore, there is a family of extensions $\mathcal{F}$ over $\mathbb{A}^1$ of $\mathcal{O}_{\mathbb{P}^1}(1)$ by $\mathcal{O}_{\mathbb{P}^1}(-1)$ with the desired property.

In both cases there is no fine moduli space for this problem. To solve these types of phenomena, one usually restricts to a nicer class of objects (we will return to this idea later on).

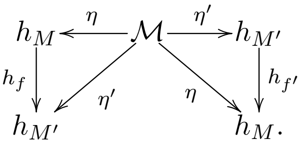
Example 2.23. We can see more directly that there is no fine moduli space for $\mathcal{E}nd_n$. Suppose M is a fine moduli space. Then we have a bijection between morphisms $S \to M$ and families over S up to equivalence. Choose any $n \times n$ matrix T, which determines a point $m \in M$. Then for $S = \mathbb{P}^1$ we have that the trivial families $(\mathcal{O}_{\mathbb{P}^1}^n, T)$ and $(\mathcal{O}_{\mathbb{P}^1}^n \otimes \mathcal{O}_{\mathbb{P}^1}(1), T \otimes \mathrm{Id}_{\mathcal{O}_{\mathbb{P}^1}(1)})$ are non-equivalent families which determine the same morphism $\mathbb{P}^1 \to M$, namely the constant morphism to the point m.

2.5. **Coarse moduli spaces.** As demonstrated by the above examples, not every moduli functor has a fine moduli space. By only asking for a natural transformation $\mathcal{M} \to h_M$ which is universal and a bijection over $\operatorname{Spec} k$ (so that the k-points of M are in bijection with the equivalence classes $\mathcal{A}/\sim$), we obtain a weaker notion of a coarse moduli space.

Definition 2.24. A coarse moduli space for a moduli functor $\mathcal{M}$ is a scheme M and a natural transformation of functors $\eta : \mathcal{M} \to h_M$ such that

(a) $\eta_{\operatorname{Spec} k} : \mathcal{M}(\operatorname{Spec} k) \to h_M(\operatorname{Spec} k)$ is bijective.
(b) For any scheme N and natural transformation $\nu : \mathcal{M} \to h_N$, there exists a unique morphism of schemes $f : M \to N$ such that $\nu = h_f \circ \eta$, where $h_f : h_M \to h_N$ is the corresponding natural transformation of presheaves.

Remark 2.25. A coarse moduli space for $\mathcal{M}$ is unique up to unique isomorphism: if (M, η) and (M', η') are coarse moduli spaces for $\mathcal{M}$, then by Property (b) there exists unique morphisms $f : M \to M'$ and $f' : M' \to M$ such that h_f and $h_{f'}$ fit into two commutative triangles:



Since $\eta = h_{f'} \circ h_f \circ \eta$ and $\eta = h_{\mathrm{id}_M} \circ \eta$, by uniqueness in (b) and the Yoneda Lemma, we have $f' \circ f = \mathrm{id}_M$ and similarly $f \circ f' = \mathrm{id}_{M'}$.

Proposition 2.26. *Let (M, η) be a coarse moduli space for a moduli problem $\mathcal{M}$. Then (M, η) is a fine moduli space if and only if*

(1) there exists a family $\mathcal{U}$ over M such that $\eta_M(\mathcal{U}) = \mathrm{id}_M$,
(2) for families $\mathcal{F}$ and $\mathcal{G}$ over a scheme S, we have $\mathcal{F} \sim_S \mathcal{G} \iff \eta_S(\mathcal{F}) = \eta_S(\mathcal{G})$.

Proof. Exercise. □

Lemma 2.27. *Let $\mathcal{M}$ be a moduli problem and suppose there exists a family $\mathcal{F}$ over $\mathbb{A}^1$ such that $\mathcal{F}_s \sim \mathcal{F}_1$ for all $s \neq 0$ and $\mathcal{F}_0 \nsim \mathcal{F}_1$. Then for any scheme M and natural transformation $\eta : \mathcal{M} \to h_M$, we have that $\eta_{\mathbb{A}^1}(\mathcal{F}) : \mathbb{A}^1 \to M$ is constant. In particular, there is no coarse moduli space for this moduli problem.*

Proof. Suppose we have a natural transformation $\eta : \mathcal{M} \to h_M$; then η sends the family $\mathcal{F}$ over $\mathbb{A}^1$ to a morphism $f : \mathbb{A}^1 \to M$. For any $s : \operatorname{Spec} k \to \mathbb{A}^1$, we have that $f \circ s = \eta_{\operatorname{Spec} k}(\mathcal{F}_s)$ and, for $s \neq 0$, $\mathcal{F}_s = \mathcal{F}_1 \in \mathcal{M}(\operatorname{Spec} k)$, so that $f|_{\mathbb{A}^1 - \{0\}}$ is a constant map. Let $m : \operatorname{Spec} k \to M$ be the point corresponding to the equivalence class for $\mathcal{F}_1$ under η. Since the k-valued points of M are closed (recall M is a scheme of finite type over an algebraically closed field), their preimages under morphisms must also be closed. Then, as $\mathbb{A}^1 - \{0\} \subset f^{-1}(m)$, the closure $\mathbb{A}^1$ of $\mathbb{A}^1 - \{0\}$ must also be contained in $f^{-1}(m)$; that is, f is the constant map to the k-valued point m of M. In particular, the map $\eta_{\operatorname{Spec} k} : \mathcal{M}(\operatorname{Spec} k) \to h_M(\operatorname{Spec} k)$ is not a bijection, as $\mathcal{F}_0 \neq \mathcal{F}_1$ in $\mathcal{M}(\operatorname{Spec} k)$, but these non-equivalent objects correspond to the same k-point m in M. □

In particular, the moduli problems of Examples 2.22 and 2.21 do not even admit coarse moduli spaces.

2.6. **The construction of moduli spaces.** The construction of many moduli spaces follows the same general pattern.

(1) Fix any discrete invariants for our objects - here the invariants should be invariant under the given equivalence relation (for example, for isomorphism classes of vector bundles on a curve, one may fix the rank and degree).
(2) Restrict to a reasonable class of objects which are bounded (otherwise, we can't find a coarse moduli space). Usually one restricts to a class of *stable* objects which are better behaved and bounded.
(3) Find a family $\mathcal{F}$ over a scheme P with the *local universal property* (i.e. locally any other family is equivalent to a pullback of this family - see below). We call P a *parameter space*, as the k-points of P surject onto $\mathcal{A}/\sim$; however, this is typically not a bijection.
(4) Find a group G acting on P such that p and q lie in the same G-orbit in P if and only if $\mathcal{F}_p \sim \mathcal{F}_q$. Then we have a bijection $P(k)/G \cong \mathcal{A}/\sim$.
(5) Typically this group action is algebraic (see Section 3) and by taking a quotient, we should obtain our moduli space. The quotient should be taken in the category of schemes (in terminology to come, it should be a *categorical quotient*) and this is done using Mumford's *Geometric Invariant Theory.*

Definition 2.28. For a moduli problem $\mathcal{M}$, a family $\mathcal{F}$ over a scheme S has the local universal property if for any other family $\mathcal{G}$ over a scheme T and for any k-point $t \in T$, there exists a neighbourhood U of t in T and a morphism $f : U \to S$ such that $\mathcal{G}|_U \sim_U f^*\mathcal{F}$.

In particular, we do not require the morphism f to be unique. We note that, for such a family to exist, we need our moduli problem to be bounded.

3. Algebraic group actions and quotients

In this section we consider group actions on algebraic varieties and also describe what type of quotients we would like to have for such group actions.

3.1. **Affine Algebraic groups.** An algebraic group (over k) is a group object in the category of schemes (over k). By a Theorem of Chevalley, every algebraic group is an extension of an abelian variety (that is, a smooth connected *projective* algebraic group) by an *affine* algebraic group (whose underlying scheme is affine) [22, Theorem 10.25]. In this course, we only work with affine algebraic groups and cover the results which are most important for our purposes. A good reference for affine algebraic group schemes is the book (in preparation) of Milne [23]. For those who are interested in discovering more about algebraic groups, see [3, 22, 11].

Definition 3.1. An *algebraic group over* k is a scheme G over k with morphisms $e : \operatorname{Spec} k \to G$ (identity element), $m : G \times G \to G$ (group law) and $i : G \to G$ (group inversion) such that we have commutative diagrams

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{\mathrm{id}\times m} & G \times G \\ {\scriptstyle m\times\mathrm{id}}\downarrow & & \downarrow{\scriptstyle m} \\ G \times G & \xrightarrow[m]{} & G \end{array} \qquad \begin{array}{ccccc} \operatorname{Spec} k \times G & \xrightarrow{e\times\mathrm{id}} & G \times G & \xleftarrow{\mathrm{id}\times e} & G \times \operatorname{Spec} k \\ & {\scriptstyle\cong}\searrow & \downarrow{\scriptstyle m} & \swarrow{\scriptstyle\cong} & \\ & & G & & \end{array}$$

$$\begin{array}{ccccc} G & \xrightarrow{(i,\mathrm{id})} & G \times G & \xleftarrow{(\mathrm{id},i)} & G \\ \downarrow & & \downarrow{\scriptstyle m} & & \downarrow \\ \operatorname{Spec} k & \xrightarrow[e]{} & G & \xleftarrow[e]{} & \operatorname{Spec} k. \end{array}$$

We say G is an *affine algebraic group* if the underlying scheme G is affine. We say G is a *group variety* if the underlying scheme G is a variety (recall in our conventions, varieties are not necessarily irreducible).

A *homomorphism* of algebraic groups G and H is a morphism of schemes $f : G \to H$ such that the following square commutes

$$\begin{array}{ccc} G \times G & \xrightarrow{m_G} & G \\ {\scriptstyle f\times f}\downarrow & & \downarrow{\scriptstyle f} \\ H \times H & \xrightarrow[m_H]{} & H. \end{array}$$

An *algebraic subgroup* of G is a closed subscheme H such that the immersion $H \hookrightarrow G$ is a homomorphism of algebraic groups. We say an algebraic group G' is an *algebraic quotient* of G if there is a homomorphism of algebraic groups $f : G \to G'$ which is flat and surjective.

Remark 3.2.

(1) The functor of points h_G of an algebraic group has a natural factorisation through the category of (abstract) groups, i.e, for every scheme X the operations m, e, i equip $\operatorname{Hom}(X, G)$ with a group structure and with this group structure, every map $h_G(f) : \operatorname{Hom}(X, G) \to \operatorname{Hom}(Y, G)$ for $f : Y \to X$ is a morphism of groups. In fact, one can show using the Yoneda lemma that there is an equivalence of categories between the category of algebraic groups and the category of functors $F : \mathrm{Sch} \to \mathrm{Grp}$ such that the composition $\mathrm{Sch} \stackrel{F}{\to} \mathrm{Grp} \to \mathrm{Set}$ is representable. When restricting to the category of affine k-schemes, this can give a very concrete description of an algebraic group, as we will see in the examples below.

(2) Let $\mathcal{O}(G) := \mathcal{O}_G(G)$ denote the k-algebra of regular functions on G. Then the above morphisms of affine varieties correspond to k-algebra homomorphisms $m^* : \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)$ (comultiplication) and $i^* : \mathcal{O}(G) \to \mathcal{O}(G)$ (coinversion) and the identity element corresponds to $e^* : \mathcal{O}(G) \to k$ (counit). These operations define a *Hopf algebra structure* on the k-algebra $\mathcal{O}(G)$. Furthermore, there is a bijection between finitely generated Hopf algebras over k and affine algebraic groups (see [23] II Theorem 5.1).

(3) By a Theorem of Cartier, every affine algebraic group over a field k of characteristic zero is smooth (see [23] VI Theorem 9.3). Moreover, in Exercise sheet 3, we see that every

algebraic group is separated. Hence, in characteristic zero, the notion of affine algebraic group and affine group variety coincide.

(4) In the definition of homomorphisms, we only require a compatibility with the group law m; it turns out that the compatibility for the identity and group inversion is then automatic. This is well known in the case of homomorphisms of abstract groups, and the algebraic case can then be deduced by applying the Yoneda lemma.

(5) For the definition of a quotient group, the condition that the homomorphism is flat is only needed in positive characteristic, as in characteristic zero this morphism is already smooth (this follows from the Theorem of Cartier mentioned above and the fact that the kernel of a homomorphism of smooth group schemes is smooth; see [22] Proposition 1.48)

Example 3.3. Many of the groups that we are already familiar with are affine algebraic groups.

(1) The additive group $\mathbb{G}_a = \operatorname{Spec} k[t]$ over k is the algebraic group whose underlying variety is the affine line $\mathbb{A}^1$ over k and whose group structure is given by addition:

$$m^*(t) = t \otimes 1 + 1 \otimes t \quad \text{and} \quad i^*(t) = -t.$$

Let us indicate how to show these operations satisfy the group axioms. We only prove the associativity, the other axioms being similar and easier. We have to show that

$$(m^* \otimes \mathrm{id}) \circ m^* = (\mathrm{id} \otimes m^*) \circ m^* : k[t] \to k[t] \otimes k[t] \otimes k[t].$$

This is a map of k-algebras, so it is enough to check it for t. We have

$$((m^* \otimes \mathrm{id}) \circ m^*)(t) = (m^* \otimes \mathrm{id})(t \otimes 1 + 1 \otimes t) = t \otimes 1 \otimes 1 + 1 \otimes t \otimes 1 + 1 \otimes 1 \otimes t$$

and similarly

$$((\mathrm{id} \otimes m^*) \circ m^*)(t) = t \otimes 1 \otimes 1 + 1 \otimes t \otimes 1 + 1 \otimes 1 \otimes t$$

which completes the proof. For a k-algebra R, we have $\mathbb{G}_a(R) = (R, +)$; this justifies the name of the 'additive group'.

(2) The multiplicative group $\mathbb{G}_m = \operatorname{Spec} k[t, t^{-1}]$ over k is the algebraic group whose underlying variety is the $\mathbb{A}^1 - \{0\}$ and whose group action is given by multiplication:

$$m^*(t) = t \otimes t \quad \text{and} \quad i^*(t) = t^{-1}.$$

For a k-algebra R, we have $\mathbb{G}_m(R) = (R^\times, \cdot)$; hence, the name of the 'multiplicative group'.

(3) The general linear group GL_n over k is an open subvariety of $\mathbb{A}^{n^2}$ cut out by the condition that the determinant is non-zero. It is an affine variety with coordinate ring $k[x_{ij} : 1 \leq i, j \leq n]_{\det(x_{ij})}$. The co-group operations are defined by:

$$m^*(x_{ij}) = \sum_{k=1}^{n} x_{ik} \otimes x_{kj} \quad \text{and} \quad i^*(x_{ij}) = (x_{ij})^{-1}_{ij}$$

where $(x_{ij})^{-1}_{ij}$ is the regular function on GL_n given by taking the (i, j)-th entry of the inverse of a matrix. For a k-algebra R, the group $\mathrm{GL}_n(R)$ is the group of invertible $n \times n$ matrices with coefficients in R, with the usual matrix multiplication.

(4) More generally, if V is a finite-dimensional vector space over k, there is an affine algebraic group $\mathrm{GL}(V)$ which is (non-canonically) isomorphic to $\mathrm{GL}_{\dim(V)}$. For a k-algebra R, we have $\mathrm{GL}(V)(R) = \mathrm{Aut}_R(V \otimes_k R)$.

(5) Let G be a finite (abstract) group. Then G can be naturally seen as an algebraic group $\underline{G}_k$ over k as follows. The group operations on G make the group algebra $k[G]$ into a Hopf algebra over k, and $\underline{G}_k := \operatorname{Spec}(k[G])$ is a 0-dimensional variety whose points are naturally identified with elements of G.

(6) Let $n \geq 1$. Put $\mu_n := \operatorname{Spec} k[t, t^{-1}]/(t^n - 1) \subset \mathbb{G}_m$, the subscheme of n-roots of unity. Write I for the ideal $(t^n - 1)$ of $R := k[t, t^{-1}]$. Then

$$m^*(t^n - 1) = t^n \otimes t^n - 1 \otimes 1 = (t^n - 1) \otimes t^n + 1 \otimes (t^n - 1) \in I \otimes R + R \otimes I$$

which implies that μ_n is an algebraic subgroup of $\mathbb{G}_m$. If n is different from $\mathrm{char}(k)$, the polynomial $X^n - 1$ is separable and there are n distinct roots in k. Then the choice of a primitive n-th root of unity in k determines an isomorphism $\mu_n \simeq \underline{\mathbb{Z}/n\mathbb{Z}}_k$. If $n = \mathrm{char}(k)$, however, we have $X^n - 1 = (X-1)^n$ in $k[X]$, which implies that the scheme μ_n is non-reduced (with 1 as only closed point). This is the simplest example of a non-reduced algebraic group.

A linear algebraic group is by definition a subgroup of GL_n which is defined by polynomial equations; for a detailed introduction to linear algebraic groups, see [1, 15, 40]. For instance, the special linear group is a linear algebraic group. In particular, any linear algebraic group is an affine algebraic group. In fact, the converse statement is also true: any affine algebraic group is a linear algebraic group (see Theorem 3.9 below).

An affine algebraic group G over k determines a group-valued functor on the category of finitely generated k-algebras given by $R \mapsto G(R)$. Similarly, for a vector space V over k, we have a group valued functor $\mathrm{GL}(V)$ given by $R \mapsto \mathrm{Aut}_R\,(V \otimes_k R)$, the group of R-linear automorphisms. If V is finite dimensional, then $\mathrm{GL}(V)$ is an affine algebraic group.

Definition 3.4. A *linear representation* of an algebraic group G on a vector space V over k is a homomorphism of group valued functors $\rho : G \to \mathrm{GL}(V)$. If V is finite dimensional, this is equivalent to a homomorphism of algebraic groups $\rho : G \to \mathrm{GL}(V)$, which we call a *finite dimensional linear representation* of G.

If G is affine, we can describe a linear representation $\rho : G \to \mathrm{GL}(V)$ more concretely in terms of its associated *co-module* as follows. The natural inclusion $\mathrm{GL}(V) \to \mathrm{End}(V)$ and $\rho : G \to \mathrm{GL}(V)$ determine a functor $G \to \mathrm{End}(V)$, such that the universal element in $G(\mathcal{O}(G))$ given by the identity morphism corresponds to an $\mathcal{O}(G)$-linear endomorphism of $V \otimes_k \mathcal{O}(G)$, which by the universality of the tensor product is uniquely determined by its restriction to a k-linear homomorphism $\rho^* : V \to V \otimes_k \mathcal{O}(G)$; this is the associated co-module. If V is finite dimensional, we can even more concretely describe the associated co-module by considering the group homomorphism $G \to \mathrm{End}(V)$ and its corresponding homomorphism of k-algebras $\mathcal{O}(V \otimes_k V^*) \to \mathcal{O}(G)$, which is determined by a k-linear homomorphism $V \otimes_k V^* \to \mathcal{O}(G)$ or equivalently by the co-module $\rho^* : V \to V \otimes_k \mathcal{O}(G)$. In particular, a linear representation of an affine algebraic group G on a vector space V is equivalent to a co-module structure on V (for the full definition of a co-module structure, see [23] Chapter 4).

3.2. **Group actions.**

Definition 3.5. An (algebraic) action of an affine algebraic group G on a scheme X is a morphism of schemes $\sigma : G \times X \to X$ such that the following diagrams commute

$$\begin{array}{ccc} \mathrm{Spec}\, k \times X & \xrightarrow{e \times \mathrm{id}_X} & G \times X \\ & \searrow^{\cong} & \downarrow \sigma \\ & & X \end{array} \qquad \begin{array}{ccc} G \times G \times X & \xrightarrow{\mathrm{id}_G \times \sigma} & G \times X \\ {\scriptstyle m_G \times \mathrm{id}_X} \downarrow & & \downarrow \sigma \\ G \times X & \xrightarrow[\sigma]{} & X. \end{array}$$

Suppose we have actions $\sigma_X : G \times X \to X$ and $\sigma_Y : G \times Y \to Y$ of an affine algebraic group G on schemes X and Y. Then a morphism $f : X \to Y$ is G-*equivariant* if the following diagram commutes

$$\begin{array}{ccc} G \times X & \xrightarrow{\mathrm{id}_G \times f} & G \times Y \\ {\scriptstyle \sigma_X} \downarrow & & \downarrow {\scriptstyle \sigma_Y} \\ X & \xrightarrow[f]{} & Y. \end{array}$$

If Y is given the trivial action $\sigma_Y = \pi_Y : G \times Y \to Y$, then we refer to a G-equivariant morphism $f : X \to Y$ as a G-*invariant morphism*.

Remark 3.6. If X is an affine scheme over k and $\mathcal{O}(X)$ denotes its algebra of regular functions, then an action of G on X gives rise to a coaction homomorphism of k-algebras:

$$\begin{array}{rccc} \sigma^* : & \mathcal{O}(X) & \to & \mathcal{O}(G \times X) \cong \mathcal{O}(G) \otimes_k \mathcal{O}(X) \\ & f & \mapsto & \sum h_i \otimes f_i. \end{array}$$

This gives rise to a homomorphism $G \to \mathrm{Aut}(\mathcal{O}(X))$ where the k-algebra automorphism of $\mathcal{O}(X)$ corresponding to $g \in G$ is given by

$$f \mapsto \sum h_i(g) f_i \in \mathcal{O}(X)$$

for $f \in \mathcal{O}(X)$ with $\sigma^*(f) = \sum h_i \otimes f_i$.

Definition 3.7. An action of an affine algebraic group G on a k-vector space V (resp. k-algebra A) is given by, for each k-algebra R, an action of $G(R)$ on $V \otimes_k R$ (resp. on $A \otimes_k R$)

$$\sigma_R : G(R) \times (V \otimes_k R) \to V \otimes_k R \quad (\text{resp. } \sigma_R : G(R) \times (A \otimes_k R) \to A \otimes_k R)$$

such that $\sigma_R(g, -)$ is a morphism of R-modules (resp. R-algebras) and these actions are functorial in R. We say that an action of G on a k-algebra A is *rational* if every element of A is contained in a finite dimensional G-invariant linear subspace of A.

Lemma 3.8. *Let G be an affine algebraic group acting on an affine scheme X. Then any $f \in \mathcal{O}(X)$ is contained in a finite dimensional G-invariant subspace of $\mathcal{O}(X)$. Furthermore, for any finite dimensional vector subspace W of $\mathcal{O}(G)$, there is a finite dimensional G-invariant vector subspace V of $\mathcal{O}(X)$ containing W.*

Proof. Let $\sigma : \mathcal{O}(X) \to \mathcal{O}(G) \otimes \mathcal{O}(X)$ denote the coaction homomorphism. Then we can write $\sigma^*(f) = \sum_{i=1}^n h_i \otimes f_i$, for $h_i \in \mathcal{O}(G)$ and $f_i \in \mathcal{O}(X)$. Then $g \cdot f = \sum_i h_i(g) f_i$ and so the vector space spanned by $f_1, \dots, f_n$ is a G-invariant subspace containing f. The second statement follows by applying the same argument to a given basis of W. □

In particular, the action of G on the k-algebra $\mathcal{O}(X)$ is *rational* (that is, every $f \in \mathcal{O}(X)$ is contained in a finite dimensional G-invariant linear subspace of $\mathcal{O}(X)$).

One of the most natural actions is the action of G on itself by left (or right) multiplication. This induces a rational action $\sigma : G \to \mathrm{Aut}(\mathcal{O}(G))$.

Theorem 3.9. *Any affine algebraic group G over k is a linear algebraic group.*

Proof. As G is an affine scheme (of finite type over k), the ring of regular functions $\mathcal{O}(G)$ is a finitely generated k-algebra. Therefore the vector space W spanned by a choice of generators for $\mathcal{O}(G)$ as a k-algebra is finite dimensional. By Lemma 3.8, there is a finite dimensional subspace V of $\mathcal{O}(G)$ which is preserved by the G-action and contains W.

Let $m^* : \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)$ denote the comultiplication; then for a basis $f_1, \dots, f_n$ of V, we have $m^*(f_i) \in \mathcal{O}(G) \otimes V$, hence we can write

$$m^*(f_i) = \sum_{j=1}^n a_{ij} \otimes f_j$$

for functions $a_{ij} \in \mathcal{O}(G)$. In terms of the action $\sigma : G \to \mathrm{Aut}(\mathcal{O}(G))$, we have that $\sigma(g, f_i) = \sum_j a_{ij}(g) f_j$. This defines a k-algebra homomorphism

$$\rho^* : \mathcal{O}(\mathrm{Mat}_{n \times n}) \to \mathcal{O}(G) \qquad x_{ij} \mapsto a_{ij}.$$

To show that the corresponding morphism of affine schemes $\rho : G \to \mathrm{Mat}_{n \times n}$ is a closed embedding, we need to show ρ^* is surjective. Note that V is contained in the image of ρ^* as

$$f_i = (\mathrm{Id}_{\mathcal{O}(G)} \otimes e^*) m^*(f_i) = (\mathrm{Id}_{\mathcal{O}(G)} \otimes e^*) \sum_{j=1}^n a_{ij} \otimes f_j = \sum_{j=1}^n e^*(f_j) a_{ij}.$$

Since V generates $\mathcal{O}(G)$ as a k-algebra, it follows that ρ^* is surjective. Hence ρ is a closed immersion.

Finally, we claim that $\rho : G \to \mathrm{Mat}_{n\times n}$ is a homomorphism of semigroups (recall that a semigroup is a group without inversion, such as matrices under multiplication) i.e. we want to show on the level of k-algebras that we have a commutative square

$$\begin{array}{ccc} \mathcal{O}(\mathrm{Mat}_{n\times n}) & \xrightarrow{m^*_{\mathrm{Mat}}} & \mathcal{O}(\mathrm{Mat}_{n\times n}) \otimes \mathcal{O}(\mathrm{Mat}_{n\times n}) \\ \downarrow{\rho^*} & & \downarrow{\rho^*\otimes\rho^*} \\ \mathcal{O}(G) & \xrightarrow[m^*_G]{} & \mathcal{O}(G) \otimes \mathcal{O}(G); \end{array}$$

that is, we want to show for the generators $x_{ij} \in \mathcal{O}(\mathrm{Mat}_{n\times n})$, we have

$$m^*_G(a_{ij}) = m^*_G(\rho^*(x_{ij})) = (\rho^* \otimes \rho^*)(m^*_{\mathrm{Mat}}(x_{ij})) = (\rho^* \otimes \rho^*)\left(\sum_k x_{ik} \otimes x_{kj}\right) = \sum_k a_{ik} \otimes a_{kj}.$$

To prove this, we consider the associativity identity $m_G \circ (\mathrm{id} \times m_G) = m_G \circ (m_G \times \mathrm{id})$ and apply this on the k-algebra level to $f_i \in \mathcal{O}(G)$ to obtain

$$\sum_{k,j} a_{ik} \otimes a_{kj} \otimes f_j = \sum_j m^*_G(a_{ij}) \otimes f_j$$

as desired. Furthermore, as G is a group rather than just a semigroup, we can conclude that the image of ρ is contained in the group GL_n of invertible elements in the semigroup $\mathrm{Mat}_{n\times n}$. □

Tori are a basic class of algebraic group which are used extensively to study the structure of more complicated algebraic groups (generalising the use of diagonal matrices to study matrix groups through eigenvalues and the Jordan normal form).

Definition 3.10. Let G be an affine algebraic group scheme over k.

(1) G is an (algebraic) torus if $G \cong \mathbb{G}_m^n$ for some $n > 0$.
(2) A torus of G is a subgroup scheme of G which is a torus.
(3) A maximal torus of G is a torus $T \subset G$ which is not contained in any other torus.

For a torus T, we have commutative groups

$$X^*(T) := \mathrm{Hom}(T, \mathbb{G}_m) \quad X_*(T) := \mathrm{Hom}(\mathbb{G}_m, T)$$

called the *character group* and *cocharacter group* respectively, where the morphisms are homomorphisms of linear algebraic groups. Let us compute $X^*(\mathbb{G}_m)$.

Lemma 3.11. *The map*

$$\begin{aligned} \theta : \mathbb{Z} &\to X^*(\mathbb{G}_m) \\ n &\mapsto (t \mapsto t^n) \end{aligned}$$

is an isomorphism of groups.

Proof. Let us first show that this is well defined. Write m^* for the comultiplication on $\mathcal{O}(\mathbb{G}_m)$. Then $m^*(t^n) = (t \otimes t)^n = t^n \otimes t^n$ shows that $\theta(n) : \mathbb{G}_m \to \mathbb{G}_m$ is a morphism of algebraic groups. Since $t^a t^b = t^{a+b}$, θ itself is a morphism of groups. It is clearly injective, so it remains to show surjectivity.

Let ϕ be an endomorphism of $\mathbb{G}_m$. Write $\phi^*(t) \in k[t, t^{-1}]$ as $\sum_{|i|<m} a_i t^i$. We have $m^*(\phi^*(t)) = \phi^*(t) \otimes \phi^*(t)$, which translates into

$$\sum_i a_i t^i \otimes t^i = \sum_{i,j} a_i a_j t^i \otimes t^j.$$

From this, we deduce that at most one a_i is non-zero, say a_n. Looking at the compatibility of ϕ with the unit, we see that necessarily $a_n = 1$. This shows that $\phi = \theta(n)$, completing the proof. □

For a general torus T, we deduce from the Lemma that the (co)character groups are finite free $\mathbb{Z}$-modules of rank $\dim T$. There is a perfect pairing between these lattices given by composition

$$< , >: X^*(T) \times X_*(T) \to \mathbb{Z}$$

where $< \chi, \lambda >:= \chi \circ \lambda$.

An important fact about tori is that their linear representations are completely reducible. We will often use this result to diagonalise a torus action (i.e. choose a basis of eigenvectors for the T-action so that the action is diagonal with respect to this basis).

Proposition 3.12. *For a finite dimensional linear representation of a torus $\rho : T \to \mathrm{GL}(V)$, there is a weight space decomposition*

$$V \cong \bigoplus_{\chi \in X^*(T)} V_\chi$$

where $V_\chi = \{v \in V : t \cdot v = \chi(t)v \ \forall t \in T\}$ are called the weight spaces *and $\{\chi : V_\chi \neq 0\}$ are called the* weights *of the action.*

Proof. To keep the notation simple, we give the proof for $T \cong \mathbb{G}_m$, where $X^*(T) \cong \mathbb{Z}$; the general case can be obtained either by adapting the proof (with further notation) or by induction on the dimension of T. The representation ρ has an associated co-module

$$\rho^* : V \to V \otimes_k \mathcal{O}(\mathbb{G}_m) \cong V \otimes k[t, t^{-1}].$$

and the diagram

$$\begin{array}{ccc} V & \xrightarrow{\ \rho\ } & V \otimes k[t, t^{-1}] \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle \mathrm{id} \otimes m^*} \\ V \otimes k[t, t^{-1}] & \xrightarrow{\ \rho \otimes \mathrm{id}\ } & V \otimes k[t, t^{-1}] \otimes k[t, t^{-1}] \end{array}$$

commutes. From this, it follows easily that, for each integer m, the space

$$V_m = \{v \in V : \rho^*(v) = v \otimes t^m\}$$

is a subrepresentation of V.

For $v \in V$, we have $\rho^*(v) = \sum_{m \in \mathbb{Z}} f_m(v) \otimes t^m$ where $f_m : V \to V$ is a linear map, and because of the compatibility with the identity element, we find that

$$v = \sum_{m \in \mathbb{Z}} f_m(v).$$

If $\rho^*(v) = \sum_{m \in \mathbb{Z}} f_m(v) \otimes t^m$, then we claim that $f_m(v) \in V_m$. From the diagram above

$$\sum_{m \in \mathbb{Z}} \rho^*(f_m(v)) \otimes t^m = (\rho^* \otimes \mathrm{Id}_{k[t,t^{-1}]})(\rho^*(v)) = (\mathrm{Id}_V \otimes m^*)(\rho^*(v)) = \sum_{m \in \mathbb{Z}} f_m(v) \otimes t^m \otimes t^m$$

and as $\{t^m\}_{m \in \mathbb{Z}}$ are linearly independent in $k[t, t^{-1}]$, the claim follows.

Let us show that in fact, the f_m form a collection of orthogonal projectors onto the subspaces V_m. Using the commutative diagram again, we get

$$\sum_{m \in \mathbb{Z}} f_m(v) \otimes t^m \otimes t^m = \sum_{m,n \in \mathbb{Z}} f_m(f_n(v)) \otimes t^m \otimes t^n,$$

which again by linear independence of the $\{t^m\}$ shows that $f_m \circ f_n$ vanishes if $m \neq n$ and is equal to f_n otherwise; this proves that they are orthogonal idempotents. Hence, the V_m are linearly independent and this completes the proof. □

This result can be phrased as follows: there is an equivalence between the category of linear representations of T and $X^*(T)$-graded k-vector spaces. We note that there are only finitely many weights of the T-action, for reasons of dimension.

3.3. **Orbits and stabilisers.**

Definition 3.13. Let G be an affine algebraic group acting on a scheme X by $\sigma : G \times X \to X$ and let x be a k-point of X.

i) The orbit $G \cdot x$ of x to be the (set-theoretic) image of the morphism $\sigma_x = \sigma(-, x) : G(k) \to X(k)$ given by $g \mapsto g \cdot x$.
ii) The stabiliser G_x of x to be the fibre product of $\sigma_x : G \to X$ and $x : \operatorname{Spec} k \to X$.

The stabiliser G_x of x is a closed subscheme of G (as it is the preimage of a closed subscheme of X under $\sigma_x : G \to X$). Furthermore, it is a subgroup of G.

Exercise 3.14. Using the same notation as above, consider the presheaf on Sch whose S-points are the set

$$\{g \in h_G(S) : g \cdot (x_S) = x_S\}$$

where $x_S : S \to X$ is the composition $S \to \operatorname{Spec} k \to X$ of the structure morphism of S with the inclusion of the point x. Describe the presheaf structure and show that this functor is representable by the stabiliser G_x.

The situation for orbits is clarified by the following result.

Proposition 3.15. *Let G be an affine algebraic group acting on a scheme X. The orbits of closed points are locally closed subsets of X, hence can be identified with the corresponding reduced locally closed subschemes.*

Moreover, the boundary of an orbit $\overline{G \cdot x} - G \cdot x$ is a union of orbits of strictly smaller dimension. In particular, each orbit closure contains a closed orbit (of minimal dimension).

Proof. Let $x \in X(k)$. The orbit $G \cdot x$ is the set-theoretic image of the morphism σ_x, hence by a theorem of Chevalley ([14] II Exercise 3.19), it is constructible, i.e., there exists a dense open subset U of $\overline{G \cdot x}$ with $U \subset G \cdot x \subset \overline{G \cdot x}$. Because G acts transitively on $G \cdot x$ through σ_x, this implies that every point of $G \cdot x$ is contained in a translate of U. This shows that $G \cdot x$ is open in $\overline{G \cdot x}$, which precisely means that $G \cdot x$ is locally closed. With the corresponding reduced scheme structure of $G \cdot x$, there is an action of G_{red} on $G \cdot x$ which is transitive on k-points. In particular, it makes sense to talk about its dimension (which is the same at every point because of the transitive action of G_{red}).

The boundary of an orbit $G \cdot x$ is invariant under the action of G and so is a union of G-orbits. Since $G \cdot x$ is locally closed, the boundary $\overline{G \cdot x} - G \cdot x$, being the complement of a dense open set, is closed and of strictly lower dimension than $G \cdot x$. This implies that orbits of minimum dimension are closed and so each orbit closure contains a closed orbit. ☐

Definition 3.16. An action of an affine algebraic group G on a scheme X is *closed* if all G-orbits in X are closed.

Example 3.17. Consider the action of $\mathbb{G}_m$ on $\mathbb{A}^2$ by $t \cdot (x, y) = (tx, t^{-1}y)$. The orbits of this action are

- conics $\{(x, y) : xy = \alpha\}$ for $\alpha \in \mathbb{A}^1 - \{0\}$,
- the punctured x-axis,
- the punctured y-axis,
- the origin.

The origin and the conic orbits are closed whereas the punctured axes both contain the origin in their orbit closures. The dimension of the orbit of the origin is strictly smaller than the dimension of $\mathbb{G}_m$, indicating that its stabiliser has positive dimension.

Example 3.18. Let $\mathbb{G}_m$ act on $\mathbb{A}^n$ by scalar multiplication: $t \cdot (a_1, \dots, a_n) = (ta_1, \dots, ta_n)$. In this case, there are two types of orbits:

- punctured lines through the origin,
- the origin.

The origin is the only closed orbit, which has dimension zero. Furthermore, every orbit contains the origin in its closure.

Exercise 3.19. In Examples 3.17 and 3.18, write down the coaction homomorphism explicitly.

Proposition 3.20. *Let G be an affine algebraic group acting on a scheme X. For $x \in X(k)$, we have*

$$\dim(G) = \dim(G_x) + \dim(G \cdot x)$$

Proof. Since the dimension is a topological invariant of a scheme, we can assume G and X are reduced. The orbit $G \cdot x$, which we see as a locally closed subscheme of X according to the previous proposition, is reduced by definition. This implies that the morphism $\sigma_x : G \to G \cdot x$ is flat at every generic point of $G \cdot x$ (every k-scheme is flat over k), hence, by the openess of the flat locus of σ_x (EGA IV_3 11.1.1), there exists a dense open set U such that $\sigma_x^{-1}(U) \to U$ is flat. Using the transitive action of G on $G \cdot x$ (which is well defined because G is reduced), we deduce that σ_x is flat. Moreover, by definition, the fibre of σ_x at x is the stabiliser G_x. We can thus apply the dimension formula for fibres of a flat morphism [14, Proposition III.9.5], which yields

$$\dim(G_x) = \dim(G) - \dim(G \cdot x)$$

as required. □

Proposition 3.21. *Let G be an affine algebraic group acting on a scheme X by a morphism $\sigma : G \times X \to X$. Then the dimension of the stabiliser subgroup (resp. orbit) viewed as a function $X \to \mathbb{N}$ is upper semi-continuous (resp. lower-semi-continuous); that is, for every n, the sets*

$$\{x \in X : \dim G_x \geq n\} \text{ and } \{x \in X : \dim(G \cdot x) \leq n\}$$

are closed in X.

Proof. Consider the graph of the action

$$\Gamma = (\mathrm{pr}_X, \sigma) : G \times X \to X \times X$$

and the fibre product P

$$\begin{array}{ccc} P & \xrightarrow{\varphi} & X \\ \downarrow & & \downarrow \Delta \\ G \times X & \xrightarrow{\Gamma} & X \times X, \end{array}$$

where $\Delta : X \to X \times X$ is the diagonal morphism; then the k-points of the fibre product P consists of pairs (g, x) such that $g \in G_x$. The function on P which sends $p = (g, x) \in P$ to the dimension of $P_{\varphi(p)} := \varphi^{-1}(\varphi(p))$ is upper semi-continuous (cf. [14] III 12.8 or EGA IV 13.1.3); that is, for all n

$$\{p \in P : \dim P_{\varphi(p)} \geq n\}$$

is closed in P. By restricting to the closed subscheme $X \cong \{(e, x) : x \in X\} \subset P$, we conclude that the dimension of the stabiliser of x is upper semi-continuous; that is,

$$\{x \in X : \dim G_x \geq n\}$$

is closed in X for all n. Using the previous proposition, we deduce the statement for dimensions of orbits. □

Lemma 3.22. *Let G be an affine algebraic group acting on a scheme X over k.*

i) If G is an affine group variety and Y and Z are subschemes of X such that Z is closed, then

$$\{g \in G : gY \subset Z\}$$

is closed.

ii) If X is a variety, then for any subgroup $H \subset G$ the fixed point locus

$$X^H = \{x \in X : H \cdot x = x\}$$

is closed in X.

Proof. Exercise. (Hint: express these subsets as intersections of preimages of closed subschemes under morphisms associated to the action.) □

3.4. **First notions of quotients.** Let G be an affine algebraic group acting on a scheme X over k. In this section and §3.5, we introduce different types of quotients for the action of G on X; the main references for these sections are [4], [25] and [31].

The orbit space $X/G = \{G \cdot x : x \in X\}$ for the G-action on X, may not always admit the structure of a scheme. Instead we ask for a universal quotient in the category of schemes (of finite type over k).

Definition 3.23. A *categorical quotient* for the action of G on X is a G-invariant morphism $\varphi : X \to Y$ of schemes which is universal; that is, every other G-invariant morphism $f : X \to Z$ factors uniquely through φ so that there exists a unique morphism $h : Y \to Z$ such that $f = \varphi \circ h$. Furthermore, if the preimage of each k-point in Y is a single orbit, then we say φ is an *orbit space*.

As φ is constant on orbits, it is also constant on orbit closures. Hence, a categorical quotient is an orbit space only if the action of G on X is closed; that is, all the orbits $G \cdot x$ are closed.

Remark 3.24. The categorical quotient has nice functorial properties in the following sense: if $\varphi : X \to Y$ is G-invariant and we have an open cover U_i of Y such that $\varphi| : \varphi^{-1}(U_i) \to U_i$ is a categorical quotient for each i, then φ is a categorical quotient.

Exercise 3.25. Let $\varphi : X \to Y$ be a categorical quotient of a G-action on X.

i) If X is connected, show that Y is connected.
ii) If X is irreducible, show that Y is irreducible.
iii) If X is reduced, show that Y is reduced.

Example 3.26. We consider the action of $\mathbb{G}_m$ on $\mathbb{A}^n$ as in Example 3.18. As the origin is in the closure of every single orbit, any G-invariant morphism $\mathbb{A}^n \to Z$ must be a constant morphism. Therefore, we claim that the categorical quotient is the structure map $\varphi : \mathbb{A}^n \to \operatorname{Spec} k$ to the point $\operatorname{Spec} k$. This morphism is clearly G-invariant and any other G-invariant morphism $f : \mathbb{A}^n \to Z$ is a constant morphism to $z \in Z(k)$. Therefore, there is a unique morphism $z : \operatorname{Spec} k \to Z$ such that $f = z \circ \varphi$.

We now see the sort of problems that may occur when we have non-closed orbits. In Example 3.18 our geometric intuition tells us that we would ideally like to remove the origin and then take the quotient of $\mathbb{G}_m$ acting on $\mathbb{A}^n - \{0\}$. In fact, we already know what we want this quotient to be: the projective space $\mathbb{P}^{n-1} = (\mathbb{A}^n - \{0\})/\mathbb{G}_m$ which is an orbit space for this action.

3.5. **Second notions of quotient.** Let G be an affine algebraic group acting on a scheme X over k. The group G acts on the k-algebra $\mathcal{O}(X)$ of regular functions on X by

$$g \cdot f(x) = f(g^{-1} \cdot x)$$

and we denote the subalgebra of invariant functions by

$$\mathcal{O}(X)^G := \{f \in \mathcal{O}(X) : g \cdot f = f \text{ for all } g \in G\}.$$

Similarly if $U \subset X$ is a subset which is invariant under the action of G (that is, $g \cdot u \in U$ for all $u \in U$ and $g \in G$), then G acts on $\mathcal{O}_X(U)$ and we write $\mathcal{O}_X(U)^G$ for the subalgebra of invariant functions.

The following notion of a good quotient came out of geometric invariant theory; more precisely, we will later see that GIT quotients are good quotients. However, it is clear that many of the properties of a good quotient are desirable. Furthermore, we will soon see that a good quotient is a categorical quotient.

Definition 3.27. A morphism $\varphi : X \to Y$ is a *good quotient* for the action of G on X if

i) φ is G-invariant.
ii) φ is surjective.
iii) If $U \subset Y$ is an open subset, the morphism $\mathcal{O}_Y(U) \to \mathcal{O}_X(\varphi^{-1}(U))$ is an isomorphism onto the G-invariant functions $\mathcal{O}_X(\varphi^{-1}(U))^G$.
iv) If $W \subset X$ is a G-invariant closed subset of X, its image $\varphi(W)$ is closed in Y.

v) If W_1 and W_2 are disjoint G-invariant closed subsets, then $\varphi(W_1)$ and $\varphi(W_2)$ are disjoint.
vi) φ is affine (i.e. the preimage of every affine open is affine).

If moreover, the preimage of each point is a single orbit then we say φ is a *geometric quotient.*

Exercise 3.28. Assuming that ii) holds, prove that conditions iv) and v) together are equivalent to:

v)′ If W_1 and W_2 are disjoint G-invariant closed subsets, then the closures of $\varphi(W_1)$ and $\varphi(W_2)$ are disjoint.

Remark 3.29. In fact, surjectivity is a consequence of iii) and iv): condition iii) shows that φ is dominant (i.e. the image of φ is dense in Y) and condition iv) shows that the image of φ is closed in Y.

Proposition 3.30. *Let G be an affine algebraic group acting on a scheme X and suppose we have a morphism $\varphi : X \to Y$ satisfying properties i), iii), iv) and v) in the definition of good quotient. Then φ is a categorical quotient. In particular, any good quotient is a categorical quotient.*

Proof. Property i) of the definition of a good quotient states that φ is G-invariant and so we need only prove that it is universal with respect to all G-invariant morphisms from X. Let $f : X \to Z$ be a G-invariant morphism; then we will construct a unique morphism $h : Y \to Z$ such that $f = h \circ \varphi$ by taking a finite affine open cover U_i of Z (we can take the cover to be finite as Z is of finite type over k), then using this cover to define a cover of Y by open subsets V_i, and finally by locally defining morphisms $h_i : V_i \to U_i$ which glue to give h.

Since $W_i := X - f^{-1}(U_i)$ is G-invariant and closed in X, its image $\varphi(W_i) \subset Y$ is closed by iv). Let $V_i := Y - \varphi(W_i)$ be the open complement; then by construction, we have an inclusion $\varphi^{-1}(V_i) \subset f^{-1}(U_i)$. As U_i cover Z, the intersection $\cap_i W_i$ is empty. We claim by property v) of the good quotient φ, we have $\cap_i \varphi(W_i) = \emptyset$; that is, V_i are an open cover of Y. To see this, suppose for a contradiction that the intersection $\cap_i \varphi(W_i)$ is non-empty; then as we are working with finite type schemes, this intersection has a closed point, which is a k-point as k is algebraically closed. Let W be a closed G-orbit in the preimage of the k-point $p \in \cap_i \varphi(W_i)$. Then by property v), we must have $W \cap W_i \neq \emptyset$ for each i, since $\varphi(W) \cap \varphi(W_i) \neq \emptyset$. Since W is a single G-orbit and each W_i is G-invariant, we must have $W \subset W_i$ and thus $W \subset \cap_i W_i$, which gives a contradiction.

Since f is G-invariant the homomorphism $\mathcal{O}_Z(U_i) \to \mathcal{O}_X(f^{-1}(U_i))$ has image in $\mathcal{O}_X(f^{-1}(U_i))^G$. Therefore, there is a unique morphism h_i^* which makes the following square commute

$$\begin{array}{ccc} \mathcal{O}_Z(U_i) & \xrightarrow{h_i^*} & \mathcal{O}_Y(V_i) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle \cong\ \varphi^*} \\ \mathcal{O}_X(f^{-1}(U_i))^G & \longrightarrow & \mathcal{O}_X(\varphi^{-1}(V_i))^G \end{array}$$

where the isomorphism on the right hand side of this square is given by property iii) of the good quotient φ. Since U_i is affine, the k-algebra homomorphism $\mathcal{O}_Z(U_i) \to \mathcal{O}_Y(V_i)$ corresponds to a morphism $h_i : V_i \to U_i$ (see [14] I Proposition 3.5). By construction

$$f|_{\varphi^{-1}(V_i)} = h_i \circ \varphi|_{\varphi^{-1}(V_i)} : \varphi^{-1}(V_i) \to U_i$$

and $h_i = h_j$ on $V_i \cap V_j$; therefore, we can glue the morphisms h_i to obtain a morphism $h : Y \to Z$ such that $f = h \circ \varphi$. Since the morphisms h_i are unique, it follows that h is also unique. □

Example 3.31. We consider the action of $\mathbb{G}_m$ on $\mathbb{A}^2$ as in Example 3.17. As the origin is in the closure of the punctured axes $\{(x, 0) : x \neq 0\}$ and $\{(0, y) : y \neq 0\}$, all three orbits will be identified by the categorical quotient. The smooth conic orbits $\{(x, y) : xy = \alpha\}$ for $\alpha \in \mathbb{A}^1 - \{0\}$ are closed. These conic orbits are parametrised by $\mathbb{A}^1 - \{0\}$ and the remaining three orbits will all be identified in the categorical quotient. Therefore, we may naturally expect that $\varphi : \mathbb{A}^2 \to \mathbb{A}^1$ given by $(x, y) \mapsto xy$ is a categorical quotient. In fact, we will prove that this is a good quotient and therefore also a categorical quotient. This morphism is clearly G-invariant and surjective, which shows parts i) and ii).

For iii), let $U \subset \mathbb{A}^1$ be an open subset and consider the morphism

$$\varphi^* : \mathcal{O}_{\mathbb{A}^1}(U) \to \mathcal{O}_{\mathbb{A}^2}(\varphi^{-1}(U)).$$

For $U = \mathbb{A}^1$, we have $\varphi^* : \mathbb{C}[z] \to \mathbb{C}[x,y]$ given by $z \mapsto xy$. We claim that this is an isomorphism on the ring of $\mathbb{G}_m$-invariant functions. The action of $t \in \mathbb{G}_m$ on $\mathcal{O}(\mathbb{A}^2) = \mathbb{C}[x,y]$ is given by

$$t \cdot \left(\sum_{i,j} a_{ij} x^i y^j \right) = \sum_{i,j} a_{ij} t^{j-i} x^i y^j.$$

Therefore, the invariant subalgebra is

$$\mathbb{C}[x,y]^{\mathbb{C}^*} = \left\{ \sum_{ij} a_{ij} x^i y^j : a_{ij} = 0 \, \forall i \neq j \right\} = \mathbb{C}[xy]$$

as required. Now suppose we have an open subset $U \subsetneq \mathbb{A}^1$; then $U = \mathbb{A}^1 - \{a_1, \dots, a_n\}$ and $\mathcal{O}_{\mathbb{A}^1}(U) = \mathbb{C}[z]_{(f)}$ where $f(z) = (z - a_1) \cdots (z - a_n) \in \mathbb{C}[z]$. Then $\varphi^{-1}(U)$ is the non-vanishing locus of $F(x,y) := f(xy) \in \mathbb{C}[x,y]$ and $\mathcal{O}_{\mathbb{A}^2}(\varphi^{-1}(U)) = \mathbb{C}[x,y]_F$. In particular, we can directly verify that

$$\mathcal{O}_{\mathbb{A}^2}(\varphi^{-1}(U))^{\mathbb{G}_m} = (\mathbb{C}[x,y]_F)^{\mathbb{G}_m} = \left(\mathbb{C}[x,y]^{\mathbb{G}_m}\right)_F = \mathbb{C}[xy]_F \cong \mathbb{C}[z]_f = \mathcal{O}_{\mathbb{A}^1}(U).$$

For v)$'$, we note that any G-invariant closed subvariety in $\mathbb{A}^2$ is either a finite union of orbit closures or the entire space $\mathbb{A}^2$. Therefore, we can assume that the disjoint G-invariant closed subsets W_1 and W_2 are both a finite union of orbit closures and even just that $W_i = \overline{G \cdot p_i}$ are disjoint for $i = 1, 2$. Since we have already determined the orbit closures, we see that there are two cases two consider: either p_1 and p_2 both do not lie on the axes in $\mathbb{A}^2$ (and so their orbits correspond to disjoint conics $\{(x,y) : xy = \alpha_i\}$ and $\varphi(W_1) = \alpha_1 \neq \alpha_2 = \varphi(W_2)$) or one of the points, say p_1 lies on an axis, so that $\varphi(W_1) = 0$, and the second point p_2 cannot also lie on an axis as we assumed the closures of the orbits were disjoint, so $\varphi(W_2) \neq 0$.

Trivially vi) holds, as any morphism of affine schemes is affine.

Finally, we note that φ is not a geometric quotient, as $\varphi^{-1}(0)$ is a union of 3 orbits.

Corollary 3.32. *Let G be an affine algebraic group acting on a scheme X and let $\varphi : X \to Y$ be a good quotient; then:*

- *a) $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \neq \phi$ if and only if $\varphi(x_1) = \varphi(x_2)$.*
- *b) For each $y \in Y$, the preimage $\varphi^{-1}(y)$ contains a unique closed orbit. In particular, if the action is closed (i.e. all orbits are closed), then φ is a geometric quotient.*

Proof. a). As φ is constant on orbit closures, it follows that $\varphi(x_1) = \varphi(x_2)$ if $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \neq \phi$. By property v) of the good quotient φ, we get the converse. For b), suppose we have two distinct closed orbits W_1 and W_2 in $\varphi^{-1}(y)$, then the fact that their images under φ are both equal to y contradicts property v) of the good quotient φ. □

Corollary 3.33. *If $\varphi : X \to Y$ is a good (resp. geometric) quotient, then for every open $U \subset Y$ the restriction $\varphi| : \varphi^{-1}(U) \to U$ is also a good (resp. geometric) quotient of G acting on $\varphi^{-1}(U)$.*

Proof. Exercise. □

Remark 3.34. The definition of good and geometric quotients are local in the target; thus if $\varphi : X \to Y$ is G-invariant and we have a cover of Y by open sets U_i such that $\varphi| : \varphi^{-1}(U_i) \to U_i$ are all good (respectively geometric) quotients, then so is $\varphi : X \to Y$. We leave the proof as an exercise.

3.6. **Moduli spaces and quotients.** Let us give one result about the construction of moduli spaces using group quotients. For a moduli problem $\mathcal{M}$, a family $\mathcal{F}$ over a scheme S has the *local universal property* if for any other family $\mathcal{G}$ over a scheme T and for any k-point $t \in T$, there exists a neighbourhood U of t in T and a morphism $f : U \to S$ such that $\mathcal{G}|_U \sim_U f^*\mathcal{F}$.

Proposition 3.35. *For a moduli problem $\mathcal{M}$, let $\mathcal{F}$ be a family with the local universal property over a scheme S. Furthermore, suppose that there is an algebraic group G acting on S such that two k-points s, t lie in the same G-orbit if and only if $\mathcal{F}_t \sim \mathcal{F}_s$. Then*

a) any coarse moduli space is a categorical quotient of the G-action on S;
b) a categorical quotient of the G-action on S is a coarse moduli space if and only if it is an orbit space.

Proof. For any scheme M, we claim that there is a bijective correspondence

$$\{\text{natural transformations } \eta : \mathcal{M} \to h_M\} \longleftrightarrow \{G\text{-invariant morphisms } f : S \to M\}$$

given by $\eta \mapsto \eta_S(\mathcal{F})$, which is G-invariant by our assumptions about the G-action on S. The inverse of this correspondence associates to a G-invariant morphism $f : S \to M$ and a family $\mathcal{G}$ over T a morphism $\eta_T(\mathcal{G}) : T \to M$ by using the local universal property of $\mathcal{F}$ over S. More precisely, we can cover T by open subsets U_i such that there is a morphism $h_i : U_i \to S$ and $h_i^*\mathcal{F} \sim_{U_i} \mathcal{G}|_{U_i}$. For $u \in U_i \cap U_j$, we have

$$\mathcal{F}_{h_i(u)} \sim (h_i^*\mathcal{F})_u \sim \mathcal{G}_u \sim (h_j^*\mathcal{F})_u \sim \mathcal{F}_{h_j(u)}$$

and so by assumption $h_i(u)$ and $h_j(u)$ lie in the same G-orbit. Since f is G-invariant, we can glue the compositions $f \circ h_i : U_i \to M$ to a morphism $\eta_T(\mathcal{G}) : T \to M$. We leave it to the reader to verify that this determines a natural transformation η (that is, this is functorial with respect to morphisms) and that these correspondences are inverse to each other.

Hence, if $(M, \eta : \mathcal{M} \to h_M)$ is a coarse moduli space, then $\eta_S(\mathcal{F}) : S \to M$ is G-invariant and universal amongst all G-invariant morphisms from S, by the universality of η. This proves statement a). Furthermore, the G-invariant morphism $\eta_S(\mathcal{F}) : S \to M$ is an orbit space if and only if $\eta_{\operatorname{Spec} k}$ is bijective. This proves statement b). □

4. Affine Geometric Invariant Theory

In this section we consider an action of an affine algebraic group G on an affine scheme X of finite type over k and show that this action has a good quotient when G is linearly reductive. The main references for this section are [25] and [31] (for further reading, see also [2], [4] and [32]).

Let X be an affine scheme of finite type over k; then the ring of regular functions $\mathcal{O}(X)$ is a finitely generated k-algebra. Conversely, for any finitely generated k-algebra A, the spectrum of prime ideals $\operatorname{Spec} A$ is an affine scheme of finite type over k.

The action of an affine algebraic group G on an affine scheme X given by a morphism

$$\sigma : G \times X \to X$$

corresponds to a homomorphism of k-algebras $\sigma^* : \mathcal{O}(X) \to \mathcal{O}(G \times X) \cong \mathcal{O}(G) \otimes_k \mathcal{O}(X)$, which gives a G-co-module structure on the (typically infinite dimensional) k-vector space $\mathcal{O}(X)$. This co-module structure in turn determines a linear representation $G \to \mathrm{GL}(\mathcal{O}(X))$. Concretely, on the level of k-points, the action of $g \in G(k)$ on $f \in \mathcal{O}(X)$ is given by

$$(g \cdot f)(x) = f(g^{-1} \cdot x).$$

The ring of G-invariant regular functions on X is

$$\mathcal{O}(X)^G := \{f \in \mathcal{O}(X) : \sigma^*(f) = 1 \otimes f\}.$$

Any G-invariant morphism $\varphi : X \to Z$ of schemes induces a homomorphism $\varphi^* : \mathcal{O}(Z) \to \mathcal{O}(X)$ whose image is contained in the subalgebra of G-invariant regular functions $\mathcal{O}(X)^G$. This leads us to an interesting problem in invariant theory which was first considered by Hilbert.

4.1. **Hilbert's 14th problem.** For a rational action of an affine algebraic group G on a finitely generated k-algebra A, Hilbert asked whether the algebra of G-invariants A^G is finitely generated.

The answer to Hilbert's 14th problem is negative in this level of generality: Nagata gave an example of an action of an affine algebraic group (constructed using copies of the additive groups) for which the ring of invariants is not finitely generated (see [27] and [29]). However, for reductive groups (which we introduce below), the answer is positive due to a Theorem of Nagata. The proof of this result is beyond the scope of this course. However, we will prove that for a rational action of a 'linearly reductive' group on an algebra, the subalgebra of invariants is finitely generated, using a Reynolds operator, which essentially mimics Hilbert's 19th century proof that, over the complex numbers, a rational action of the general linear group GL_n on an algebra has a finitely generated invariant subalgebra.

4.2. **Reductive groups.** In this section, we will give the definition of a reductive group, a linearly reductive group and a geometrically reductive group, and explain the relationship between these different notions of reductivity.

Our starting point is the Jordan decomposition for affine algebraic groups over k. We first recall the Jordan decomposition for GL_n: an element $g \in \mathrm{GL}_n(k)$ has a decomposition

$$g = g_{ss}g_u = g_u g_{ss}$$

where g_{ss} is semisimple (or, equivalently, diagonalisable, as k is algebraically closed) and g_u is unipotent (that is, $g - I_n$ is nilpotent).

For any affine algebraic group G, we would like to have an analogous decomposition, and we can hope to make use of the fact that G admits a faithful linear representation $G \hookrightarrow \mathrm{GL}_n$. However, this would require the decomposition to be functorial with respect to closed immersions of groups.

Definition 4.1. Let G be an affine algebraic group over k. An element g is *semisimple* (resp. *unipotent*) if there is a faithful linear representation $\rho : G \hookrightarrow \mathrm{GL}_n$ such that $\rho(g)$ is diagonalisable (resp. unipotent).

Theorem 4.2 (Jordan decomposition, see [23] X Theorem 2.8 and 2.10)**.** *Let G be an affine algebraic group over k. For every $g \in G(k)$, there exists a unique semisimple element g_{ss} and a unique unipotent element g_u such that*

$$g = g_{ss}g_u = g_u g_{ss}.$$

Furthermore, this decomposition is functorial with respect to group homomorphisms. In particular, if $g \in G(k)$ is semisimple (resp. unipotent), then for all linear representations $\rho : G \to \mathrm{GL}_n$, the element $\rho(g)$ is semisimple (resp. unipotent).

Let $\rho : G \to \mathrm{GL}(V)$ be a linear representation of an affine algebraic group G on a vector space V and let $\rho^* : V \to \mathcal{O}(G) \otimes_k V$ denote the associated co-module. Then a vector subspace $V' \subset V$ is *G-invariant* if $\rho^*(V') \subset \mathcal{O}(G) \otimes_k V'$ and a vector $v \in V$ is *G-invariant* if $\rho^*(v) = 1 \otimes v$. We let V^G denote the subspace of G-invariant vectors.

Definition 4.3. An affine algebraic group G is *unipotent* if every non-trivial linear representation $\rho : G \to \mathrm{GL}(V)$ has a non-zero G-invariant vector.

Proposition 4.4. *For an affine algebraic group G, the following statements are equivalent.*

i) G is unipotent.
ii) For every representation $\rho : G \to \mathrm{GL}(V)$ there is a basis of V such that $\rho(G)$ is contained in the subgroup $\mathbb{U} \subset \mathrm{GL}(V)$ consisting of upper triangular matrices with diagonal entries equal to 1.
iii) G is isomorphic to a subgroup of a standard unipotent group $\mathbb{U}_n \subset \mathrm{GL}_n$ consisting of upper triangular matrices with diagonal entries equal to 1.

Proof. i) $\iff$ ii): If $e_1, \dots, e_n$ is a basis of V such that $\rho(G) \subset \mathbb{U}$, then e_1 is fixed by ρ. Conversely if $\rho : G \to \mathrm{GL}(V)$ is a representation of a unipotent group G, then we can proceed by induction on the dimension of V. As U is unipotent, the linear subspace of G-fixed points V^G is non-zero; let $e_1, \cdots e_m$ be a basis of V^G. Then there is a basis $\overline{e}_{m+1}, \dots, \overline{e}_n$ of V/V^G such that the induced representation has image in the upper triangular matrices with diagonal entries equal to 1. By choosing lifts $e_{m+i} \in V$ of $\overline{e}_{m+i}$, we get the desired basis of V.

ii) $\iff$ iii): As every affine algebraic group G has a faithful representation $\rho : G \to \mathrm{GL}_n$, we see that ii) implies iii). Conversely, any subgroup of $\mathbb{U}_n$ is unipotent (see [23] XV Theorem 2.4). □

Remark 4.5. If G is a unipotent affine algebraic group, then every $g \in G(k)$ is unipotent. The converse is true if in addition G is smooth (for example, see [23] XV Corollary 2.6 or SGA3 XVII Corollary 3.8).

Example 4.6.

(1) The additive group $\mathbb{G}_a$ is unipotent, as we have an embedding $\mathbb{G}_a \hookrightarrow \mathbb{U}_2$ given by
$$c \mapsto \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}.$$
(2) In characteristic p, there is a finite subgroup $\alpha_p \subset \mathbb{G}_a$ where we define the functor of points of α_p by associating to a k-algebra R,
$$\alpha_p(R) := \{c \in \mathbb{G}_a(R) : c^p = 0\}.$$
This is represented by the scheme $\operatorname{Spec} k[t]/(t^p)$ and so α_p is a unipotent group which is not smooth.

Definition 4.7. An algebraic subgroup H of an affine algebraic group G is *normal* if the conjugation action $H \times G \to G$ given by $(h, g) \mapsto ghg^{-1}$ factors through $H \hookrightarrow G$.

Definition 4.8. An affine algebraic group G over k is *reductive* if it is smooth and every smooth unipotent normal algebraic subgroup of G is trivial.

Remark 4.9. In fact, one can define reductivity by saying that the unipotent radical of G (which is the maximal connected unipotent normal algebraic subgroup of G) is trivial; however, to define the unipotent radical carefully, we would need to prove that, for a group G, the subgroup generated by two smooth algebraic subgroups of G is also algebraic (see [22] Proposition 2.24).

Exercise 4.10. Show that the general linear group GL_n and the special linear group SL_n are reductive. [Hint: if we have a non-trivial smooth connected unipotent normal algebraic subgroup $U \subset \mathrm{GL}_n$, then there exists $g \in U(k) \subset \mathrm{GL}_n(k)$ whose Jordan normal form has a $r \times r$ Jordan block for $r > 1$ (as g is unipotent). Using normality of U, find another element $g' \in U(k)$ such that the product gg' is not unipotent.]

Definition 4.11. An affine algebraic group G is

(1) *linearly reductive* if every finite dimensional linear representation $\rho : G \to \mathrm{GL}(V)$ is completely reducible; that is the representation decomposes as a direct sum of irreducibles.
(2) *geometrically reductive* if, for every finite dimensional linear representation $\rho : G \to \mathrm{GL}(V)$ and every non-zero G-invariant point $v \in V$, there is a G-invariant non-constant homogeneous polynomial $f \in \mathcal{O}(V)$ such that $f(v) \neq 0$.

Example 4.12. Any algebraic torus $(\mathbb{G}_m)^r$ is linearly reductive by Proposition 3.12.

Exercise 4.13. Show directly that the additive group $\mathbb{G}_a$ is not geometrically reductive. [Hint: there is a representation $\rho : \mathbb{G}_a \to \mathrm{GL}_2$ and a G-invariant point $v \in \mathbb{A}^2$ such that every non-constant G-invariant homogeneous polynomial in two variables vanishes at v].

Proposition 4.14. *For an affine algebraic group G, the following statements are equivalent.*

i) G is linearly reductive.

ii) For any finite dimensional linear representation $\rho : G \to \mathrm{GL}(V)$, any G-invariant subspace $V' \subset V$ admits a G-stable complement (i.e. there is a subrepresentation $V'' \subset V$ such that $V = V' \oplus V''$).

iii) For any surjection of finite dimensional G-representations $\phi : V \to W$, the induced map on G-invariants $\phi^G : V^G \to W^G$ is surjective.

iv) For any finite dimensional linear representation $\rho : G \to \mathrm{GL}(V)$ and every non-zero G-invariant point $v \in V$, there is a G-invariant linear form $f : V \to k$ such that $f(v) \neq 0$.

v) For any finite dimensional linear representation $\rho : G \to \mathrm{GL}(V)$ and any surjective G-invariant linear form $f : V \to k$, there is $v \in V^G$ such that $f(v) \neq 0$.

Proof. The equivalence i) $\iff$ ii) is clear, as we are working with finite dimensional representations.

ii) $\implies$ iii): Let $f : V \to W$ be a surjection of finite dimensional G-representations and $V' := \ker(f) \subset V$. Then, by assumption, V' has a G-stable complement $V'' \cong W$. Since both V' and V'' are G-invariant, $V^G = (V')^G \oplus (V'')^G$ and so $f^G : V^G \to (V'')^G \cong W^G$ is surjective.

iii) $\implies$ ii): Let $\rho : G \to \mathrm{GL}(V)$ be a finite dimensional linear representation and $V' \subset V$ a G-invariant subspace. Then we have a surjection

$$\phi : \mathrm{Hom}(V, V') \to \mathrm{Hom}(V', V')$$

of finite dimensional G-representations and so by iii) the identity map id'_V lifts to G-equivariant morphism $f : V \to V'$ splitting the inclusion $V' \subset V$. More precisely, V' has G-stable complement $V'' := \ker f$.

iv) $\iff$ v): We can identify V^G with the space of G-invariant linear forms $V^\vee \to k$

$$V^G = \mathrm{Hom}_G(k, V) = \mathrm{Hom}_G(V^\vee, k).$$

iii) $\implies$ iv): Let V be a finite dimensional linear G-representation and $v \in V^G$ be a non-zero G-invariant vector. Then v determines a G-invariant linear form $\phi : V^\vee \to k$. By letting G act trivially on k, we can view ϕ as a surjection of G-representations and so by iii), the fixed point $1 \in k = k^G$ has a lift $f \in (V^\vee)^G = \mathrm{Hom}_G(V, k)$ such that $f(v) = 1$.

iv) $\implies$ iii): Let $\phi : V \to W$ be a finite dimensional G-representation. Then we want to prove that ϕ^G is surjective: i.e. lift any non-zero $w \in W^G$ to a point $v \in V^G$. By iv), there exists a G-invariant form $f : W \to k$ such that $f(w) \neq 0$. Then $f \circ \phi : V \to k$ is a G-invariant surjective form on V and so by v) $\iff$ iv), there exists $v \in V^G$ such that $(f \circ \phi)(v) \neq 0$. By suitably rescaling $v \in V^G$ so that $(f \circ \phi)(v) = f(w)$, we get the desired lift. □

Exercise 4.15. Prove that any finite group of order not divisible by the characteristic of k is linearly reductive. [Hint: consider averaging over the group.]

We summarise the main results relating the different notions of reductivity in the following theorem, whose proof is beyond the scope of this course.

Theorem 4.16. *(Weyl, Nagata, Mumford, Haboush)*

i) Every linearly reductive group is geometrically reductive.

ii) In characteristic zero, every reductive group is linearly reductive.

iii) A smooth affine algebraic group is reductive if and only if it is geometrically reductive.

In particular, for smooth affine algebraic group schemes, we have

$$\textit{linearly reductive} \implies \textit{geometrically reductive} \iff \textit{reductive}$$

and all three notions coincide in characteristic zero.

Statement i) follows immediately from the definition of geometrically reductive and Proposition 4.14. There are several proofs of Statement ii); the earliest goes back to Weyl, where he first reduces to $k = \mathbb{C}$, and then uses the representation theory of compact Lie groups (this argument is known an Weyl's unitary trick; see Proposition 4.18). An alternative approach is to use Lie algebras (for example, see the proof that SL_n is linearly reductive in characteristic zero in [24] Theorem 4.43). Statement iii) was conjectured by Mumford after Nagata proved that every geometrically reductive group is reductive [29], and the converse statement was proved by Haboush [12].

Remark 4.17. In positive characteristic, the groups GL_n, SL_n and PGL_n are not linearly reductive for $n > 1$; see [28].

We will now sketch the proof that over the complex numbers every reductive group is linearly reductive.

Proposition 4.18. *Every reductive group G over $\mathbb{C}$ is linearly reductive.*

Proof. We let $K \subset G(\mathbb{C})$ be a maximal compact subgroup.
Step 1. For a compact Lie subgroup K, we claim that every finite dimensional representation of the Lie group K is completely reducible. Let us sketch the proof of this claim. Let V be a finite dimensional representation of K (i.e. there is a morphism $\rho : K \to \mathrm{GL}(V)$ of Lie groups); then analogously to Proposition 4.14 above, it suffices to prove that every K-invariant subspace $W \subset V$ has a K-stable complement. There is a K-invariant Hermitian inner product on V, as we can take any Hermitian inner product h on V and integrate over the compact group K using a Haar measure $d\mu$ on K to obtain a K-invariant Hermitian inner product

$$h^K(v_1, v_2) := \int_K h(k \cdot v_1, k \cdot v_2) d\mu(k).$$

Then, we define the K-stable complement of $W \subset V$ to be the orthogonal complement of $W \subset V$ with respect to this K-invariant Hermitian inner product.
Step 2. For G reductive and a maximal compact subgroup $K \subset G(\mathbb{C})$, the elements of K are Zariski dense in G. We prove this statement in Lemma 4.19 below. The proof works with the Lie algebras $\mathfrak{k}$ and $\mathfrak{g}(\mathbb{C})$, using the fact that the exponential map $\exp : \mathfrak{g}(\mathbb{C}) \to G(\mathbb{C})$ is holomorphic, the fact that $\mathfrak{g}(\mathbb{C}) = \mathfrak{k}_{\mathbb{C}}$ as $G(\mathbb{C})$ is reductive (for a proof see, for example, [34] Theorem 2.7) and the Identity Theorem from complex analysis.
Step 3. For any finite dimensional linear representation $\rho : G \to \mathrm{GL}(V)$, we claim that $V^G = V^K$, where K is a maximal compact of G. As $K \subset G$ is a subgroup, we have $V^G \subset V^K$. To prove the reverse inclusion, let $v \in V^K$ and consider the morphism

$$\sigma : G \to V$$

given by $g \mapsto \rho(g) \cdot v$. Then $\sigma^{-1}(v) \subset G$ is Zariski closed. Since $v \in V^K$, we have $K \subset \sigma^{-1}(v)$ and so also $\overline{K} \subset \sigma^{-1}(v)$. However, as $K \subset G$ is Zariski dense, it follows that $G \subset \sigma^{-1}(v)$; that is, $v \in V^G$ as required.
Step 4. The reductive group G is linearly reductive. By Proposition 4.14, it suffices to show for every surjective homomorphism of finite dimensional linear G-representations $\phi : V \to W$, the induced homomorphism ϕ^G on invariant subspaces is also surjective. By Step 3, this is equivalent to showing that ϕ^K is surjective, which follows by Step 1. □

Lemma 4.19. *Over the complex numbers, let G be a reductive group and $K \subset G(\mathbb{C})$ be a maximal compact subgroup. Then the elements of K are Zariski dense in G.*

Proof. If this is not the case, then there exists a function $f \in \mathcal{O}(G)$ which is not identically zero such that $f(K) = 0$. On the level of Lie algebras, as $G(\mathbb{C})$ is a complex reductive group and $K \subset G(\mathbb{C})$ a maximal compact subgroup, we have

$$\mathfrak{g}(\mathbb{C}) = \mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C}$$

(see [34] Theorem 2.7). Furthermore, the exponential map $\exp : \mathfrak{g}(\mathbb{C}) \to G(\mathbb{C})$ is holomorphic and maps $\mathfrak{k}$ to K. Therefore, $h := f \circ \exp : \mathfrak{g}(\mathbb{C}) \to \mathbb{C}$ is holomorphic and vanishes on $\mathfrak{k}$. However, if V is a real vector space and $l : V \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C}$ is holomorphic with $l(V \otimes_{\mathbb{R}} \mathbb{R}) = 0$, then l is identically zero (the proof of this follows from the Identity Theorem in complex analysis when V has dimension 1 and, for higher dimensional V, we can view l as a function in a single variable x_i by fixing all other variables and by applying this argument for each i, we deduce that $l = 0$). In particular, $h : \mathfrak{g}(\mathbb{C}) \to \mathbb{C}$ is identically zero and, as the exponential map is a local homeomorphism, we deduce that f is identically zero which is a contradiction. □

4.3. **Nagata's theorem.** In this section, when we talk of a group G acting on a k-algebra A, we will always mean that the group G acts by k-algebra homomorphisms. We recall that a G-action on a k-algebra A is rational if every element of A is contained in a finite dimensional G-invariant linear subspace of A. In particular, if $A = \mathcal{O}(X)$ and the G-action on A comes from an algebraic action of an affine algebraic group G on X, then this action is rational by Lemma 3.8.

Theorem 4.20 (Nagata)**.** *Let G be a geometrically reductive group acting rationally on a finitely generated k-algebra A. Then the G-invariant subalgebra A^G is finitely generated.*

As every reductive group is geometrically reductive, we can use Nagata's theorem for reductive groups. In the following section, we will prove this result for linearly reductive groups using Reynolds operators (so in characteristic zero this also proves Nagata's theorem). Nagata also gave a counterexample of a non-reductive group action for which the ring of invariants is not finitely generated (see [27] and [29]).

4.4. **Reynolds operators.** Given a linearly reductive group G, for any finite dimensional linear representation $\rho : G \to \mathrm{GL}(V)$, we can write $V = V^G \oplus W$ where W is the direct sum of all non-trivial irreducible subrepresentations. This gives a canonical G-complement W to V^G and a unique projection $p_V : V \to V^G$. This projection motivates the following definition.

Definition 4.21. For a group G acting on a k-algebra A, a linear map $R_A : A \to A^G$ is called a *Reynolds operator* if it is a projection onto A^G and, for $a \in A^G$ and $b \in A$, we have $R_A(ab) = aR_A(b)$.

Lemma 4.22. *Let G be a linearly reductive group acting rationally on a finitely generated k-algebra A; then there exists a Reynolds operator $R_A : A \to A^G$.*

Proof. Since A is finitely generated, it has a countable basis. Therefore, we can write A as an increasing union of finite dimensional G-invariant vector subspaces $A_n \subset A$ using the fact that the action is rational. More precisely, if we label our basis elements $a_1, a_2, \dots,$ then we iteratively construct the subsets A_n by letting A_n be the finite dimensional G-invariant subspace containing $a_1, \dots, a_n$ and a basses of A_{n-1} and $a_j \cdot A_{n-1}$ for $j = 1, \dots, n$. Then $A = \bigcup_{n \geq 1} A_n$. Since G is linearly reductive and each A_n is a finite dimensional G-representation, we can write

$$A_n = A_n^G \oplus A_n'$$

where A_n' is the direct sum of all non-trivial irreducible G-subrepresentations of A_n. We let $R_n : A_n \to A_n^G$ be the canonical projection onto the direct factor A_n^G.

For $m > n$, we have a commutative square

$$\begin{array}{ccc} A_n & \xrightarrow{R_n} & A_n^G \\ \downarrow & & \downarrow \\ A_m & \xrightarrow{R_m} & A_m^G, \end{array}$$

as we have $A_n' \subset A_m'$ and $A_n^G \subset A_m^G$. Hence, we have a linear map $R_A : A \to A^G$ given by the compatible projections $R_n : A_n \to A_n^G$ for each n.

It remains to check that for $a \in A^G$ and $b \in A$, we have $R_A(ab) = aR_A(b)$. Pick n such that $a, b \in A_n$ and pick $m \geq n$ such that $a(A_n) \subset A_m$. Then consider the homomorphism of G-representations given by left multiplication by a

$$l_a : A_n \to A_m.$$

We can write $A_n = A_n^G \oplus A_n'$, where $A_n' = W_1 \oplus \cdots \oplus W_r$ is a direct sum of non-trivial irreducible subrepresentations $W_i \subset A_n$. Since G acts by algebra homomorphisms and $a \in A^G$, we have $l_a(A_n^G) \subset A_m^G$. By Schur's Lemma, the image of each irreducible W_i under l_a is either zero or isomorphic to W_i. Therefore, we have $l_a(W_i) \subset A_m'$ and so $l_a(A_n') \subset A_m'$. In particular, if we write $b = b^G + b'$ for $b^G \in A_n^G$ and $b' \in A'_n$, then $ab = l_a(b) = l_a(b^G) + l_a(b')$, where $l_a(b^G) = ab^G \in A_m^G$ and $l_a(b') = ab' \in A'_m$. Hence,

$$R_A(ab) = ab^G = aR_A(b)$$

as required. □

In fact, the arguments used in the final part of this proof, give the following result.

Corollary 4.23. *Let A and B be k-algebras with a rational action of a linearly reductive group G, which have Reynolds operators $R_A : A \to A^G$ and $R_B : B \to B^G$. Then any G-equivariant homomorphism $h : A \to B$ of these k-algebras commutes with the Reynolds operators: $R_B \circ h = h \circ R_A$.*

Lemma 4.24. *Let A be a k-algebra with a rational G-action and suppose that A has a Reynolds operator $R_A : A \to A^G$. Then for any ideal $I \subset A^G$, we have $IA \cap A^G = I$. More generally, if $\{I_j\}_{j \in J}$ are a set of ideals in A^G, then we have*

$$(\sum_{j \in J} I_j A) \cap A^G = \sum_{j \in J} I_j.$$

In particular, if A is noetherian, then so is A^G.

Proof. Clearly, $I \subset IA \cap A^G$. Conversely, let $x \in IA \cap A^G$; then we can write $x = \sum_{l=1}^n i_l x_l$ for $i_l \in I$ and $x_l \in A$. As R_A is a Reynolds operator,

$$x = R_A(x) = R_A\left(\sum_{l=1}^n i_l x_l\right) = \sum_{l=1}^n i_l R_A(x_l) \in I.$$

Now suppose that A is Noetherian and consider a chain $I_1 \subset I_2 \subset \cdots$ of ascending ideals in A^G. Then $I_1 A \subset I_2 A \subset \cdots$ is a chain of ascending ideals in A and so must stabilise as A is Noetherian. However, as $I_n = I_n A \cap A^G$, it follows that the chain of ideals I_n must also stabilise. □

Theorem 4.25 (Hilbert, Mumford)**.** *Let G be a linearly reductive group acting rationally on a finitely generated k-algebra A. Then A^G is finitely generated.*

Proof. Let us first reduce to the case where A is a polynomial algebra and the G-action is linear. Let V be a finite dimensional G-invariant vector subspace of A containing a set of generators for A as a k-algebra; the existence of V is guaranteed as our action is rational. As V contains generators for A as an algebra, we have a G-equivariant surjection of k-algebras

$$\mathcal{O}(V^\vee) = \mathrm{Sym}^*(V) \to A.$$

Since G is linearly reductive, both algebras admit a Reynolds operator by Lemma 4.22 and, moreover, these Reynolds operators commute with this surjection by Corollary 4.23. Therefore, we have a surjection $(\mathrm{Sym}^*(V))^G \to A^G$ and so to prove A^G is finitely generated, it suffices to assume that A is a polynomial algebra with a linear G-action.

Let $A = \mathrm{Sym}^*(V)$ where V is a finite dimensional G-representation. Then A is naturally a graded k-algebra, where the grading is by homogeneous degree $A = \oplus_n A_n = \oplus_{n \geq 0} \mathrm{Sym}^n V$. As the G-action on A is linear, the invariant subalgebra A^G is also graded $A^G = \oplus_n A_n^G$. By Hilbert's basis theorem, A is Noetherian and so by Lemma 4.24, the invariant ring A^G is also Noetherian. Hence, the ideal $A_+^G = \oplus_{n>0} A_n^G \subset A^G$ is finitely generated. We then use the following technical but not difficult result: for a graded k-algebra $B = \oplus_{n \geq 0} B_n$ and $b_1, \dots, b_m \in B$ homogeneous elements of positive degree, the following statements are equivalent:

(1) B is generated by $b_1, \dots, b_m$ as a B_0-algebra; that is, $B = B_0[b_1, \dots b_m]$;
(2) $B_+ := \oplus_{n>0} B_n$ is generated by $b_1, \dots, b_m$ as an ideal; that is $B_+ = Bb_1 + \cdots + Bb_m$.

By applying this to A^G and the finitely generated ideal $A_+^G = \oplus_{n>0} A_n^G$, we deduce that A^G is a finitely generated k-algebra. □

Nagata gave an example of an action of a product of additive groups $\mathbb{G}_a^r$ on an affine space $\mathbb{A}^n$ such that the algebra of invariants fails to be finitely generated; see [27] and [29]. From this example, one can produce an affine scheme X with a $\mathbb{G}_a$-action such that $\mathcal{O}(X)^{\mathbb{G}_a}$ is not finitely generated. More generally, a theorem of Popov states that for any non-reductive group G there is an affine scheme X such that $\mathcal{O}(X)^G$ is not finitely generated. Let us quickly outline the proof following [25] Theorem A.1.0. As G is non-reductive, we can pick a surjective homomorphism from the unipotent radical $R_u(G)$ of G onto $\mathbb{G}_a$, which defines an action of $R_u(G)$ on X such that the algebra of invariants is not finitely generated. Then we can take the Borel construction associated to $R_u(G) \subset G$

$$Y := G \times^{R_u(G)} X := (G \times X)/R_u(G)$$

which is locally trivial over $G/R_u(G)$ with fibre X and there is a natural G-action on Y where

$$\mathcal{O}(Y)^G \cong \mathcal{O}(X)^{R_u(G)}$$

is not finitely generated. In fact Y is affine (and so $\mathcal{O}(Y)$ is finitely generated) as $G \to G/R_u(G)$ has a local section by a result of Rosenlicht and so the fibre bundle $Y \to G/R_u(G)$ also has a local section.

Theorem 4.26 (Popov)**.** *An affine algebraic group G over k is reductive if and only if for every rational G-action on a finitely generated k-algebra A, the subalgebra A^G of G-invariants is finitely generated.*

4.5. **Construction of the affine GIT quotient.** Let G be a reductive group acting on an affine scheme X. We have seen that this induces an action of G on the coordinate ring $\mathcal{O}(X)$, which is a finitely generated k-algebra. By Nagata's Theorem, the subalgebra of invariants $\mathcal{O}(X)^G$ is finitely generated.

Definition 4.27. The *affine GIT quotient* is the morphism $\varphi : X \to X/\!/G := \operatorname{Spec} \mathcal{O}(X)^G$ of affine schemes associated to the inclusion $\varphi^* : \mathcal{O}(X)^G \hookrightarrow \mathcal{O}(X)$.

Remark 4.28. The double slash notation $X/\!/G$ used for the GIT quotient is a reminder that this quotient is not necessarily an orbit space and so it may identify some orbits. In nice cases, the GIT quotient is also a geometric quotient and in this case we shall often write X/G instead to emphasise the fact that it is an orbit space.

We will soon prove that the reductive GIT quotient is a good quotient. In preparation for proving that the GIT quotient is a good quotient, we need the following lemma.

Lemma 4.29. *Let G be a geometrically reductive group acting on an affine scheme X. If W_1 and W_2 are disjoint G-invariant closed subsets of X, then there is an invariant function $f \in \mathcal{O}(X)^G$ which separates these sets i.e.*

$$f(W_1) = 0 \quad \text{and} \quad f(W_2) = 1.$$

Proof. As W_i are disjoint and closed, we have

$$(1) = I(\emptyset) = I(W_1 \cap W_2) = I(W_1) + I(W_2)$$

and so we can write $1 = f_1 + f_2$, where $f_i \in I(W_i)$. Then $f_1(W_1) = 0$ and $f_1(W_2) = 1$. By Lemma 3.8, the function f_1 is contained in a finite dimensional G-invariant linear subspace V of $\mathcal{O}(X)$; therefore, so we can choose a basis $h_1, \dots, h_n$ of V. This basis defines a morphism $h : X \to \mathbb{A}^n$ by

$$h(x) = (h_1(x), \dots, h_n(x)).$$

For each i, the function h_i is a linear combination of translates of f_1, so we have

$$h_i = \sum_{l=1}^{n_i} c_{il}\, g_{il} \cdot f_1$$

for constants c_{il} and group elements g_{il}. Then $h_i(x) = \sum_{l=1}^{n_i} c_{il}\, f_1(g_{il}^{-1} \cdot x)$ and, as W_i are G-invariant subsets and f_1 takes the value 0 (resp. 1) on W_1 (resp. W_2), it follows that $h(W_1) = 0$ and $h(W_2) = v \neq 0$.

As the functions $g \cdot h_i$ also belong to V, we can write them in terms of our given basis as

$$g \cdot h_i = \sum_{j=1}^{n} a_{ij}(g) h_j.$$

This defines a representation $G \to \mathrm{GL}_n$ given by $g \mapsto (a_{ij}(g))$ such that $h : X \to \mathbb{A}^n$ is G-equivariant with respect to the G-action on X and the G-action on $\mathbb{A}^n$ via this representation $G \to \mathrm{GL}_n$. Therefore $v = h(W_2)$ is a non-zero G-invariant point. Since G is geometrically reductive, there is a non-constant homogeneous polynomial $P \in k[x_1, \dots, x_n]^G$ such that $P(v) \neq 0$ and $P(0) = 0$. Then $f = cP \circ h$ is the desired invariant function where $c = 1/P(v)$. □

Theorem 4.30. *Let G be a reductive group acting on an affine scheme X. Then the affine GIT quotient $\varphi : X \to X//G$ is a good quotient and, moreover, $X//G$ is an affine scheme.*

Proof. As G is reductive and so also geometrically reductive, it follows from Nagata's Theorem that the algebra of G-invariant regular functions on X is a finitely generated k-algebra. Hence $Y := X//G = \operatorname{Spec} \mathcal{O}(X)^G$ is an affine scheme of finite type over k. Since the affine GIT quotient is defined by taking the morphism of affine schemes associated to the inclusion $\mathcal{O}(X)^G \hookrightarrow \mathcal{O}(X)$, it is G-invariant and affine: this gives part i) and vi) in the definition of good quotient.

To prove ii), we take $y \in Y(k)$ and want to construct $x \in X(k)$ whose image under $\varphi : X \to Y$ is y. Let $\mathfrak{m}_y$ be the maximal ideal in $\mathcal{O}(Y) = \mathcal{O}(X)^G$ of the point y and choose generators $f_1, \dots, f_m$ of $\mathfrak{m}_y$. Since G is reductive, we claim that it follows that

$$\sum_{i=1}^{m} f_i \mathcal{O}(X) \neq \mathcal{O}(X).$$

For a linearly reductive group, this claim follows from Lemma 4.24 as

$$\left(\sum_{i=1}^{m} f_i \mathcal{O}(X) \right) \cap \mathcal{O}(X)^G = \sum_{i=1}^{m} f_i \mathcal{O}(X)^G \neq \mathcal{O}(X)^G.$$

For a proof for geometrically reductive groups, see [31] Lemma 3.4.2. Then, as $\sum_{i=1}^{m} f_i \mathcal{O}(X)$ is not equal to $\mathcal{O}(X)$, it is contained in some maximal idea $\mathfrak{m}_x \subset \mathcal{O}(X)$ corresponding to a closed point $x \in X(k)$. In particular, we have that $f_i(x) = 0$ for $i = 1, \dots, m$ and so $\varphi(x) = y$ as required. Therefore, every closed point is in the image of φ and as the image of φ is a constructible subset by Chevalley's Theorem, we can conclude that φ is surjective.

For $f \in \mathcal{O}(X)^G$, the open sets $U = Y_f$ form a basis of the open subsets of Y. Therefore, to prove iii), it suffices to consider open sets U of the form Y_f for $f \in \mathcal{O}(X)^G$. Let $f \in \mathcal{O}(X)^G$; then $\mathcal{O}_Y(Y_f) = (\mathcal{O}(X)^G)_f$ is the localisation of $\mathcal{O}(X)^G$ with respect to f and

$$\mathcal{O}_X(\varphi^{-1}(Y_f))^G = \mathcal{O}_X(X_f)^G = (\mathcal{O}(X)_f)^G = (\mathcal{O}(X)^G)_f = \mathcal{O}_Y(Y_f)$$

as localisation with respect to an invariant function commutes with taking G-invariants. Hence the image of the inclusion homomorphism $\mathcal{O}_Y(Y_f) = (\mathcal{O}(X)^G)_f \to \mathcal{O}_X(\varphi^{-1}(Y_f)) = \mathcal{O}(X)_f$ is $\mathcal{O}_X(\varphi^{-1}(Y_f))^G = (\mathcal{O}(X)_f)^G$ which proves iii).

By Remark 3.28, given the surjectivity of φ, properties iv) and v) are equivalent to v)$'$ and so it suffices to prove v)$'$. By Lemma 4.29, for any two disjoint G-invariant closed subsets W_1 and W_2 in X, there is a function $f \in \mathcal{O}(X)^G$ such that $f(W_1) = 0$ and $f(W_2) = 1$. Since $\mathcal{O}(X)^G = \mathcal{O}(Y)$, we can view f as a regular function on Y with $f(\varphi(W_1)) = 0$ and $f(\varphi(W_2)) = 1$. Hence, it follows that

$$\overline{\varphi(W_1)} \cap \overline{\varphi(W_2)} = \emptyset$$

which finishes the proof. □

Corollary 4.31. *Suppose a reductive group G acts on an affine scheme X and let $\varphi : X \to Y := X//G$ be the affine GIT quotient. Then*

$$\varphi(x) = \varphi(x') \iff \overline{G \cdot x} \cap \overline{G \cdot x'} \neq \emptyset.$$

Furthermore, the preimage of each point $y \in Y$ contains a unique closed orbit. In particular, if the action of G on X is closed, then φ is a geometric quotient.

Proof. As φ is a good quotient, this follows immediately from Corollary 3.32 □

Example 4.32. Consider the action of $G = \mathbb{G}_m$ on $X = \mathbb{A}^2$ by $t \cdot (x, y) = (tx, t^{-1}y)$ as in Example 3.17. In this case $\mathcal{O}(X) = k[x, y]$ and $\mathcal{O}(X)^G = k[xy] \cong k[z]$ so that $Y = \mathbb{A}^1$ and the GIT quotient $\varphi : X \to Y$ is given by $(x, y) \mapsto xy$. The three orbits consisting of the punctured axes and the origin are all identified and so the quotient is not a geometric quotient.

Example 4.33. Consider the action of $G = \mathbb{G}_m$ on $\mathbb{A}^n$ by $t \cdot (x_1, \dots, x_n) = (tx_1, \dots, tx_n)$ as in Example 3.18. Then $\mathcal{O}(X) = k[x_1, \dots, x_n]$ and $\mathcal{O}(X)^G = k$ so that $Y = \operatorname{Spec} k$ is a point and the GIT quotient $\varphi : X \to Y = \operatorname{Spec} k$ is given by the structure morphism. In this case all orbits are identified and so this good quotient is not a geometric quotient.

Remark 4.34. We note that the fact that G is reductive was used several times in the proof, not just to show the ring of invariants is finitely generated. In particular, there are non-reductive group actions which have finitely generated invariant rings but for which other properties listed in the definition of good quotient fail. For example, consider the additive group $\mathbb{G}_a$ acting on $X = \mathbb{A}^4$ by the linear representation $\rho : \mathbb{G}_a \to \mathrm{GL}_4$

$$s \mapsto \begin{pmatrix} 1 & s & & \\ & 1 & & \\ & & 1 & s \\ & & & 1 \end{pmatrix}.$$

Even though $\mathbb{G}_a$ is non-reductive, the invariant ring is finitely generated: one can prove that

$$k[x_1, x_2, x_3, x_4]^{\mathbb{G}_a} = k[x_2, x_4, x_1x_4 - x_2x_3].$$

However the GIT 'quotient' map $X \to X//\mathbb{G}_a = \mathbb{A}^3$ is not surjective: its image misses the punctured line $\{(0, 0, \lambda) : \lambda \in k^*\} \subset \mathbb{A}^3$. For further differences, see [6].

4.6. **Geometric quotients on open subsets.** As we saw above, when a reductive group G acts on an affine scheme X in general a geometric quotient (i.e. orbit space) does not exist because the action is not necessarily closed. For finite groups G, every good quotient is a geometric quotient as the action of a finite group is always closed. In this section, we define an open subset X^s of 'stable' points in X for which there is a geometric quotient.

Definition 4.35. We say $x \in X$ is *stable* if its orbit is closed in X and $\dim G_x = 0$ (or equivalently, $\dim G \cdot x = \dim G$). We let X^s denote the set of stable points.

Proposition 4.36. *Suppose a reductive group G acts on an affine scheme X and let $\varphi : X \to Y := X//G$ be the affine GIT quotient. Then $X^s \subset X$ is an open and G-invariant subset, $Y^s := \varphi(X^s)$ is an open subset of Y and $X^s = \varphi^{-1}(Y^s)$. Moreover, $\varphi : X^s \to Y^s$ is a geometric quotient.*

Proof. We first show that X^s is open by showing for every $x \in X^s(k)$ there is an open neighbourhood of x in X^s. By Lemma 3.21, the set $X_+ := \{x \in X : \dim G_x > 0\}$ of points with positive dimensional stabilisers is a closed subset of X. If $x \in X^s$, then by Lemma 4.29 there is a function $f \in \mathcal{O}(X)^G$ such that

$$f(X_+) = 0 \quad \text{and} \quad f(G \cdot x) = 1.$$

Then $x \in X_f(k)$ and we claim that $X_f \subset X^s$ so that X_f is an open neighbourhood of x in X^s. Since all points in X_f have stabilisers of dimension zero, it remains to check that their orbits are closed. Suppose $z \in X_f(k)$ has a non-closed orbit so $w \notin G \cdot z$ belongs to the orbit closure of z; then $w \in X_f(k)$ too as f is G-invariant and so w must have stabiliser of dimension zero. However, by Proposition 3.15 the boundary of the orbit $G \cdot z$ is a union of orbits of strictly lower dimension and so the orbit of w must be of dimension strictly less than that of z which contradicts the fact that w has zero dimensional stabiliser. Hence, X^s is an open subset of X, and is covered by sets of the form X_f as above.

Since $\varphi(X_f) = Y_f$ is open in Y and also $X_f = \varphi^{-1}(Y_f)$, it follows that Y^s is open in Y and also $X^s = \varphi^{-1}(\varphi(X^s))$. Then $\varphi : X^s \to Y^s$ is a good quotient by Corollary 3.33. Furthermore, the action of G on X^s is closed and so $\varphi : X^s \to Y^s$ is a geometric quotient by Corollary 3.32. □

Example 4.37. We can now calculate the stable set for the action of $G = \mathbb{G}_m$ on $X = \mathbb{A}^2$ as in Examples 3.17 and 4.32. The closed orbits are the conics $\{xy = \alpha\}$ for $\alpha \in \mathbb{A}^1 - \{0\}$ and the origin, but the origin has a positive dimensional stabiliser. Thus

$$X^s = \{(x, y) \in \mathbb{A}^2 : xy \neq 0\} = X_{xy}.$$

In this example, it is clear why we need to insist that $\dim G_x = 0$ in the definition of stability: so that the stable set is open. In fact this requirement should also be clear from the proof of Proposition 4.36.

Example 4.38. We may also consider which points are stable for the action of $G = \mathbb{G}_m$ on $\mathbb{A}^n$ as in Examples 3.18 and 4.33. The only closed orbit is the origin, whose stabiliser is positive dimensional, and so $X^s = \emptyset$. In particular, this example shows that the stable set may be empty.

Example 4.39. Consider $G = \mathrm{GL}_2$ acting on the space $M_{2\times2}$ of 2×2 matrices with k-coefficients by conjugation. The characteristic polynomial of a matrix A is given by

$$\mathrm{char}_A(t) = \det(xI - A) = x^2 + c_1(A)x + c_2(A)$$

where $c_1(A) = -\mathrm{Tr}(A)$ and $c_2(A) = \det(A)$ and is well defined on the conjugacy class of a matrix. The Jordan canonical form of a matrix is obtained by conjugation and so lies in the same orbit of the matrix. The theory of Jordan canonical forms says there are three types of orbits:

- matrices with characteristic polynomial with distinct roots α, β. These matrices are diagonalisable with Jordan canonical form
$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.$$
These orbits are closed and have dimension 2. The stabiliser of the above matrix is the subgroup of diagonal matrices which is 2 dimensional.
- matrices with characteristic polynomial with repeated root α for which the minimum polynomial is equal to the characteristic polynomial. These matrices are not diagonalisable and their Jordan canonical form is
$$\begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}.$$
These orbits are also 2 dimensional but are not closed: for example
$$\lim_{t\to0} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}.$$
- matrices with characteristic polynomial with repeated root α for which the minimum polynomial is $x - \alpha$. These matrices have Jordan canonical form
$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}.$$
Since scalar multiples of the identity commute with everything, their stabilisers are equal to the full group GL_2 and their orbits are simply a point, which is closed and zero dimensional.

We note that every orbit of the second type contains an orbit of the third type and so will be identified in the quotient. Clearly there are two G-invariant functions on $M_{2\times2}$: the trace and determinant, and so

$$k[\mathrm{tr}, \det] \subset \mathcal{O}(M_{2\times2})^{\mathrm{GL}_2}.$$

We claim that these are the only G-invariant functions on $M_{2\times 2}$. To see this we note that from the above discussion about Jordan normal forms and orbit closures, a G-invariant function on $M_{2\times 2}$ is completely determined by its values on the diagonal matrices $D_2 \subset M_{2\times 2}$. Hence the ring of GL_2-invariants on $M_{2\times 2}$ is contained in the ring $\mathcal{O}(D_2) \cong k[x_{11}, x_{22}]$. In fact, using the GL_2-action we can permute the diagonal entries; therefore,

$$\mathcal{O}(M_{2\times 2})^{\mathrm{GL}_2} \subset k[x_{11}, x_{22}]^{S_2} = k[x_{11} + x_{22}, x_{11}x_{22}],$$

as the symmetric polynomials are generated by the elementary symmetric polynomials. These elementary symmetric polynomials correspond to the trace and determinant respectively, and we see there are no additional invariants. Hence

$$k[\mathrm{tr}, \det] = \mathcal{O}(M_{2\times 2})^{\mathrm{GL}_2}$$

and the affine GIT quotient is given by

$$\varphi = (\mathrm{tr}, \det) : M_{2\times 2} \to \mathbb{A}^2.$$

The subgroup $\mathbb{G}_m I_2$ fixes every point and so there are no stable points for this action.

Example 4.40. More generally, we can consider $G = \mathrm{GL}_n$ acting on $M_{n\times n}$ by conjugation. If A is an $n \times n$ matrix, then the coefficients of its characteristic polynomial

$$\mathrm{char}_A(t) = \det(tI - A) = t^n + c_1(A)t^{n-1} + \cdots + c_n(A)$$

are all G-invariant functions. As in Example 4.39 above, we can use the theory of Jordan normal forms as above to describe the different orbits and the closed orbits correspond to diagonalisable matrices. By a similar argument to above, we have

$$k[c_1, \ldots, c_n] \subset \mathcal{O}(M_{n\times n})^{\mathrm{GL}_n} \subset \mathcal{O}(D_n)^{S_n} \cong k[x_{11}, \ldots, x_{nn}]^{S_n} = k[\sigma_1, \ldots, \sigma_n]$$

where σ_i is the ith elementary symmetric polynomial in the x_js. Hence, we conclude these are all equalities and the affine GIT quotient is given by

$$\varphi : M_{n\times n} \to \mathbb{A}^n$$
$$A \mapsto (c_1(A), \ldots, c_n(A)).$$

Again as every orbit contains a copy of $\mathbb{G}_m$ in its stabiliser subgroup, there are no stable points.

Remark 4.41. In situations where there is a non-finite subgroup $H \subset G$ which is contained in the stabiliser subgroup of every point for a given action of G on X, the stable set is automatically empty. Hence, for the purposes of GIT, it is better to work with the induced action of the group G/H. In the above example, this would be equivalent to considering the action of the special linear group on the space of $n \times n$ matrices by conjugation.

5. Projective GIT quotients

In this section we extend the theory of affine GIT developed in the previous section to construct GIT quotients for reductive group actions on projective schemes. The idea is that we would like construct our GIT quotient by gluing affine GIT quotients. In order to do this we would like to cover our scheme X by affine open subsets which are invariant under the group action and glue the affine GIT quotients of these affine open subsets of X. However, it may not be possible to cover all of X by such compatible open invariant affine subsets.

For a projective scheme X with an action of a reductive group G, there is not a canonical way to produce an open subset of X which is covered by open invariant affine subsets. Instead, this will depend on a choice of an equivariant projective embedding $X \hookrightarrow \mathbb{P}^n$, where G acts on $\mathbb{P}^n$ by a linear representation $G \to \mathrm{GL}_{n+1}$. We recall that a projective embedding of X corresponds to a choice of a (very) ample line bundle $\mathcal{L}$ on X. We will shortly see that equivariant projective embeddings are given by an ample *linearisation* of the G-action on X, which is a lift of the G-action to a ample line bundle on X such that the projection to X is equivariant and the action on the fibres is linear.

In this section, we will show for a reductive group G acting on a projective scheme X and a choice of ample linearisation of the action, there is a good quotient of an open subset of *semistable* points in X. Furthermore, this quotient is itself projective and restricts to a geometric quotient on an open subset of *stable* points. The main reference for the construction of the projective GIT quotient is Mumford's book [25] and other excellent references are [4, 24, 31, 32, 42].

5.1. **Construction of the projective GIT quotient.**

Definition 5.1. Let X be a projective scheme with an action of an affine algebraic group G. A linear G-equivariant projective embedding of X is a group homomorphism $G \to \mathrm{GL}_{n+1}$ and a G-equivariant projective embedding $X \hookrightarrow \mathbb{P}^n$. We will often simply say that the G-action on $X \hookrightarrow \mathbb{P}^n$ is linear to mean that we have a linear G-equivariant projective embedding of X as above.

Suppose we have a linear action of a reductive group G on a projective scheme $X \subset \mathbb{P}^n$. Then the action of G on $\mathbb{P}^n$ lifts to an action of G on the affine cone $\mathbb{A}^{n+1}$ over $\mathbb{P}^n$. Since the projective embedding $X \subset \mathbb{P}^n$ is G-equivariant, there is an induced action of G on the affine cone $\tilde{X} \subset \mathbb{A}^{n+1}$ over $X \subset \mathbb{P}^n$. More precisely, we have

$$\mathcal{O}(\mathbb{A}^{n+1}) = k[x_0, \dots, x_n] = \bigoplus_{r\geq 0} k[x_0, \dots, x_n]_r = \bigoplus_{r \geq 0} H^0(\mathbb{P}^n, \mathcal{O}_X(r))$$

and if $X \subset \mathbb{P}^n$ is the closed subscheme associated to a homogeneous ideal $I(X) \subset k[x_0, \dots, x_n]$, then $\tilde{X} = \operatorname{Spec} R(X)$ where $R(X) = k[x_0, \dots, x_n]/I(X)$.

The k-algebras $\mathcal{O}(\mathbb{A}^{n+1})$ and $R(X)$ are graded by homogeneous degree and, as the G-action on $\mathbb{A}^{n+1}$ is linear it preserves the graded pieces, so that the invariant subalgebra

$$\mathcal{O}(\mathbb{A}^{n+1})^G = \bigoplus_{r \geq 0} k[x_0, \dots, x_n]_r^G$$

is a graded algebra and, similarly, $R(X)^G = \oplus_{r\geq 0} R(X)_r^G$. By Nagata's theorem, $R(X)^G$ is finitely generated, as G is reductive. The inclusion of finitely generated graded k-algebras $R(X)^G \hookrightarrow R(X)$ determines a rational morphism of projective schemes

$$X \dashrightarrow \operatorname{Proj} R(X)^G$$

whose indeterminacy locus is the closed subscheme of X defined by the homogeneous ideal $R(X)_+^G := \oplus_{r>0} R(X)_r^G$.

Definition 5.2. For a linear action of a reductive group G on a projective scheme $X \subset \mathbb{P}^n$, we define the *nullcone* N to be the closed subscheme of X defined by the homogeneous ideal $R(X)_+^G$ in $R(X)$ (strictly speaking the nullcone is really the affine cone $\tilde{N}$ over N). We define the *semistable set* $X^{ss} = X - N$ to be the open subset of X given by the complement to the nullcone. More precisely, $x \in X$ is *semistable* if there exists a G-invariant homogeneous function $f \in R(X)_r^G$ for $r > 0$ such that $f(x) \neq 0$. By construction, the semistable set is the open subset which is the domain of definition of the rational map

$$X \dashrightarrow \operatorname{Proj} R(X)^G.$$

We call the morphisms $X^{ss} \to X//G := \operatorname{Proj} R(X)^G$ the *GIT quotient* of this action.

Theorem 5.3. *For a linear action of a reductive group G on a projective scheme $X \subset \mathbb{P}^n$, the GIT quotient $\varphi : X^{ss} \to X//G$ is a good quotient of the G-action on the open subset X^{ss} of semistable points in X. Furthermore, $X//G$ is a projective scheme.*

Proof. We let $\varphi : X^{ss} \to Y := X//G$ denote the projective GIT quotient. By construction $X//G$ is the projective spectrum of the finitely generated graded k-algebra $R(X)^G$. We claim that $\operatorname{Proj} R(X)^G$ is projective over $\operatorname{Spec} R(X)_0^G = \operatorname{Spec} k$. If $R(X)^G$ is finitely generated by $R(X)_1^G$ as a k-algebra, this result follows immediately from [14] II Corollary 5.16. If not, then as $R(X)^G$ is a finitely generated k-algebra, we can pick generators $f_1, \dots, f_r$ in degrees $d_1, \dots, d_r$. Let $d := d_1 \cdot \dots \cdot d_r$; then

$$(R(X)^G)^{(d)} = \bigoplus_{l \geq 0} R(X)_{dl}^G$$

is finitely generated by $(R(X)^G)_1^{(d)}$ as k-algebra and so $\operatorname{Proj}\left((R(X)^G)^{(d)}\right)$ is projective over $\operatorname{Spec} k$. Since $X//G := \operatorname{Proj} R(X)^G \cong \operatorname{Proj}\left((R(X)^G)^{(d)}\right)$ (see [14] II Exercise 5.13), we can conclude that $X//G$ is projective.

For $f \in R_+^G$, the open affine subsets $Y_f \subset Y$ form a basis of the open sets on Y. Since $f \in R(X)_+^G \subset R(X)_+$, we can also consider the open affine subset $X_f \subset X$ and, by construction of φ, we have that $\varphi^{-1}(Y_f) = X_f$. Let $\tilde{X}_f$ (respectively $\tilde{Y}_f$) denote the affine cone over X_f (respectively Y_f). Then

$$\mathcal{O}(Y_f) \cong \mathcal{O}(\tilde{Y}_f)_0 \cong ((R(X)^G)_f)_0 \cong ((R(X)_f)_0)^G \cong (\mathcal{O}(\tilde{X}_f)_0)^G \cong \mathcal{O}(X_f)^G$$

and so the corresponding morphism of affine schemes $\varphi_f : X_f \to Y_f \cong \operatorname{Spec} \mathcal{O}(X_f)^G$ is an affine GIT quotient, and so also a good quotient by Theorem 4.30. The morphism $\varphi : X^{ss} \to Y$ is obtained by gluing the good quotients $\varphi_f : X_f \to Y_f$. Since Y_f cover Y (and X_f cover X^{ss}) and being a good quotient is local on the target Remark 3.34, we can conclude that φ is also a good quotient. □

We recall that as $\varphi : X^{ss} \to X//G$ is a good quotient, for two semistable points x_1, x_2 in X^{ss}, we have

$$\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \cap X^{ss} \neq \emptyset \iff \varphi(x_1) = \varphi(x_2).$$

Furthermore, the preimage of each point in $X//G$ contains a unique closed orbit. The presence of non-closed orbits in the semistable locus will prevent the good quotient $\varphi : X^{ss} \to X//G$ from being a geometric quotient.

We can now ask if there is an open subset X^s of X^{ss} on which this quotient becomes a geometric quotient. For this we want the action to be closed on X^s. This motivates the definition of stability (see also Definition 4.35).

Definition 5.4. Consider a linear action of a reductive group G on a closed subscheme $X \subset \mathbb{P}^n$. Then a point $x \in X$ is

(1) *stable* if $\dim G_x = 0$ and there is a G-invariant homogeneous polynomial $f \in R(X)_+^G$ such that $x \in X_f$ and the action of G on X_f is closed.
(2) *unstable* if it is not semistable.

We denote the set of stable points by X^s and the set of unstable points by $X^{us} := X - X^{ss} = N$.

We emphasise that, somewhat confusingly, unstable does not mean not stable, but this terminology has long been accepted by the mathematical community.

Lemma 5.5. *The stable and semistable sets X^s and X^{ss} are open in X.*

Proof. By construction, the semistable set is open in X as it is the complement to the nullcone N, which is closed. To prove that the stable set is open, we consider the subset $X_c := \cup X_f$ where the union is taken over $f \in R(X)_+^G$ such that the action of G on X_f is closed; then X_c is open in X and it remains to show X^s is open in X_c. By Proposition 3.21, the function $x \mapsto \dim G_x$ is an upper semi-continuous function on X and so the set of points with zero dimensional stabiliser is open. Hence, we have open inclusions $X^s \subset X_c \subset X$. □

Theorem 5.6. *For a linear action of a reductive group G on a closed subscheme $X \subset \mathbb{P}^n$, let $\varphi : X^{ss} \to Y := X//G$ denote the GIT quotient. Then there is an open subscheme $Y^s \subset Y$ such that $\varphi^{-1}(Y^s) = X^s$ and that the GIT quotient restricts to a geometric quotient $\varphi : X^s \to Y^s$.*

Proof. Let Y_c be the union of Y_f for $f \in R(X)_+^G$ such that the G-action on X_f is closed and let X_c be the union of X_f over the same index set so that $X_c = \varphi^{-1}(Y_c)$. Then $\varphi : X_c \to Y_c$ is constructed by gluing $\varphi_f : X_f \to Y_f$ for $f \in R(X)_+^G$ such that the G-action on X_f is closed. Each φ_f is a good quotient and as the action on X_f is closed, φ_f is also a geometric quotient by Corollary 3.32. Hence $\varphi : X_c \to Y_c$ is a geometric quotient by Remark 3.34.

By definition, X^s is the open subset of X_c consisting of points with zero dimensional stabilisers and we let $Y^s := \varphi(X^s) \subset Y_c$. It remains to prove that Y^s is open. As $\varphi : X_c \to Y_c$ is a geometric quotient and X^s is a G-invariant subset of X, $\varphi^{-1}(Y^s) = X^s$ and also $Y_c - Y^s = \varphi(X_c - X^s)$. As

$X_c - X^s$ is closed in X_c, property iv) of good quotient gives that $\varphi(X_c - X^s) = Y_c - Y^s$ is closed in Y_c and so Y^s is open in Y_c. Since Y_c is open in Y, we can conclude that $Y^s \subset Y$ is open. Finally, the geometric quotient $\varphi : X_c \to Y_c$ restricts to a geometric quotient $\varphi : X^s \to Y^s$ by Corollary 3.33. □

Remark 5.7. We see from the proof of this theorem that to get a geometric quotient we do not have to impose the condition $\dim G_x = 0$ and in fact in Mumford's original definition of stability this condition was omitted. However, the modern definition of stability, which asks for zero dimensional stabilisers, is now widely accepted. One advantage of the modern definition is that if the semistable set is nonempty, then the dimension of the geometric quotient equals its expected dimension. A second advantage of the modern definition of stability is that it is better suited to moduli problems.

Example 5.8. Consider the linear action of $G = \mathbb{G}_m$ on $X = \mathbb{P}^n$ by

$$t \cdot [x_0 : x_1 : \cdots : x_n] = [t^{-1}x_0 : tx_1 : \cdots : tx_n].$$

In this case $R(X) = k[x_0, \dots, x_n]$ which is graded into homogeneous pieces by degree. It is easy to see that the functions $x_0x_1, \dots, x_0x_n$ are all G-invariant. In fact, we claim that these functions generate the ring of invariants. To prove the claim, suppose we have $f \in R(X)$; then

$$f = \sum_{\underline{m}=(m_0,\dots,m_n)} a(\underline{m}) x_0^{m_0} x_1^{m_1} \dots x_n^{m_n}$$

and, for $t \in \mathbb{G}_m$,

$$t \cdot f = \sum_{\underline{m}=(m_0,\dots,m_n)} a(\underline{m}) t^{m_0 - m_1 - \cdots - m_n} x_0^{m_0} x_1^{m_1} \dots x_n^{m_n}.$$

Then f is G-invariant if and only if $a(\underline{m}) = 0$ for all $\underline{m} = (m_0, \dots, m_n)$ such that $m_0 \neq \sum_{i=1}^n m_i$. If m satisfies $m_0 = \sum_{i=1}^n m_i$, then

$$x_0^{m_0} x_1^{m_1} \dots x_n^{m_n} = (x_0x_1)^{m_1} \dots (x_0x_n)^{m_n};$$

that is, if f is G-invariant, then $f \in k[x_0x_1, \dots x_0x_n]$. Hence

$$R(X)^G = k[x_0x_1, \dots, x_0x_n] \cong k[y_0, \dots, y_{n-1}]$$

and after taking the projective spectrum we obtain the projective variety $X//G = \mathbb{P}^{n-1}$. The explicit choice of generators for $R(X)^G$ allows us to write down the rational morphism

$$\varphi : X = \mathbb{P}^n \dashrightarrow X//G = \mathbb{P}^{n-1}$$

$$[x_0 : x_1 : \cdots : x_n] \mapsto [x_0x_1 : \cdots : x_0x_n]$$

and its clear from this description that the nullcone

$$N = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n : x_0 = 0 \text{ or } (x_1, \cdots, x_n) = 0\}$$

is the projective variety defined by the homogeneous ideal $I = (x_0x_1, \cdots, x_0x_n)$. In particular,

$$X^{ss} = \bigcup_{i=1}^n X_{x_0x_i} = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n : x_0 \neq 0 \text{ and } (x_1, \dots, x_n) \neq 0\} \cong \mathbb{A}^n - \{0\}$$

where we are identifying the affine chart on which $x_0 \neq 0$ in $\mathbb{P}^n$ with $\mathbb{A}^n$. Therefore

$$\varphi : X^{ss} = \mathbb{A}^n - \{0\} \dashrightarrow X//G = \mathbb{P}^{n-1}$$

is a good quotient of the action on X^{ss}. As the preimage of each point in $X//G$ is a single orbit, this is also a geometric quotient. Moreover, every semistable point is stable as all orbits are closed in $\mathbb{A}^n - \{0\}$ and have zero dimensional stabilisers.

In general it can be difficult to determine which points are semistable and stable as it is necessary to have a description of the graded k-algebra of invariant functions. The ideal situation is as above where we have an explicit set of generators for the invariant algebra which allows us to write down the quotient map. However, finding generators and relations for the invariant algebra in general can be hard. We will soon see that there are other criteria that we can use to determine (semi)stability of points.

Lemma 5.9. *Let G be a reductive group acting linearly on $X \subset \mathbb{P}^n$. A k-point $x \in X(k)$ is stable if and only if x is semistable and its orbit $G \cdot x$ is closed in X^{ss} and its stabiliser G_x is zero dimensional.*

Proof. Suppose x is stable and $x' \in \overline{G \cdot x} \cap X^{ss}$; then $\varphi(x') = \varphi(x)$ and so $x' \in \varphi^{-1}(\varphi(x)) \subset \varphi^{-1}(Y^s) = X^s$. As G acts on X^s with zero-dimensional stabiliser, this action must be closed as the boundary of an orbit is a union of orbits of strictly lower dimension. Therefore, $x' \in G \cdot x$ and so the orbit $G \cdot x$ is closed in X^{ss}.

Conversely, we suppose x is semistable with closed orbit in X^{ss} and zero dimensional stabiliser. As x is semistable, there is a homogeneous $f \in R(X)^G_+$ such that $x \in X_f$. As $G \cdot x$ is closed in X^{ss}, it is also closed in the open affine set $X_f \subset X^{ss}$. By Proposition 3.21, the G-invariant set

$$Z := \{z \in X_f : \dim G_z > 0\}$$

is closed in X_f. Since Z is disjoint from $G \cdot x$ and both sets are closed in the affine scheme X_f, by Lemma 4.29, there exists $h \in \mathcal{O}(X_f)^G$ such that

$$h(Z) = 0 \text{ and } h(G \cdot x) = 1.$$

We claim that from the function h, we can produce a G-invariant homogeneous polynomial $h' \in R(X)^G_+$ such that $x \in X_{fh'}$ and $X_{fh'}$ is disjoint from Z, as then all orbits in $X_{fh'}$ have zero dimensional stabilisers and so must be closed in $X_{fh'}$ (as the closure of an orbit is a union of lower dimensional orbits), in which case we can conclude that x is stable. The proof of the above claim follows from Lemma 5.10 below and uses the fact that G is geometrically reductive. More precisely, we have that $\mathcal{O}(X_f) = \mathcal{O}(\tilde{X}_f)_0$ is a quotient of $A := (k[x_0, \dots, x_n]_f)_0$ and we take I to be the kernel. Then $h^r = h'/f^s \in A^G/(I \cap A^G)$ for some homogeneous polynomial h' and positive integers r and s. □

Lemma 5.10. *Let G be a geometrically reductive group acting rationally on a finitely generated k-algebra A. For a G-invariant ideal I of A and $a \in (A/I)^G$, there is a positive integer r such that $a^r \in A^G/(I \cap A^G)$.*

Proof. Let $b \in A$ be an element whose image in A/I is a and we can assume $a \neq 0$. As the action is rational, b is contained in a finite dimensional G-invariant linear subspace $V \subset A$ spanned by the translates $g \cdot b$. Then $b \notin V \cap I$ as $a \neq 0$; however, $g \cdot b - b \in V \cap I$ for all $g \in G$ as a is G-invariant. Therefore $\dim V = \dim(V \cap I) + 1$ and every element in V can be uniquely written as $\lambda b + b'$ for $\lambda \in k$ and $b' \in V \cap I$. Consider the linear projection $l : V \to k$ onto the line spanned by b, which is G-equivariant. In terms of the dual representation $V^\vee$, the projection l corresponds to a non-zero fixed point l^* and so, as G is geometrically reductive, there exists a G-invariant homogeneous function $F \in \mathcal{O}(V^\vee)$ of positive degree r which is not vanishing at l^*. We can take a basis of V (and dual basis of $V^\vee$) where the first basis vector corresponds to b. Then the coefficient λ of x_1^r in F is non-zero. Consider the algebra homomorphism

$$\mathcal{O}(V^\vee) = \operatorname{Sym}^* V \to A$$

and let $b_0 \in A^G$ be the image of $F \in \mathcal{O}(V^\vee)^G$. Then $b_0 - \lambda b^r \in I$, as this belongs to the ideal generated by a choice of basis vectors for $V \cap I$. Hence $a^r \in A^G/(I \cap A^G)$ as required. □

Remark 5.11. If G is linearly reductive, then taking G-invariants is exact, and so we can take $r = 1$ in the above lemma.

5.2. **A description of the k-points of the GIT quotient.**

Definition 5.12. Let G be a reductive group acting linearly on $X \subset \mathbb{P}^n$. A k-point $x \in X(k)$ is said to be *polystable* if it is semistable and its orbit is closed in X^{ss}. We say two semistable k-points are *S-equivalent* if their orbit closures meet in X^{ss}. We write this equivalence relation on $X^{ss}(k)$ as $\sim_{\text{S-equiv.}}$ and let $X^{ss}(k)/\sim_{\text{S-equiv.}}$ denote the S-equivalence classes of semistable k-points.

By Lemma 5.9 above, every stable k-point is polystable.

Lemma 5.13. *Let G be a reductive group acting linearly on $X \subset \mathbb{P}^n$ and let $x \in X(k)$ be a semistable k-point; then its orbit closure $\overline{G \cdot x}$ contains a unique polystable orbit. Moreover, if x is semistable but not stable, then this unique polystable orbit is also not stable.*

Proof. The first statement follows from Corollary 3.32: φ is constant on orbit closures and the preimage of a k-point under φ contains a orbit which is closed in X^{ss}; this is the polystable orbit. For the second statement we note that if a semistable orbit $G \cdot x$ is not closed, then the unique closed orbit in $\overline{G \cdot x}$ has dimension strictly less than $G \cdot x$ by Proposition 3.15 and so cannot be stable. □

Corollary 5.14. *Let G be a reductive group acting linearly on $X \subset \mathbb{P}^n$. For two semistable points $x, x' \in X^{ss}$, we have $\varphi(x) = \varphi(x')$ if and only if x and x' are S-equivalent. Moreover, there is a bijection of sets*

$$X//G(k) \cong X^{ps}(k)/G(k) \cong X^{ss}(k)/\sim_{\text{S-equiv.}}$$

where $X^{ps}(k)$ is the set of polystable k-points.

5.3. **Linearisations.** An abstract projective scheme X does not come with a pre-specified embedding in a projective space. However, an ample line bundle L on X (or more precisely some power of L) determines an embedding of X into a projective space. More precisely, the projective scheme X and ample line bundle L, determine a finitely generated graded k-algebra

$$R(X, L) := \bigoplus_{r \geq 0} H^0(X, L^{\otimes r}).$$

We can choose generators of this k-algebra: $s_i \in H^0(X, L^{\otimes r_i})$ for $i = 0, , ..n$, where $r_i \geq 1$. Then these sections determine a closed immersion

$$X \hookrightarrow \mathbb{P}(r_0, \dots, r_n)$$

into a weighted projective space, by evaluating each point of X at the sections s_i. In fact, if we replace L by $L^{\otimes m}$ for m sufficiently large, then we can assume that the generators s_i of the finitely generated k-algebra

$$R(X, L^{\otimes m}) = \bigoplus_{r \geq 0} H^0(X, L^{\otimes mr})$$

all lie in degree 1. In this case, the sections s_i of the line bundle $L^{\otimes m}$ determine a closed immersion

$$X \hookrightarrow \mathbb{P}^n = \mathbb{P}(H^0(X, L^{\otimes m})^*)$$

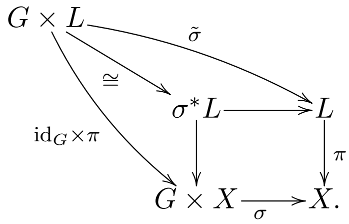
given by evaluation $x \mapsto (s \mapsto s(x))$.

Now suppose we have an action of an affine algebraic group G on X; then we would like to do everything above G-equivariantly, by lifting the G-action to L such that the above embedding is equivariant and the action of G on $\mathbb{P}^n$ is linear. This idea is made precise by the notion of a linearisation.

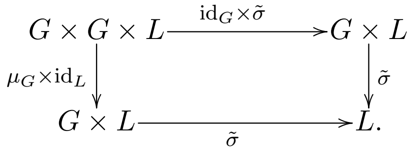
Definition 5.15. Let X be a scheme and G be an affine algebraic group acting on X via a morphism $\sigma : G \times X \to X$. Then a *linearisation* of the G-action on X is a line bundle $\pi : L \to X$ over X with an isomorphism of line bundles

$$\pi_X^* L = G \times L \cong \sigma^* L,$$

where $\pi_X : G \times X \to X$ is the projection, such that the induced bundle homomorphism $\tilde{\sigma} : G \times L \to L$ defined by



induces an action of G on L; that is, we have a commutative square of bundle homomorphisms



We say that a linearisation is *(very) ample* if the underlying line bundle is (very) ample.

Let us unravel this definition a little. Since $\tilde{\sigma} : G \times L \to L$ is a homomorphism of vector bundles, we have

i) the projection $\pi : L \to X$ is G-equivariant,
ii) the action of G on the fibres of L is linear: for $g \in G$ and $x \in X$, the map on the fibres $L_x \to L_{g \cdot x}$ is linear.

Remark 5.16.

(1) The notion of a linearisation can also be phrased sheaf theoretically: a linearisation of a G-action on X on an invertible sheaf $\mathcal{L}$ is an isomorphism
$$\Phi : \sigma^*\mathcal{L} \to \pi_X^*\mathcal{L},$$
where $\pi_X : G \times X \to X$ is the projection map, which satisfies the cocycle condition:
$$(\mu \times \mathrm{id}_X)^*\Phi = \pi_{23}^*\Phi \circ (\mathrm{id}_G \times \sigma)^*\Phi$$
where $\pi_{23} : G \times G \times X \to G \times X$ is the projection onto the last two factors. If $\pi : L \to X$ denotes the line bundle associated to the invertible sheaf $\mathcal{L}$, then the isomorphism Φ determines a bundle isomorphism of line bundles over $G \times X$:
$$\Phi : (G \times X) \times_{\pi_X, X, \pi} L \to (G \times X) \times_{\sigma, X, \pi} L$$
and then we obtain $\tilde{\sigma} := \pi_X \circ \Phi$. The cocycle condition ensures that $\tilde{\sigma}$ is an action.
(2) The above notion of a linearisation of a G-action on X can be easily modified to larger rank vector bundles (or locally free sheaves) over X. However, we will only work with linearisations for line bundles (or equivalently invertible sheaves).

Exercise 5.17. For an action of an affine algebraic group G on a scheme X, the tensor product of two linearised line bundles has a natural linearisation and the dual of a linearised line bundle also has a natural linearisation. By an isomorphism of linearisations, we mean an isomorphism of the underlying line bundles that is G-equivariant; that is, commutes with the actions of G on these line bundles. We let $\mathrm{Pic}^G(X)$ denote the group of isomorphism classes of linearisations of a G-action on X. There is a natural forgetful map $\alpha : \mathrm{Pic}^G(X) \to \mathrm{Pic}(X)$.

Example 5.18. (1) Let us consider $X = \mathrm{Spec}\, k$ with necessarily the trivial G-action. Then there is only one line bundle $\pi : \mathbb{A}^1 \to \mathrm{Spec}\, k$ over $\mathrm{Spec}\, k$, but there are many linearisations. In fact, the group of linearisations of X is the character group of G. If $\chi : G \to \mathbb{G}_m$ is a character of G, then we define an action of G on $\mathbb{A}^1$ by acting by $G \times \mathbb{A}^1 \to \mathbb{A}^1$. Conversely, any linearisation is given by a linear action of G on $\mathbb{A}^1$; that is, by a group homomorphism $\chi : G \to \mathrm{GL}_1 = \mathbb{G}_m$.

(2) For any scheme X with an action of an affine algebraic group G and any character $\chi : G \to \mathbb{G}_m$, we can construct a linearisation on the trivial line bundle $X \times \mathbb{A}^1 \to X$ by

$$g \cdot (x, z) = (g \cdot x, \chi(g)z).$$

More generally, for any linearisation $\tilde{\sigma}$ on $L \to X$, we can twist the linearisation by a character $\chi : G \to \mathbb{G}_m$ to obtain a linearisation $\tilde{\sigma}^\chi$.

(3) Not every linearisation on a trivial line bundle comes from a character. For example, consider $G = \mu_2 = \{\pm 1\}$ acting on $X = \mathbb{A}^1 - \{0\}$ by $(-1) \cdot x = x^{-1}$. Then the linearisation on $X \times \mathbb{A}^1 \to X$ given by $(-1) \cdot (x, z) = (x^{-1}, xz)$ is not isomorphic to a linearisation given by a character, as over the fixed points $+1$ and -1 in X, the action of $-1 \in \mu_2$ on the fibres is given by $z \mapsto z$ and $z \mapsto -z$ respectively.

(4) The natural actions of GL_{n+1} and SL_{n+1} on $\mathbb{P}^n$ inherited from the action of GL_{n+1} on $\mathbb{A}^{n+1}$ by matrix multiplication can be naturally linearised on the line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$. To see why, we note that the trivial rank $n+1$-vector bundle on $\mathbb{P}^n$ has a natural linearisation of GL_{n+1} (and also SL_{n+1}). The tautological line bundle $\mathcal{O}_{\mathbb{P}^n}(-1) \subset \mathbb{P}^n \times \mathbb{A}^{n+1}$ is preserved by this action and so we obtain natural linearisations of these actions on $\mathcal{O}_{\mathbb{P}^n}(\pm 1)$. However, the action of PGL_{n+1} on $\mathbb{P}^n$ does not admit a linearisation on $\mathcal{O}_{\mathbb{P}^n}(1)$ (see Exercise Sheet 9), but we can always linearise any G-action on $\mathbb{P}^n$ to $\mathcal{O}_{\mathbb{P}^n}(n+1)$ as this is isomorphic to the nth exterior power of the cotangent bundle, and we can lift any action on $\mathbb{P}^n$ to its cotangent bundle.

Lemma 5.19. *Let G be an affine algebraic group acting on a scheme X via $\sigma : G \times X \to X$ and let $\tilde{\sigma} : G \times L \to L$ be a linearisation of the action on a line bundle L over X. Then there is a natural linear representation $G \to \mathrm{GL}(H^0(X, L))$.*

Proof. We construct the co-module $H^0(X, L) \to \mathcal{O}(G) \otimes_k H^0(X, L)$ defining this representation by the composition

$$H^0(X, L) \xrightarrow{\sigma^*} H^0(G \times X, \sigma^* L) \cong H^0(G \times X, G \times L) \cong H^0(G, \mathcal{O}_G) \otimes H^0(X, L)$$

where the final isomorphism follows from the Künneth formula and the middle isomorphism is defined using the isomorphism $G \times L \cong \sigma^* L$. □

Remark 5.20. Suppose that X is a projective scheme and L is a very ample linearisation. Then the natural evaluation map

$$H^0(X, L) \otimes_k \mathcal{O}_X \to L$$

is G-equivariant. Moreover, this evaluation map induces a G-equivariant closed embedding

$$X \hookrightarrow \mathbb{P}(H^0(X, L)^*)$$

such that L is isomorphic to the pullback of the Serre twisting sheaf $\mathcal{O}(1)$ on this projective space. In this case, we see that we have an embedding of X as a closed subscheme of a projective space $\mathbb{P}(H^0(X, L)^*)$ such that the action of G on X comes from a linear action of G on $H^0(X, L)^*$. In particular, we see that a linearisation naturally generalises the setting of G acting linearly on $X \subset \mathbb{P}^n$.

5.4. **Projective GIT with respect to an ample linearisation.** Let G be a reductive group acting on a projective scheme X and let L be an ample linearisation of the G-action on X. Then consider the graded finitely generated k-algebra

$$R := R(X, L) := \bigoplus_{r \geq 0} H^0(X, L^{\otimes^r})$$

of sections of powers of L. Since each line bundle $L^{\otimes r}$ has an induced linearisation, there is an induced action of G on the space of sections $H^0(X, L^{\otimes^r})$ by Lemma 5.19. We consider the graded algebra of G-invariant sections

$$R^G = \bigoplus_{r \geq 0} H^0(X, L^{\otimes^r})^G.$$

The subalgebra of invariant sections R^G is a finitely generated k-algebra and $\operatorname{Proj} R^G$ is projective over $R_0^G = k^G = k$ following a similar argument to above.

Definition 5.21. For a reductive group G acting on a projective scheme X with respect to an ample line bundle, we make the following definitions.

1) A point $x \in X$ is *semistable* with respect to L if there is an invariant section $\sigma \in H^0(X, L^{\otimes r})^G$ for some $r > 0$ such that $\sigma(x) \neq 0$.
2) A point $x \in X$ is *stable* with respect to L if $\dim G \cdot x = \dim G$ and there is an invariant section $\sigma \in H^0(X, L^{\otimes r})^G$ for some $r > 0$ such that $\sigma(x) \neq 0$ and the action of G on $X_\sigma := \{x \in X : \sigma(x) \neq 0\}$ is closed.

We let $X^{ss}(L)$ and $X^s(L)$ denote the open subset of semistable and stable points in X respectively. Then we define the *projective GIT quotient with respect to* L to be the morphism

$$X^{ss} \to X//_L G := \operatorname{Proj} R(X, L)^G$$

associated to the inclusion $R(X, L)^G \hookrightarrow R(X, L)$.

Exercise 5.22. We have already defined notions of semistability and stability when we have a linear action of G on $X \subset \mathbb{P}^n$. In this case, the action can naturally be linearised using the line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$. Show that the two notions of semistability agree; that is,

$$X^{(s)s} = X^{(s)s}(\mathcal{O}_{\mathbb{P}^n}(1)|_X).$$

Theorem 5.23. *Let G be a reductive group acting on a projective scheme X and L be an ample linearisation of this action. Then the GIT quotient*

$$\varphi : X^{ss}(L) \to X//_L G = \operatorname{Proj} \bigoplus_{r \geq 0} H^0(X, L^{\otimes r})^G$$

is a good quotient and $X//_L G$ is a projective scheme with a natural ample line bundle L' such that $\varphi^* L' = L^{\otimes n}$ for some $n > 0$. Furthermore, there is an open subset $Y^s \subset X//_L G$ such that $\varphi^{-1}(Y^s) = X^s(L)$ and $\varphi : X^s(L) \to Y^s$ is a geometric quotient for the G-action on $X^s(L)$.

Proof. As L is ample, for each $\sigma \in R(X, L)_+^G$, the open set X_σ is affine and the above GIT quotient is obtained by gluing affine GIT quotients (we omit the proof as it is very similar to that of Theorem 5.3 and Theorem 5.6). □

Remark 5.24. In fact, the graded homogeneous ring $R(X, L)^G$ also determines an ample line bundle L' on its projectivisation $X//_L G$ such that $R(X//_L G, L') \cong R(X, L)^G$. Furthermore, $\phi^*(L') = L^{\otimes r}$ for some $r > 0$ (for example, see [4] Theorem 8.1 for a proof of this statement).

Remark 5.25 (Variation of geometric invariant theory quotient)**.** We note that the GIT quotient depends on a choice of linearisation of the action. One can study how the semistable locus $X^{ss}(L)$ and the GIT quotient $X//_L G$ vary with the linearisation L; this area is known as variation of GIT. A key result in this area is that there are only finitely many distinct GIT quotients produced by varying the ample linearisation of a fixed G-action on a projective normal variety X (for example, see [5] and [41]).

Remark 5.26. For an ample linearisation L, we know that some positive power of L is very ample. By definition $X^{ss}(L) = X^{ss}(L^{\otimes n})$ and $X^s(L) = X^s(L^{\otimes n})$ and $X//_L G \cong X//_{L^{\otimes n}} G$ (as abstract projective schemes), we can assume without loss of generality that L is very ample and so $X \subset \mathbb{P}^n$ and G acts linearly. However, we note that the induced ample line bundles on $X//_L G$ and $X//_{L^{\otimes n}} G$ are different, and so these GIT quotients come with different embeddings into (weighted) projective spaces.

Definition 5.27. We say two semistable k-points x and x' in X are S-equivalent if the orbit closures of x and x' meet in the semistable subset $X^{ss}(L)$. We say a semistable k-point is polystable if its orbit is closed in the semistable locus $X^{ss}(L)$.

Corollary 5.28. *Let x and x' be k-points in $X^{ss}(L)$; then $\varphi(x) = \varphi(x')$ if and only if x and x' are S-equivalent. Moreover, we have a bijection of sets*

$$(X/\!/_L G)(k) \cong X^{ps}(L)(k)/G(k) \cong X^{ss}(L)(k)/\sim_{S\text{-equiv.}}$$

where $X^{ps}(L)(k)$ is the set of polystable k-points.

5.5. **GIT for general varieties with linearisations.** In this section, we give a more general theorem of Mumford for constructing GIT quotients of reductive group actions on quasi-projective schemes with respect to (not necessarily ample) linearisations.

Definition 5.29. Let X be a quasi-projective scheme with an action by a reductive group G and L be a linearisation of this action.

1) A point $x \in X$ is *semistable* with respect to L if there is an invariant section $\sigma \in H^0(X, L^{\otimes r})^G$ for some $r > 0$ such that $\sigma(x) \neq 0$ and $X_\sigma = \{x \in X : \sigma(x) \neq 0\}$ is affine.
2) A point $x \in X$ is *stable* with respect to L if $\dim G \cdot x = \dim G$ and there is an invariant section $\sigma \in H^0(X, L^{\otimes r})^G$ for some $r > 0$ such that $\sigma(x) \neq 0$ and X_σ is affine and the action of G on X_σ is closed.

The open subsets of stable and semistable points with respect to L are denoted $X^s(L)$ and $X^{ss}(L)$ respectively.

Remark 5.30. If X is projective and L is ample, then this agrees with Definition 5.21 as X_σ is affine for any non-constant section σ (see [14] III Theorem 5.1 and II Proposition 2.5).

In this setting, the GIT quotient $X/\!/_L G$ is defined by taking the projective spectrum of the ring $R(X, L)^G$ of G-invariant sections of powers of L. One proves that $\varphi : X^{ss}(L) \to Y := X/\!/_L G$ is a good quotient by locally showing that this morphism is obtained by gluing affine GIT quotients $\varphi_\sigma : X_\sigma \to Y_\sigma$ in exactly the same way as Theorem 5.3. Then similarly to Theorem 5.6, one proves that this restricts to a geometric quotient on the stable locus. In particular, we have the following result.

Theorem 5.31. *(Mumford) Let G be a reductive group acting on a quasi-projective scheme X and L be a linearisation of this action. Then there is a quasi-projective scheme $X/\!/_L G$ and a good quotient $\varphi : X^{ss}(L) \to X/\!/_L G$ of the G-action on $X^{ss}(L)$. Furthermore, there is an open subset $Y^s \subset X/\!/_L G$ such that $\varphi^{-1}(Y^s) = X^s(L)$ and $\varphi : X^s(L) \to Y^s$ is a geometric quotient for the G-action on $X^s(L)$.*

The only part of this theorem which remains to be proved is the statement that the GIT quotient $X/\!/_L G$ is quasi-projective. To prove this, one notes that the GIT quotient comes with an ample line bundle L' which can be used to give an embedding of X into a projective space.

6. Criteria for (semi)stability

Let us suppose that we have a reductive group G acting on a projective scheme X with respect to an ample linearisation L. In order to determine the GIT semistable locus $X^{ss}(L) \subset X$, we need to calculate the algebra of G-invariant sections of all powers of L. In practice, there are very few examples in which one can compute these rings of invariants by hand (or even with the aid of a computer). In this section, we will give alternative criteria for determining the semistability of a point. The main references for the material covered in this section are [4], [25], [31] and [42].

We first observe that we can simplify our situation by assuming that $X \subset \mathbb{P}^n$ and the G-action is linear. Indeed, by replacing L by some power $L^{\otimes r}$, we get an embedding

$$X \subset \mathbb{P}^n = \mathbb{P}(H^0(X, L^{\otimes r})^*)$$

such that $\mathcal{O}_{\mathbb{P}^n}(1)|_X = L^{\otimes r}$ and G acts linearly on $\mathbb{P}^n$. Furthermore, by Remark 5.26, we have an agreement of (semi)stable sets $X^{(s)s}(L) = X^{(s)s}(L^{\otimes r})$.

6.1. **A topological criterion.** Let G be a reductive group acting linearly on a projective scheme $X \subset \mathbb{P}^n$. Then as G acts via $G \to \mathrm{GL}_{n+1}$, the action of G lifts to the affine cones $\tilde{X} \subset \mathbb{A}^{n+1}$. We let $R(X) = \mathcal{O}(\tilde{X})$ denote the homogeneous coordinate ring of X.

Proposition 6.1. *Let $x \in X(k)$ and choose a non-zero lift $\tilde{x} \in \tilde{X}(k)$ of x. Then:*

i) x is semistable if and only if $0 \notin \overline{G \cdot \tilde{x}}$;

ii) x is stable if and only if $\dim G_{\tilde{x}} = 0$ and $G \cdot \tilde{x}$ is closed in $\tilde{X}$.

Proof. i) If x is semistable, then there is a G-invariant homogeneous polynomial $f \in R(X)^G$ which is non-zero at x. We can view f as a G-invariant function on $\tilde{X}$ such that $f(\tilde{x}) \neq 0$. As invariant functions are constant on orbits and also their closures we see that $f(\overline{G \cdot \tilde{x}}) \neq 0$ and so there is a function which separates the closed subschemes $\overline{G \cdot \tilde{x}}$ and 0; therefore, these closed subschemes are disjoint.

For the converse, suppose that $\overline{G \cdot \tilde{x}}$ and 0 are disjoint. Then as these are both G-invariant closed subsets of the affine variety $\tilde{X}$ and G is geometrically reductive, there exists a G-invariant polynomial $f \in \mathcal{O}(\tilde{X})^G$ which separates these subsets

$$f(\overline{G \cdot \tilde{x}}) = 1 \quad \text{and} \quad f(0) = 0$$

by Lemma 4.29. In fact, we can take f to be homogeneous: if we decompose f into homogeneous elements $f = f_0 + \cdots + f_r$, then as the action is linear, each f_i must be G-invariant and, in particular, there is at least one G-invariant homogeneous polynomial f_i which does not vanish on $\overline{G \cdot \tilde{x}}$. Hence, x is semistable.

ii) If x is stable, then $\dim G_x = 0$ and there is a G-invariant homogeneous polynomial $f \in R(X)^G$ such that $x \in X_f$ and $G \cdot x$ is closed in X_f. Since $G_{\tilde{x}} \subset G_x$, the stabiliser of $\tilde{x}$ is also zero dimensional. We can view f as a function on $\tilde{X}$ and consider the closed subscheme

$$Z := \{z \in \tilde{X} : f(z) = f(\tilde{x})\}$$

of $\tilde{X}$. It suffices to show that $G \cdot \tilde{x}$ is a closed subset of Z. The projection map $\tilde{X} - \{0\} \to X$ restricts to a surjective finite morphism $\pi : Z \to X_f$. The preimage of the closed orbit $G \cdot x$ in X_f under π is closed and G-invariant and, as π is also finite, the preimage $\pi^{-1}(G \cdot x)$ is a finite number of G-orbits. Since π is finite, the finite number of G-orbits in the preimage of $G \cdot x$ all have dimension equal to $\dim G$, and so these orbits must be closed in the preimage (see Proposition 3.15). Hence $G \cdot \tilde{x}$ is closed in Z.

Conversely suppose that $\dim G_{\tilde{x}} = 0$ and $G \cdot \tilde{x}$ is closed in $\tilde{X}$; then $0 \notin \overline{G \cdot \tilde{x}} = G \cdot \tilde{x}$ and so x is semistable by i). As x is semistable there is a non-constant G-invariant homogeneous polynomial f such that $f(x) \neq 0$. As above, we consider the finite surjective morphism

$$\pi : Z := \{z \in \tilde{X} : f(z) = f(\tilde{x})\} \to X_f.$$

Since $\pi(G \cdot \tilde{x}) = G \cdot x$ and π is finite, x has zero dimensional stabiliser group and $G \cdot x$ is closed in X_f. Since this holds for all f such that $f(x) \neq 0$, it follows that $G \cdot x$ is closed in $X^{ss} = \cup_f X_f$. Hence x is stable by Lemma 5.9. □

6.2. **The Hilbert–Mumford Criterion.** Suppose we have a linear action of a reductive group G on a projective scheme $X \subset \mathbb{P}^n$ as above. In this section, we give a numerical criterion which can be used to determine (semi)stability of a point x.

Following the topological criterion above, we see that to determine semistability, it is important to understand the closure of an orbit. One way to study the closure of an orbit is by using 1-parameter subgroups of G.

Definition 6.2. A 1-parameter subgroup (1-PS) of G is a non-trivial group homomorphism $\lambda : \mathbb{G}_m \to G$.

Fix $x \in X(k)$ and a 1-PS $\lambda : \mathbb{G}_m \to G$. Then we let $\lambda_x : \mathbb{G}_m \to X$ be the morphism given by

$$\lambda_x(t) = \lambda(t) \cdot x.$$

We have a natural embedding of $\mathbb{G}_m = \mathbb{A}^1 - \{0\} \hookrightarrow \mathbb{P}^1$ given by $t \mapsto [1 : t]$. Since X is projective, it is proper over $\operatorname{Spec} k$ and so, by the valuative criterion for properness, the morphism $\lambda_x : \mathbb{G}_m \to X$ extends uniquely to a morphism $\hat{\lambda}_x : \mathbb{P}^1 \to X$:

$$\begin{array}{ccc} \mathbb{G}_m & \xrightarrow{\lambda_x} & X \\ \downarrow & \nearrow_{\exists ! \hat{\lambda}_x} & \downarrow \\ \mathbb{P}^1 & \longrightarrow & \operatorname{Spec} k. \end{array}$$

We use suggestive notation for the specialisations of this extended morphism at the zero and infinity points of $\mathbb{P}^1$:

$$\lim_{t \to 0} \lambda(t) \cdot x := \hat{\lambda}_x([1:0]) \quad \text{and} \quad \lim_{t \to \infty} \lambda(t) \cdot x := \hat{\lambda}_x([0:1]).$$

In fact, we can focus on the specialisation at zero, as

$$\lim_{t \to \infty} \lambda(t) \cdot x = \lim_{t \to 0} \lambda^{-1}(t) \cdot x.$$

Let $y := \lim_{t \to 0} \lambda(t) \cdot x$; then y is fixed by the action of $\lambda(\mathbb{G}_m)$; therefore, on the fibre over y of the line bundle $\mathcal{O}(1) := \mathcal{O}_{\mathbb{P}^n}(1)|_X$, the group $\lambda(\mathbb{G}_m)$ acts by a character $t \mapsto t^r$.

Definition 6.3. We define the *Hilbert-Mumford weight* of the action of the 1-PS λ on $x \in X(k)$ to be

$$\mu^{\mathcal{O}(1)}(x, \lambda) = r$$

where r is the weight of the $\lambda(\mathbb{G}_m)$ on the fibre $\mathcal{O}(1)_y$ over $y := \lim_{t \to 0} \lambda(t) \cdot x$.

From this definition, it is not so straight forward to compute this Hilbert–Mumford weight; therefore, we will rephrase this in terms of the weights for the action on the affine cone. Recall that $\mathcal{O}_{\mathbb{P}^n}(1)$ is the dual of the tautological line bundle on $\mathbb{P}^n$. Let $\mathbb{A}^{n+1}$ be the affine cone over $\mathbb{P}^n$; then $\mathcal{O}_{\mathbb{P}^n}(-1)$ is the blow up of $\mathbb{A}^{n+1}$ at the origin. Pick a non-zero lift $\tilde{x} \in \tilde{X}$ of $x \in X$. Then we can consider the morphism

$$\lambda_{\tilde{x}} := \lambda(-) \cdot \tilde{x} : \mathbb{G}_m \to \tilde{X}$$

which may no longer extend to $\mathbb{P}^1$, as $\tilde{X}$ is not proper. If it extends to zero (or infinity), we will denote the limits by

$$\lim_{t \to 0} \lambda(t) \cdot \tilde{x} \quad (\text{or} \lim_{t \to \infty} \lambda(t) \cdot \tilde{x}).$$

Any point in the boundary $\overline{\lambda_{\tilde{x}}(\mathbb{G}_m)} - \lambda_{\tilde{x}}(\mathbb{G}_m)$ must be equal to either of these limit points.

The action of the 1-PS $\lambda(\mathbb{G}_m)$ on the affine cone $\mathbb{A}^{n+1}$ is linear, and so diagonalisable by Proposition 3.12; therefore, we can pick a basis $e_0, ..., e_n$ of k^{n+1} such that

$$\lambda(t) \cdot e_i = t^{r_i} e_i \quad \text{for } r_i \in \mathbb{Z}.$$

We call the integers r_i the λ-weights of the action on $\mathbb{A}^{n+1}$. For $x \in X(k)$ we can pick $\tilde{x} \in \tilde{X}(k)$ lying above this point and write $\tilde{x} = \sum_{i=0}^n x_i e_i$ with respect to this basis; then

$$\lambda(t) \cdot \tilde{x} = \sum_{i=0}^n t^{r_i} x_i e_i$$

and we let $\lambda\text{-wt}(x) := \{r_i : x_i \neq 0\}$ be the λ-weights of x (note that this does not depend on the choice of lift $\tilde{x}$).

Definition 6.4. We define the Hilbert-Mumford weight of x at λ to be

$$\mu(x, \lambda) := -\min\{r_i : x_i \neq 0\}.$$

We will soon show that this definition agrees with the above definition. However, we first note some useful properties of the Hilbert–Mumford weight.

Exercise 6.5. Show that the Hilbert–Mumford weight has the following properties.

(1) $\mu(x, \lambda)$ is the unique integer μ such that $\lim_{t \to 0} t^{\mu} \lambda(t) \cdot \tilde{x}$ exists and is non-zero.

(2) $\mu(x, \lambda^n) = n\mu(x, \lambda)$ for positive n.
(3) $\mu(g \cdot x, g\lambda g^{-1}) = \mu(x, \lambda)$ for all $g \in G$.
(4) $\mu(x, \lambda) = \mu(y, \lambda)$ where $y = \lim_{t \to 0} \lambda(t) \cdot x$.

Lemma 6.6. *The two definitions of the Hilbert–Mumford weight agree:*

$$\mu^{\mathcal{O}(1)}(x, \lambda) = \mu(x, \lambda).$$

Proof. Pick a non-zero lift $\tilde{x}$ in the affine cone which lies over x. Then we assume that we have taken coordinates on $\mathbb{A}^{n+1}$ as above so that the action of $\lambda(t)$ is given by

$$\lambda(t) \cdot \tilde{x} = \lambda(t) \cdot (x_0, \dots, x_n) = (t^{r_0} x_0, \dots, t^{r_n} x_n).$$

Since $\mu(x, \lambda) + r_i \geq 0$ for all i such that $x_i \neq 0$, with equality for at least one i with $x_i \neq 0$, we see that

$$\tilde{y} := \lim_{t \to 0} t^{\mu(x,\lambda)} \lambda(t) \cdot \tilde{x} = (y_0, \dots, y_n)$$

exists and is non-zero. More precisely, we have

$$y_i = \begin{cases} x_i & \text{if } r_i = -\mu(x, \lambda) \\ 0 & \text{else.} \end{cases}$$

Therefore, $\lambda(t) \cdot \tilde{y} = t^{-\mu(x,\lambda)} \tilde{y}$. Furthermore, $\tilde{y}$ lies over $y := \lim_{t \to 0} \lambda(t) \cdot x$ and the weight of the λ-action on $\tilde{y}$ is $-\mu(x, \lambda)$. Since $\mathcal{O}_{\mathbb{P}^n}(-1)$ is the blow up of $\mathbb{A}^{n+1}$ at 0, we see that $-\mu(x, \lambda)$ is the weight of the $\lambda(\mathbb{G}_m)$-action on $\mathcal{O}(-1)_y$. Hence, the weight of the $\lambda(\mathbb{G}_m)$-action on $\mathcal{O}(1)_y$ is $\mu(x, \lambda)$ and this completes the proof of the claim. □

From the second definition of the Hilbert–Mumford weight, we easily deduce the following lemma.

Lemma 6.7. *Let λ be a 1-PS of G and let $x \in X(k)$. We diagonalise the $\lambda(\mathbb{G}_m)$-action on the affine cone as above and let $\tilde{x} = \sum_{i=0}^n x_i e_i$ be a non-zero lift of x.*

i) $\mu(x, \lambda) < 0 \iff \tilde{x} = \sum_{r_i > 0} x_i e_i \iff \lim_{t \to 0} \lambda(t) \cdot \tilde{x} = 0$.
ii) $\mu(x, \lambda) = 0 \iff \tilde{x} = \sum_{r_i \geq 0} x_i e_i$ *and there exists* $r_i = 0$ *such that* $x_i \neq 0 \iff \lim_{t \to 0} \lambda(t) \cdot \tilde{x}$ *exists and is non-zero.*
iii) $\mu(x, \lambda) > 0 \iff \tilde{x} = \sum_{r_i} x_i e_i$ *and there exists* $r_i < 0$ *such that* $x_i \neq 0 \iff \lim_{t \to 0} \lambda(t) \cdot \tilde{x}$ *does not exist.*

Remark 6.8. We can use λ^{-1} to study $\lim_{t \to \infty} \lambda(t) \cdot \tilde{x}$ as

$$\lim_{t \to 0} \lambda^{-1}(t) \cdot \tilde{x} = \lim_{t \to \infty} \lambda(t) \cdot \tilde{x}.$$

Then it follows that

i) $\mu(x, \lambda^{-1}) < 0 \iff \tilde{x} = \sum_{r_i < 0} x_i e_i \iff \lim_{t \to \infty} \lambda(t) \cdot \tilde{x} = 0$.
ii) $\mu(x, \lambda^{-1}) = 0 \iff \tilde{x} = \sum_{r_i \leq 0} x_i e_i$ and there exists $r_i = 0$ such that $x_i \neq 0 \iff \lim_{t \to \infty} \lambda(t) \cdot \tilde{x}$ exists and is non-zero.
iii) $\mu(x, \lambda^{-1}) > 0 \iff \tilde{x} = \sum_{r_i} x_i e_i$ and there exists $r_i > 0$ such that $x_i \neq 0 \iff \lim_{t \to \infty} \lambda(t) \cdot \tilde{x}$ does not exist.

Following the discussion above and the topological criterion (see Proposition 6.1), we have the following results for (semi)stability with respect to the action of the subgroup $\lambda(\mathbb{G}_m) \subset G$.

Lemma 6.9. *Let G be a reductive group acting linearly on a projective scheme $X \subset \mathbb{P}^n$. Suppose $x \in X(k)$; then*

i) x is semistable for the action of $\lambda(\mathbb{G}_m)$ if and only if $\mu(x, \lambda) \geq 0$ and $\mu(x, \lambda^{-1}) \geq 0$.
ii) x is stable for the action of $\lambda(\mathbb{G}_m)$ if and only if $\mu(x, \lambda) > 0$ and $\mu(x, \lambda^{-1}) > 0$.

Proof. For i), by the topological criterion x is semistable for $\lambda(\mathbb{G}_m)$ if and only if $0 \notin \overline{\lambda(\mathbb{G}_m) \cdot \tilde{x}}$, where and $\tilde{x} \in \tilde{X}(k)$ is a point lying over x. Since any point in the boundary $\overline{\lambda(\mathbb{G}_m) \cdot \tilde{x}} - \lambda(\mathbb{G}_m) \cdot \tilde{x}$ is either

$$\lim_{t \to 0} \lambda(t) \cdot \tilde{x} \quad \text{or} \quad \lim_{t \to \infty} \lambda(t) \cdot \tilde{x} = \lim_{t \to 0} \lambda^{-1}(t) \cdot \tilde{x},$$

it follows from Lemma 6.7 that x is semistable if and only if

$$\mu(x,\lambda) \geq 0 \quad \text{and} \quad \mu(x,\lambda^{-1}) \geq 0.$$

For ii), by the topological criterion x is stable for $\lambda(\mathbb{G}_m)$ if and only if $\dim \lambda(\mathbb{G}_m)_{\tilde{x}} = 0$ and $\lambda(\mathbb{G}_m)\cdot\tilde{x}$ is closed. The orbit is closed if and only if the boundary is empty; that is, if and only if both limits

$$\lim_{t\to 0} \lambda(t)\cdot\tilde{x} \quad \text{and} \quad \lim_{t\to\infty} \lambda(t)\cdot\tilde{x} = \lim_{t\to 0} \lambda^{-1}(t)\cdot\tilde{x}$$

do not exist, i.e.

$$\mu(x,\lambda) > 0 \quad \text{and} \quad \mu(x,\lambda^{-1}) > 0.$$

Furthermore, if these inequalities hold, then $\lambda(\mathbb{G}_m)$ cannot fix $\tilde{x}$ (as otherwise the above limits would both exist) and so we must have that $\dim \lambda(\mathbb{G}_m)_{\tilde{x}} = 0$. □

Exercise 6.10. Let $\mathbb{G}_m$ act on $\mathbb{P}^2$ by $t\cdot[x:y:z] = [tx:y:t^{-1}z]$. For every point $x \in \mathbb{P}^2$ and the 1-PS $\lambda(t) = t$, calculate $\mu(x,\lambda^{\pm 1})$ and then by using Lemma 6.9 above or otherwise, determine X^s and X^{ss}.

If x is (semi)stable for G, then it is (semi)stable for all subgroups H of G as every G-invariant function is also H-invariant. Hence, for a k-point x, we have

$$x \text{ is semistable} \implies \mu(x,\lambda) \geq 0 \,\forall \text{ 1-PS } \lambda \text{ of } G,$$

$$x \text{ is stable} \implies \mu(x,\lambda) > 0 \,\forall \text{ 1-PS } \lambda \text{ of } G.$$

The Hilbert-Mumford criterion gives the converse to these statements; the idea is that because G is reductive it has enough 1-PSs to detect points in the closure of an orbit (see Theorem 6.13 below).

Theorem 6.11. *(Hilbert–Mumford Criterion) Let G be a reductive group acting linearly on a projective scheme $X \subset \mathbb{P}^n$. Then, for $x \in X(k)$, we have*

$$x \in X^{ss} \iff \mu(x,\lambda) \geq 0 \text{ for all 1-PSs } \lambda \text{ of } G,$$
$$x \in X^{s} \iff \mu(x,\lambda) > 0 \text{ for all 1-PSs } \lambda \text{ of } G.$$

Remark 6.12. A 1-PS is *primitive* if it is not a multiple of any other 1-PS. By Exercise 6.5 ii), it suffices to check the Hilbert–Mumford criterion for primitive 1-PSs of G.

It follows from the topological criterion given in Proposition 6.1 and also from Lemma 6.9, that the Hilbert–Mumford criterion is equivalent to the following fundamental theorem in GIT.

Theorem 6.13. *[Fundamental Theorem in GIT] Let G be a reductive group acting on an affine space $\mathbb{A}^{n+1}$. If $x \in \mathbb{A}^{n+1}$ is a closed point and $y \in \overline{G\cdot x}$, then there is a 1-PS λ of G such that $\lim_{t\to 0}\lambda(t)\cdot x = y$.*

The proof of the above fundamental theorem relies on a decomposition theorem of Iwahori which roughly speaking says there is an abundance of 1-PSs of reductive groups [17]. The proof of this theorem essentially follows from ideas of Mumford [25] §2.1 and we delay the proof until the end of this section.

Example 6.14. We consider the action of $G = \mathbb{G}_m$ on $X = \mathbb{P}^n$ as in Example 5.8. As the group is a 1-dimensional torus, we need only calculate $\mu(x,\lambda)$ and $\mu(x,\lambda^{-1})$ for $\lambda(t) = t$ as was the case in Lemma 6.9. Suppose $\tilde{x} = (x_0,\dots,x_n)$ lies over $x = [x_0:\dots:x_n] \in \mathbb{P}^n$. Then

$$\lim_{t\to 0} \lambda(t)\cdot\tilde{x} = (t^{-1}x_0, tx_1\dots,tx_n)$$

exists if and only if $x_0 = 0$. If $x_0 = 0$, then $\mu(x,\lambda) = -1$ and otherwise $\mu(x,\lambda) > 0$. Similarly

$$\lim_{t\to 0} \lambda^{-1}(t)\cdot\tilde{x} = (tx_0, t^{-1}x_1\dots,t^{-1}x_n)$$

exists if and only if $x_1 = \dots = x_n = 0$. If $x_1 = \dots = x_n = 0$, then $\mu(x,\lambda) = -1$ and otherwise $\mu(x,\lambda) > 0$. Therefore, the GIT semistable set and stable coincide:

$$X^{ss} = X^s = \{[x_0:\dots:x_n] : x_0 \neq 0 \text{ and } (x_1,\dots,x_n) \neq 0\} \subset \mathbb{P}^n.$$

6.3. **The Hilbert–Mumford Criterion for ample linearisations.** In this section we consider the following more general set up: suppose X is a projective scheme with an action by a reductive group G and ample linearisation L.

Definition 6.15. The *Hilbert–Mumford weight* of a 1-PS λ and $x \in X(k)$ with respect to L is

$$\mu^L(x, \lambda) := r$$

where r is the weight of the $\lambda(\mathbb{G}_m)$-action on the fibre L_y over the fixed point $y = \lim_{t \to 0} \lambda(t) \cdot x$.

Remark 6.16. We note that when $X \subset \mathbb{P}^n$ and the action of G is linear that this definition is consistent with the old definition; that is,

$$\mu^{\mathcal{O}_{\mathbb{P}^n}(1)|_X}(x, \lambda) = \mu(x, \lambda).$$

Exercise 6.17. Fix $x \in X$ and a 1-PS λ of G; then show $\mu^{\bullet}(x, \lambda) : \mathrm{Pic}^G(X) \to \mathbb{Z}$ is a group homomorphism where $\mathrm{Pic}^G(X)$ is the group of G-linearised line bundles on X.

Theorem 6.18. *(Hilbert–Mumford Criterion for ample linearisations) Let G be a reductive group acting on a projective scheme X and L be an ample linearisation of this action. Then, for $x \in X(k)$, we have*

$$x \in X^{ss}(L) \iff \mu^L(x, \lambda) \geq 0 \text{ for all 1-PSs } \lambda \text{ of } G,$$
$$x \in X^{s}(L) \iff \mu^L(x, \lambda) > 0 \text{ for all 1-PSs } \lambda \text{ of } G.$$

Proof. (Assuming Theorem 6.11) As L is ample, there is $n > 0$ such that $L^{\otimes n}$ is very ample. Then since

$$\mu^{L^{\otimes n}}(x, \lambda) = n\mu^L(x, \lambda)$$

it suffices to prove the statement for L very ample. If L is very ample then it induces a G-equivariant embedding $i : X \hookrightarrow \mathbb{P}^n$ such that $L \cong i^*\mathcal{O}_{\mathbb{P}^n}(1)$. Then we can just apply the first version of the Hilbert–Mumford criterion (cf. Theorem 6.11 and Remark 6.16). ☐

6.4. **Proof of the Fundamental Theorem in GIT.** In order to complete our proof of the Hilbert–Mumford Theorem, it suffices to prove the following slightly weaker version of the Fundamental Theorem in GIT.

Theorem 6.19. *Let G be a reductive group acting linearly on $\mathbb{A}^n$ and let $z \in \mathbb{A}^n$ be a k-point. If 0 lies in the orbit closure of z, then there exists a 1-PS λ of G such that* $\lim_{t \to 0} \lambda(t) \cdot z = 0$.

Proof. Suppose that $0 \in \overline{G \cdot z}$; then we will split the proof into 6 steps.

Step 1. We claim there is an irreducible (but not complete and not necessarily smooth) curve $C_1 \subset G \cdot z$ which contains 0 in its closure. To prove the existence of this curve, we use an argument similar to Bertini's Theorem and obtain the curve by intersecting hyperplanes in a projective completion $\mathbb{P}^n$ of $\mathbb{A}^n$; the argument is given in Lemma 6.20 below.

Step 2. We claim that there is a smooth projective curve C, a rational map $p : C \dashrightarrow G$ and a k-point $c_0 \in C$ such that $\lim_{c \to c_0} p(c) \cdot z = 0$. To prove this claim, we consider the action morphism $\sigma_z : G \to \mathbb{A}^n$ given by $g \mapsto g \cdot z$ and find a curve C_2 in G which dominates C_1 under σ_z (see Lemma 6.21 below) and then let C be a projective completion of the normalisation $\tilde{C}_2 \to C_2$; then the rational map $p : C \dashrightarrow G$ is defined by the morphism $\tilde{C}_2 \to C_2 \to G$. Finally, as the morphism $\tilde{C}_2 \to C_1$ is dominant it extends to their smooth projective completions and, as 0 lies in the closure of C_1, we can take a preimage $c_0 \in C$ of zero under this extension. Then $\lim_{c \to c_0} p(c) \cdot z = \lim_{c \to c_0} \sigma_z(p(c)) = 0$.

Step 3. Since C is a smooth proper curve, the completion of the local ring $\mathcal{O}_{C,c_0}$ of the curve at c_0 is isomorphic to the formal power series ring $k[[t]]$, whose field of fractions is the field of Laurent series $k((t))$. As the rational map $p : C \dashrightarrow G$ is defined in a punctured neighbourhood of c_0, it induces a morphism

$$q : K := \mathrm{Spec}\, k((t)) \cong \mathrm{Spec}\, \mathrm{Frac}\hat{\mathcal{O}}_{C,c_0} \to \mathrm{Spec}\, \mathrm{Frac}\mathcal{O}_{C,c_0} \to G$$

such that $\lim_{t \to 0}[q(t) \cdot z] = 0$. In Step 5, we will relate this K-valued point of G to a 1-PS.

Step 4. Let $R := \mathrm{Spec}\, k[[t]]$ and $K := \mathrm{Spec}\, k((t))$; then there is a natural morphism $K \to R$ and so the R-valued points of G form a subgroup of the K-valued points (i.e. $G(R) \subset G(K)$)

whose limit as $t \to 0$ exists. More precisely, the natural map $\operatorname{Spec} k \to R$ induces a morphism $G(R) \to G(k)$ given by taking the specialisation as $t \to 0$.

There is a morphism $K \to \mathbb{G}_m = \operatorname{Spec} k[s, s^{-1}]$ induced by the homomorphism $k[s, s^{-1}] \to k((t))$ given by $s \mapsto t$. For a 1-PS λ, we define its Laurent series expansion $< \lambda > \in G(K)$ to be the composition of the natural morphism $K \to \mathbb{G}_m$ with λ.

Step 5. We will use without proof the Cartan-Iwahori decomposition for G which states that every double coset in $G(K)$ for the subgroup $G(R)$ is represented by a Laurent series expansion $< \lambda >$ of 1-PS of G (for example, see [25] §2.1). Therefore, as $q \in G(K)$, there exists $l_i \in G(R)$ for $i = 1, 2$ and a 1-PS λ of G such that

$$l_1 \cdot q =< \lambda > \cdot l_2$$

and the 1-PS λ is non-trivial, as q is not an R-valued point of G.

Step 6. Let $g_i := l_i(0) \in G$; then following the equality in Step 5, we have

$$0 = g_1 \cdot 0 = \lim_{t \to 0} l_1(t) \cdot \lim_{t \to 0} (q(t) \cdot z) = \lim_{t \to 0} [(< \lambda > \cdot l_2)(t) \cdot z] .$$

We claim that $\lim_{t \to 0} \lambda(t) \cdot g_2 \cdot z = 0$ and so $\lambda' := g_2^{-1} \lambda g_2$ is a 1-PS of G with $\lim_{t \to 0} \lambda'(t) \cdot z = 0$, which would complete the proof of the theorem. To prove the claim, we use the fact that the action of the 1-PS λ on $V = \mathbb{A}^n$ decomposes into weight spaces V_r for $r \in \mathbb{Z}$. Since $l_2 \in G(R)$ and $g_2 = \lim_{t \to 0} l_2(0)$, we can write $l_2(t) \cdot z = g_2 \cdot z + \epsilon(t)$, where $\epsilon(t)$ only involves strictly positive powers of t. Then with respect to the weight space decomposition, we have

$$g_2 \cdot z + \epsilon(t) = \sum_{r \in \mathbb{Z}} (g_2 \cdot z)_r + \epsilon(t)_r .$$

Since $\lim_{t \to 0} [(< \lambda > \cdot l_2)(t) \cdot z] = 0$, it follows that $(g_2 \cdot z)_r = 0$ for $r \leq 0$, which proves the claim and completes our proof. □

Lemma 6.20. *With the notation and assumptions of the previous theorem, there exists an irreducible curve $C_1 \subset G \cdot z$ which contains the origin in its closure.*

Proof. Fix an embedding $\mathbb{A}^n \hookrightarrow \mathbb{P}^n$ and let $p \in \mathbb{P}^n$ denote the image of the origin. Let Y denote the closure of $G \cdot z$ in $\mathbb{P}^n$. We claim there exists a complete curve C_1' in Y containing the point $p \in \mathbb{P}^n$ and which is not contained entirely in the boundary $Z := Y - G \cdot z$. Assuming this claim, we obtain the desired curve $C_1 \subset G \cdot z$, by removing points of C_1' that lie in Z. To prove the claim, let $d = \dim Y$; then we can assume $d > 1$ as otherwise Y is already a curve. Then also $n > 1$. In the following section, we will see that hyperplanes in $\mathbb{P}^n$ are parametrised by $\mathbb{P}^n = \mathbb{P}(k[x_0, \dots, x_n]_1)$ and the space of hyperplanes containing p is a closed codimension 1 subspace $\mathcal{H}_p \subset \mathbb{P}^n$. Let $\mathcal{H}$ be the non-empty open subset of the product of $(d-1)$-copies of $\mathcal{H}_p$ consisting of hypersurfaces $(H_1, \dots, H_{d-1})$ such that

(1) $\cap_i H_i \cap Y$ is a curve (generically, $\dim \cap_{i=1}^{d-1} H_i \cap Y = \dim Y - (d-1) = 1$ and so this is a non-empty open condition), and
(2) $\cap_i H_i \cap Y$ is not entirely contained in Z (this is also a non-empty open condition, as $Z \subsetneq Y$ is a closed subscheme).

Hence, $\mathcal{H}$ is a non-empty open subset of $(\mathcal{H}_p)^{d-1}$, which has dimension $(n-1)(d-1) > 0$, and so the desired curve exists: we take $C_1' := \cap_i H_i \cap Y$, for $(H_1, \dots, H_{d-1}) \in \mathcal{H}_p \neq \emptyset$. □

Lemma 6.21. *With the notation and assumptions of the previous theorem, there exists a curve $C_2 \subset G$ that dominates the curve $C_1 \subset G \cdot z$ under the action morphism $\sigma_z : G \to G \cdot z$.*

Proof. Let η be the generic point of C_1. As η is not a geometric point and the above arguments about the existence of curves requires an algebraically closed field, we pick a geometric point $\overline{\eta}$ over η corresponding to a choice of an algebraically closed finite field extension of $k(C_1)$. We let $\sigma_z^{-1}(C_1)_\eta$ and $\sigma_z^{-1}(C_1)_{\overline{\eta}}$ be the base change of the preimage to $k(C_1)$ and its fixed algebraic closure. Then by Lemma 6.20, there exists a curve $C_2' \subset \sigma_z^{-1}(C_1)_{\overline{\eta}}$. The curve C_2' maps to a curve $C_2 \subset \sigma_z^{-1}(C_1)_\eta$ under the finite map $\sigma_z^{-1}(C_1)_{\overline{\eta}} \to \sigma_z^{-1}(C_1)_\eta$. By construction, C_2 is a curve in $\sigma_z^{-1}(C_1) \subset G$ which dominates C_1 under σ_z. □

7. Moduli of projective hypersurfaces

In this section, we will consider the moduli problem of classifying hypersurfaces of a fixed degree d in a projective space $\mathbb{P}^n$ up to linear change of coordinates on $\mathbb{P}^n$; that is, up to the action of the automorphism group PGL_{n+1} of $\mathbb{P}^n$. To avoid some difficulties associated with fields of positive characteristic, we assume that the characteristic of k is coprime to d.

7.1. **The moduli problem.** A non-zero homogeneous degree d polynomial F in $n+1$ variables $x_0, \dots, x_n$ determines a projective degree d hypersurface $(F = 0)$ in $\mathbb{P}^n$. If F is irreducible then the associated hypersurface is an irreducible closed subvariety of $\mathbb{P}^n$ of codimension 1. If F is reducible, then the associated hypersurface is a union of irreducible subvarieties of $\mathbb{P}^n$ of codimension 1 counted with multiplicities. For example, the polynomial $F(x_0, x_1) = x_0^d$ gives a degree d reducible hypersurface in $\mathbb{P}^1$: the d-fold point.

Hypersurfaces of degree d in $\mathbb{P}^n$ are parametrised by points in the space $k[x_0, \dots, x_n]_d - \{0\}$ of non-zero degree d homogeneous polynomials in $n+1$ variables. This variety has dimension

$$\begin{pmatrix} n+d \\ d \end{pmatrix}.$$

As any non-zero scalar multiple of a homogeneous polynomial F defines the same hypersurface, the projectivisation of this space

$$Y_{d,n} = \mathbb{P}(k[x_0, \dots, x_n]_d)$$

is a smaller dimensional parameter space for these hypersurfaces.

The automorphism group PGL_{n+1} of $\mathbb{P}^n$ acts naturally on $Y_{d,n} = \mathbb{P}(k[x_0, \dots, x_n]_d)$ as follows. The linear representation $\mathrm{GL}_{n+1} \to \mathrm{GL}(k^{n+1})$ given by acting by left multiplication induces a linear action of GL_{n+1} on $\mathbb{P}^n$. Consequently, there is an induced GL_{n+1}-action on the homogeneous coordinate ring $R(\mathbb{P}^n) = k[x_0, \dots, x_n]$ which preserves the graded pieces $k[x_0, \dots, x_n]_d$. This determines a linear action of GL_{n+1} on $\mathbb{P}(k[x_0, \dots, x_n]_d)$ by

$$(g \cdot F)(p) = F(g^{-1} \cdot p)$$

for $g \in \mathrm{GL}_{n+1}$, $F \in k[x_0, \dots, x_n]_d$ and $p \in \mathbb{A}^{n+1}$ (we note that the inverse here makes this a left action). This descends to an action

$$\mathrm{PGL}_{n+1} \times \mathbb{P}(k[x_0, \dots, x_n]_d) \to \mathbb{P}(k[x_0, \dots, x_n]_d).$$

One may expect that a moduli space for degree d hypersurfaces in $\mathbb{P}^n$ is given by a categorical quotient of this action and we will soon show that this is the case, by proving that $Y_{d,n}$ parametrises a family with the local universal property. However, the PGL_{n+1}-action on $Y_{n,d}$ is not linear, but the actions of GL_{n+1} and SL_{n+1} are both linear. Since we have a surjection $\mathrm{SL}_{n+1} \to \mathrm{PGL}_{n+1}$ with finite kernel, the SL_{n+1}-orbits are the same as the PGL_{n+1}-orbits, and the only small changes is that for SL_{n+1} there is now a global finite stabiliser group, but from the perspective of GIT finite groups do not matter. Therefore, we will work with the SL_{n+1}-action.

To prove the tautological family over $Y_{d,n}$ has the local universal property in order to apply Proposition 3.35, we need to introduce a notion of families of hypersurfaces. Let us start formulating a reasonable notion of families of hypersurfaces. One natural idea for a family of hypersurfaces over S is that we have a closed subscheme $X \subset S \times \mathbb{P}^n$ such that $X_s = X \cap \{s\} \times \mathbb{P}^n$ is a degree d hypersurface. For $S = \mathbb{A}^r = \operatorname{Spec} k[z_1, \dots, z_r]$, this is given by $H \in k[z_1, \dots, z_r, x_0, \dots, x_n]$ which is homogeneous of degree d in the variables $x_0, \dots, x_n$ and is non-zero at each point $s \in S$. In this case, a family of hypersurfaces is given by a degree d homogeneous polynomial in $n+1$ variables with coefficients in $\mathcal{O}(S)$. In fact, we can take this as a local definition for our families and generalise this notion to allow coefficients in an arbitrary line bundle L over S.

Definition 7.1. A *family of degree d hypersurfaces in $\mathbb{P}^n$ over S* is a line bundle L over S and a tuple of sections

$$\sigma := (\sigma_{i_0 \dots i_n} : i_j \geq 0, \sum_{j=0}^n i_j = d)$$

of L such that for each k-point $s \in S$, the polynomial

$$F(L, \sigma, s) := \sum_{i_0 \dots i_n} \sigma_{i_0 \dots i_n}(s) x_0^{i_0} \dots x_n^{i_n}$$

is non-zero.

We note that to make sense of this final sentence, we must trivialise L locally at s. Then the tuple of constants $\sigma(s)$ are determined up to multiplication by a non-zero scalar. In particular, we can determine whether $F(L, \sigma, s)$ is non-zero and the associated hypersurface is uniquely determined. We denote the family by (L, σ) and the hypersurface over a k-point s by $(L, \sigma)_s$: $F(L, \sigma, s) = 0$.

Definition 7.2. We say *two families (L, σ) and (L', σ') of degree d hypersurfaces in $\mathbb{P}^n$ over S are equivalent over S* if there exists an isomorphism $\phi : L \to L'$ of line bundles and $g \in \mathrm{GL}_{n+1}$ such that $\phi \circ \sigma = g \cdot \sigma'$.

We note that with this definition of equivalence the families (L, σ) and $(L, \lambda\sigma)$ are equivalent for any non-zero scalar λ.

Exercise 7.3. Show that $Y_{d,n} = \mathbb{P}(k[x_0, \dots, x_n]_d)$ parametrises a tautological family of degree d hypersurfaces in $\mathbb{P}^n$ with the local universal property. Deduce that any coarse moduli space for hypersurfaces is a categorical quotient of SL_{n+1} acting on $Y_{d,n}$ as above.

Since SL_{n+1} is reductive, we can take a projective GIT quotient of the action on $Y_{d,n}$ which is a good (and categorical) quotient of the semistable locus $Y_{d,n}^{ss}$. There are now two problems to address:

(1) determine the (semi)stable points in $Y_{d,n}$;
(2) geometrically interpret (semi)stability of points in terms of properties of the corresponding hypersurfaces.

For small values of d and n, we shall see that it is possible to give a full solution to the above two problems, although as both values get larger the problem becomes increasingly difficult.

7.2. Singularities of hypersurfaces.

Definition 7.4. A point p in $\mathbb{P}^n$ is a *singular point* of a projective hypersurface defined by a polynomial $F \in k[x_0, \dots, x_n]_d$ if

$$F(\tilde{p}) = 0 \text{ and } \frac{\partial F}{\partial x_i}(\tilde{p}) = 0 \text{ for } i = 0, \dots, n,$$

where $\tilde{p} \in \mathbb{A}^{n+1} - \{0\}$ is a lift of $p \in \mathbb{P}^n$. We say a hypersurface is *non-singular (or smooth)* if it has no singular points.

Remark 7.5.

(1) By using the Euler formula

$$\sum_{i=0}^{n} x_i \frac{\partial F}{\partial x_i} = d\, F$$

and the fact that d is coprime to the characteristic of k, we see that $p \in \mathbb{P}^n$ is a singular point of F if and only if all partial derivatives $\partial F / \partial x_i$ vanish at p.

(2) If we consider F as a function $F : \mathbb{A}^{n+1} \to k$, then we can consider its derivative $d_{\tilde{p}} F : T_{\tilde{p}} \mathbb{A}^{n+1} \to T_{F(\tilde{p})} k \cong k$ at $\tilde{p} \in \mathbb{A}^{n+1} - \{0\}$. The corresponding point $p \in \mathbb{P}^n$ is a singular point of F if and only if this derivative $d_{\tilde{p}} F$ is zero.

(3) Let $\sigma_g : \mathbb{A}^{n+1} \to \mathbb{A}^{n+1}$ denote the action of $g \in G$. By the chain rule, we have $d_{g \cdot \tilde{p}}(g \cdot F) = d_{\tilde{p}} F \circ d_{g \cdot \tilde{p}} \sigma_{g^{-1}}$, where $d_p \sigma_{g^{-1}}$ is invertible (as σ is an action). Hence $d_{g \cdot \tilde{p}}(g \cdot F) = 0$ if and only if $d_{\tilde{p}} F$; in other words p is a singular point of the hypersurface $F = 0$ if and only if $g \cdot p$ is a singular point of the hypersurface $g \cdot F = 0$ for any $g \in G$.

The resultant polynomial of a collection of polynomials is a function in the coefficients of these polynomials which vanishes if and only if these polynomials all have a common root; for the existence of the resultant and how to compute it, see [7] Chapter 13 1.A.

Definition 7.6. For a polynomial $F \in k[x_0, \dots, x_n]_d$, we define the *discriminant* $\Delta(F)$ of F to be the resultant of the polynomials $\partial F / \partial x_i$.

Then Δ is a homogeneous polynomial in $R(Y_{d,n})$ and is non-zero at F if and only if F defines a smooth hypersurface. It follows from Remark 7.5 that Δ is SL_{n+1}-invariant.

Example 7.7. If $d = 1$, then $Y_{1,n} \cong (\mathbb{P}^n)^\vee$ and as the only SL_{n+1}-invariant homogeneous polynomials are the constants:

$$k[x_0, \dots, x_n]^{\mathrm{SL}_{n+1}} = k,$$

there are no semistable points for the action of SL_{n+1} on $Y_{1,n}$. In particular, the discriminant Δ is constant on $Y_{1,n}$. Alternatively, as the action of SL_{n+1} on $\mathbb{P}^n$ is transitive, to show $Y_{1,n}^{ss} \cong (\mathbb{P}^n)^{ss} = \emptyset$, it suffices to show a single point $x = [1 : 0 : \cdots : 0] \in \mathbb{P}^n$ is unstable. For this, one can use the Hilbert-Mumford criterion: it is easy to check that if $\lambda(t) = \mathrm{diag}(t, t^{-1}, 1, \dots, 1)$, then $\mu(x, \lambda) < 0$.

For $d > 1$, the discriminant is a non-constant SL_{n+1}-invariant homogeneous polynomial on $Y_{d,n}$ and as it is non-zero for all smooth hypersurfaces we have:

Proposition 7.8. *For $d > 1$, every smooth degree d hypersurface in $\mathbb{P}^n$ is semistable for the action of SL_{n+1} on $Y_{d,n}$.*

To determine whether a semistable point is stable we can check whether its stabiliser subgroup is finite.

Example 7.9. If $d = 2$, then we are considering the space $Y_{2,n}$ of quadric hypersurfaces in $\mathbb{P}^n$. Given $F = \sum_{i,j} a_{ij} x_i x_j \in k[x_0, \dots, x_n]_2$, we can associate to F a symmetric $(n+1) \times (n+1)$ matrix $B = (b_{ij})$ where $b_{ij} = b_{ji} = a_{ij}$ and $b_{ii} = 2a_{ii}$. This procedure defines an isomorphism between $Y_{2,n}$ and the space $\mathbb{P}(\mathrm{Sym}_{(n+1)\times(n+1)}(k))$ where $\mathrm{Sym}_{(n+1)\times(n+1)}(k)$ denotes the space of symmetric $(n+1)\times(n+1)$ matrices. The discriminant Δ on $Y_{2,n}$ corresponds to the determinant on $\mathbb{P}(\mathrm{Sym}_{(n+1)\times(n+1)}(k))$; thus F is smooth if and only if its associated matrix is invertible. In fact if F corresponds to a matrix B of rank $r + 1$, then F is projectively equivalent to the quadratic form

$$x_0^2 + \cdots + x_r^2.$$

As all non-singular quadratic forms $F(x_0, \dots, x_n)$ are equivalent to $x_0^2 + \cdots + x_n^2$ (after a change of coordinates), we see that these points cannot be stable: the stabiliser of $x_0^2 + \cdots + x_n^2$ is equal to the special orthogonal group $\mathrm{SO}(n + 1)$ which is positive dimensional. Moreover, the discriminant generates the ring of invariants (for example, see [31] Example 4.2) and so the semistable locus is just the set of non-singular quadratic forms. In this case, the GIT quotient consists of a single point and this represents the fact that all non-singular quadratic forms are projectively equivalent to $x_0^2 + \cdots + x_n^2$.

The projective automorphism group of a hypersurface is the subgroup of the automorphism group PGL_{n+1} of $\mathbb{P}^n$ which leaves this hypersurface invariant. For $d > 2$, the projective automorphism group of any irreducible degree d hypersurface is finite; this is a classical but non-trivial result (see [20] Lemma 14.2). As PGL_{n+1} is a quotient of SL_{n+1} by a finite subgroup, this implies the stabiliser subgroup of a point in $Y_{d,n}$ corresponding to an irreducible hypersurface is finite dimensional. Since every smooth hypersurface is irreducible, the stabiliser group of a smooth hypersurface is finite. In fact, one can also check that for $d > 2$, the orbit of a smooth hypersurface is closed and so the following result holds.

Proposition 7.10 ([25] §4.3)**.** *For $d > 2$, every degree d smooth hypersurface is stable.*

7.3. **The Hilbert–Mumford criterion for hypersurfaces.** To determine the (semi)stable points for the action of SL_{n+1} on $Y_{d,n}$, we can use the Hilbert–Mumford criterion. Any 1-PS of SL_{n+1} is conjugate to a 1-PS of the form

$$\lambda(t) = \begin{pmatrix} t^{r_0} & & & \\ & t^{r_1} & & \\ & & \ddots & \\ & & & t^{r_n} \end{pmatrix}$$

where r_i are integers such that $\sum_{i=0}^{n} r_i = 0$ and $r_0 \geq r_1 \geq \cdots \geq r_n$. Then the action of λ is diagonal with respect to the basis of the affine cone over $Y_{d,n}$ given by the monomials

$$x_I = x_0^{i_0} x_1^{i_1} \dots x_n^{i_n},$$

for $I = (i_0, \dots, i_n)$ a tuple of non-negative integers which sum to d. Furthermore, the weight of each monomial x_I for the action of λ is $-\sum_{j=0}^{n} r_j i_j$, where the negative sign arises as we act by the inverse of $\lambda(t)$.

Let $F = \sum a_I x_I \in k[x_0, \dots, x_n]_d - \{0\}$, where $I = (i_0, \dots, i_n)$ is a tuple of non-negative integers which sum to d and x_I, and let $p_F \in Y_{d,n}$ be the corresponding class. Then

$$\begin{aligned} \mu(p_F, \lambda) &= -\min\{-\sum_{j=0}^{n} r_j i_j : I = (i_0, \dots, i_n) \text{ and } a_I \neq 0\} \\ &= \max\{\sum_{j=0}^{n} r_j m_j : I = (i_0, \dots, i_n) \text{ and } a_I \neq 0\}. \end{aligned}$$

For general (d, n), there is not always a clean description of the semistable locus. However for certain small values, we shall see that this has a nice description. In §7.4 below we discuss the case when $n = 1$; in this case, a degree d hypersurface corresponds to d unordered points (counted with multiplicity) on $\mathbb{P}^1$. Then in §7.5 we discuss the case when $(d, n) = (3, 2)$; that is, cubic curves in the projective plane $\mathbb{P}^2$. Both of these classical examples were studied by Hilbert and can also be found in [25] and [31].

7.4. **Binary forms of degree** d**.** A binary form of degree d is a degree d homogeneous polynomial in two variables x, y. The set of zeros of a binary form F determine d points (counted with multiplicity) in $\mathbb{P}^1$. In this section we study the action of SL_2 on

$$Y_{d,1} = \mathbb{P}(k[x, y]_d) \cong \mathbb{P}^d.$$

Our aim is to describe the (semi)stable locus and the GIT quotient.

One method to determine the semistable and stable locus is to compute the ring of invariants $R(Y_{d,1})^{\mathrm{SL}_2}$ for this action. For $d \leq 6$, the ring of invariants is known due to classical computations in invariant theory going back to Hilbert and later work of Schur. For general values of d, the ring of invariants is still unknown today, which shows how difficult it can be in general to determine the ring of invariants. For $d = 8$, a list of generators of the ring of invariants is given by work of von Gall (1880) and Shioda (1967) [38]. For $d = 9, 10$, generators for the ring of invariants were calculated by Brouwer and Popoviciu in (2010). Instead, we will use the Hilbert–Mumford criterion to obtain a complete description of the semistable locus, which bypasses the need to calculate the ring of invariants.

Remark 7.11. If $d = 1$, then this corresponds to the action of SL_2 on $\mathbb{P}^1$, for which there are no semistable points as the only invariant functions are constant (see also Example 7.7).

Henceforth, we assume $d \geq 2$ and use the Hilbert–Mumford criterion for semistability. We fix the maximal torus $T \subset \mathrm{SL}_2$ given by the diagonal matrices

$$T = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t \in \mathbb{C}^* \right\}.$$

Any primitive 1-PS of G is conjugate to the 1-PS of T given by

$$\lambda(t) = \left(\begin{array}{cc} t & 0 \\ 0 & t^{-1} \end{array} \right).$$

If $F(x,y) = \sum_i a_i x^{d-i} y^i \in k[x,y]_d - \{0\}$ lies over $p_F \in Y_{d,1}$, then

$$\lambda(t) \cdot F(x,y) = \sum t^{2i-d} a_i x^{d-i} y^i$$

and

$$\mu(p_F, \lambda) = -\min\{2i - d : a_i \neq 0\} = \max\{d - 2i : a_i \neq 0\} = d - 2i_0,$$

where i_0 is the smallest integer for which $a_i \neq 0$. Hence

(1) $\mu(p_F, \lambda) \geq 0$ if and only if $i_0 \leq d/2$ if and only if $[1:0]$ occurs with multiplicity at most $d/2$.

(2) $\mu(p_F, \lambda) > 0$ if and only if $i_0 < d/2$ if and only if $[1:0]$ occurs as a root with multiplicity strictly less than $n/2$.

By the Hilbert–Mumford criterion, $p_F \in Y_{d,1}$ is semistable if and only if $\mu(p_F, \lambda') \geq 0$ for all 1-PSs λ'. For a general 1-PS λ' we can write $\lambda = g^{-1}\lambda' g$, then

$$\mu(p_F, \lambda') = \mu(g \cdot p_F, \lambda).$$

If F has roots $p_1, \dots, p_d \in \mathbb{P}^1$, then $g \cdot F$ has roots $g \cdot p_1, \dots, g \cdot p_d$. As SL_2 acts transitively on $\mathbb{P}^1$, we deduce the following result.

Proposition 7.12. *Let $F \in k[x,y]_d$ lie over $p_F \in Y_{d,1}$; then:*

i) p_F is semistable if and only if all roots of F in $\mathbb{P}^1$ have multiplicity less than or equal to $d/2$.

ii) p_F is stable if and only if all roots of F in $\mathbb{P}^1$ have multiplicity strictly less than $d/2$.

In particular, if d is odd then $Y_{d,1}^{ss} = Y_{d,1}^{s}$ and the GIT quotient is a projective variety which is a geometric quotient of the space of stable degree d hypersurfaces in $\mathbb{P}^1$.

Example 7.13. If $d = 2$, then the semistable locus corresponds to binary forms F with two distinct roots and the stable locus is empty. Given any two distinct points (p_1, p_2) on $\mathbb{P}^1$, there is a mobius transformation taking these points to any other two distinct points (q_1, q_2). However this mobius transformation is far from unique; in fact given points p_3 distinct from (p_1, p_2) and q_3 distinct from (q_1, q_2), there is a unique mobius transformation taking p_i to q_i. Hence all semistable points have positive dimensional stabilisers and so can never be stable. As the action on the semistable locus is transitive, the GIT quotient is just the point $\operatorname{Spec} k$.

Example 7.14. If $d = 3$, then the stable locus (which coincides with the stable locus) consists of forms with 3 distinct roots. We recall that given any 3 distinct points (p_1, p_2, p_3) on $\mathbb{P}^1$, there is a unique mobius transformation taking these points to any other 3 distinct points. Hence the GIT quotient is the projective variety $\mathbb{P}^0 = \operatorname{Spec} k$. In fact, the SL_2-invariants have a single generator: the discriminant

$$\Delta\left(\sum a_i x^{d-i} y^i\right) := 27a_0^2a_3^2 - a_1^2a_2^2 - 18a_0a_1a_2a_3 + 4a_0a_2^3 + 4a_1^3a_3$$

which is zero if and only if there is a repeated root.

Example 7.15. If $d = 4$, then we are considering binary quartics. In this case the semistable locus is the set of degree 4 binary forms F with at most 2 repeated roots and the stable locus is the set of points in which all 4 roots are distinct. Given 4 distinct ordered points $(p_1, \dots, p_4)$ there is a unique mobius transformation which takes this ordered set of points to $(0, 1, \infty, \lambda)$ where $\lambda \in \mathbb{A}^1 - \{0, 1\}$ is the cross-ratio of these points. However, the points in our case do not have a natural ordering and so there are 6 possible values of the cross-ratio depending on how we choose to order our points:

$$\lambda, 1 - \lambda, \frac{1}{\lambda}, \frac{\lambda - 1}{\lambda}, \frac{\lambda}{\lambda - 1}, \frac{1}{1 - \lambda}.$$

The morphism $f : Y_{4,1}^{s} \to \mathbb{A}^1$ given by

$$\left(\frac{(2\lambda-1)(\lambda-2)(\lambda+1)}{\lambda(\lambda-1)}\right)^3$$

is symmetric in the six possible values of the cross-ratio, and so is SL_2-invariant. It is easy to check that f is surjective and in fact an orbit space: for each value of f in $\mathbb{A}^1 - \{0, -27\}$, there are six distinct possible choices for λ as above and so this corresponds to a unique stable orbit. For the values 0 (resp. -27), there are 3 (resp. 2) possible values for λ and these correspond to a unique stable orbit.

The strictly semistable points have either one or two double roots and so correspond to two orbits. The orbit consisting of one double root is not closed: its closure contains the orbit of points with two double roots (imagine choosing a family of mobius transformations h_t that sends (p, p, q, r) to $(1, 1, 0, t)$, then as $t \to 0$, we see that the point $(1, 1, 0, 0)$ lies in this orbit closure). This suggests that the GIT quotient $Y_{4,1}//\mathrm{SL}_2$ is $\mathbb{P}^1$, the single point compactification of $\mathbb{A}^1$.

In fact, this is true: there are two independent generators for the SL_2-invariants of binary quartics (called the I and J invariants - for example, see [31] Example 4.5 or [4], where they are called S and T) and the good quotient is $\varphi : Y_{4,1}^{ss} \to \mathbb{P}^1$.

7.5. **Plane cubics.** In this section, we study moduli of degree 3 hypersurfaces in $\mathbb{P}^2$; that is, plane cubic curves. We write a degree 3 homogeneous polynomial F in variables x, y, z as

$$F(x, y, z) = \sum_{i=0}^{3} \sum_{j=0}^{3-i} a_{ij} x^{3-i-j} y^i z^j.$$

We want to describe all plane cubic curves up to projective equivalence; that is, describe the quotient for the action of SL_3 on $Y_{3,2}$. For simplicity, we assume that the characteristic of k is not equal to 2 or 3.

An important classical result about the intersection of plane curves is Bézout's Theorem, which says for two projective plane curves C_1 and C_2 in $\mathbb{P}^2$ with no common components, the number of points of intersection of C_1 and C_2 counted with multiplicities is equal to the product of the degrees of these curves. The fact that k is algebraically closed is crucial for this result. For a basic introduction to algebraic curves and an elementary proof of Bézout's Theorem, see [19]. In this section, we will use without proof the following easy applications of Bézout's Theorem.

Proposition 7.16. *(1) Any non-singular projective plane curve $C \subset \mathbb{P}^2$ is irreducible.*
(2) Any irreducible projective plane curve $C \subset \mathbb{P}^2$ has at most finitely many singular points.

Furthermore, Bézout's Theorem can be used to obtain a classification of plane curves of low degree.

Lemma 7.17. *Any irreducible plane conic $C \subset \mathbb{P}^2$ is projectively equivalent to the conic defined by $x^2 + yz = 0$, which is isomorphic to $\mathbb{P}^1$.*

Proof. By the above proposition, C has only finitely many singular points, and so we can choose coordinates so that $[0 : 1 : 0] \in C$ is non-singular and the tangent line to the curve at this point is the line $z = 0$. Then we must have that C is the zero locus of a polynomial

$$P(x, y, z) = ayz + bx^2 + cxz + dz^2$$

and, as P is irreducible, we must have $b \neq 0$. Since $\partial P(p)/\partial z \neq 0$, we have $a \neq 0$. Then the change of coordinates $x' := \sqrt{b}x$, $y' := ay + cx + dz$, $z' := z$ transforms the above conic into the desired form.

Finally, for $C : (x^2 + yz = 0)$, we have an isomorphism $f : C \to \mathbb{P}^1$ given by

$$f([x : y : z]) = \begin{cases} [x : y] & \text{if } y \neq 0 \\ [-z : x] & \text{if } z \neq 0. \end{cases}$$

The inverse of f is $f^{-1} : \mathbb{P}^1 \to C$ given by $[u : v] \mapsto [uv : v^2 : -u^2]$. □

This enables us to easily classify all reducible plane cubics up to projective equivalence, as any reducible plane conic is either the union of an irreducible conic with a line or a union of three lines. In fact, one can also prove that two reducible plane cubics are projectively equivalent if and only if they are isomorphic. If the reducible plane cubic curve is a union of a line and a conic, then the line can either meet the conic at two distinct points or a single point (so that the line is tangent to the conic). By the above lemma, the irreducible conic is projectively equivalent to $y^2 + xz = 0$. As the projective automorphism group of this conic acts transitively on the set of tangents lines to this conic and the set of lines meeting the conic at two distinct points, any reducible cubic which is a union of a conic and a line is projectively equivalent to either

- $(xz + y^2)y = 0$, where the line meets the conic in two distinct points, or
- $(xz + y^2)z = 0$, where the line meets the conic tangentially.

If the reducible cubic curve is a union of three lines, there are four possibilities: one line occurring with multiplicity three; a union of a double line with another distinct line; a union of three lines meeting in a single intersection point; a union of three lines which meet in three intersection points. Since the group of projective transformations acts transitively on the space of 3 lines, we see that a reducible cubic curve which is a union of three lines is projectively equivalent to either

- $y^3 = 0$ (a triple line), or
- $y^2(y + z) = 0$ (a union of a double line with a distinct line), or
- $yz(y + z) = 0$ (three concurrent lines), or
- $xyz = 0$ (three non-concurrent lines).

The above reducible plane cubics contain a singular point at $[1 : 0 : 0]$. In fact, we can define a notion of multiplicities for singularities to distinguish between different types of singularities. For a plane cubic, all points have multiplicity at most 3.

Definition 7.18. A singular point at p of cubic curved defined by $F(x, y, z) = 0$ is a *triple point* if all second order partial derivatives of F vanish at p; otherwise we say p is a *double point.* A non-singular point is called a single point or point of multiplicity 1.

Example 7.19. The cubics defined by $y^3 = 0$ (a triple line), $y^2(y+z) = 0$ (a union of a double line with a distinct line), $yz(y + z) = 0$ (three concurrent lines) all contain a triple point at $[1 : 0 : 0]$. The cubic defined by $xyz = 0$ (three non-concurrent lines) has three double points: $[1 : 0 : 0]$, $[0 : 1 : 0]$ and $[0 : 0 : 1]$. The cubic defined by $(xz+y^2)y = 0$ (a union of an irreducible conic with a non-tangential line) has two double points: $[1 : 0 : 0]$ and $[0 : 0 : 1]$. The cubic defined by $(xz + y^2)z = 0$ (a union of an irreducible conic with a tangential line) has a single double point at $[1 : 0 : 0]$ (with a single tangent direction).

Since tangent lines will play an important role in the classification of semistable plane cubics, we recall their definition. Every non-singular point has a single tangent line, whereas singular points have multiple tangents.

Definition 7.20. Let $p = [p_0 : p_1 : p_2]$ be a point of a plane algebraic curve $C : (F(x, y, z) = 0)$.

(1) If p is a non-singular point, then the *tangent line* to C at p is given by
$$\frac{\partial F(\tilde{p})}{\partial x}x + \frac{\partial F(\tilde{p})}{\partial y}y + \frac{\partial F(\tilde{p})}{\partial z}z = 0$$
where $\tilde{p} = (p_0, p_1, p_2)$.

(2) If $p = [p_0 : p_1 : p_2]$ is a double point of C; then the *tangent lines* to C at p are given by the degree 2 homogeneous polynomial
$$0 = (x - p_0, y - p_1, z - p_2) \begin{pmatrix} \frac{\partial^2 F(\tilde{p})}{\partial x^2} & \frac{\partial^2 F(\tilde{p})}{\partial y \partial x} & \frac{\partial^2 F(\tilde{p})}{\partial z \partial x} \\ \frac{\partial^2 F(\tilde{p})}{\partial x \partial y} & \frac{\partial^2 F(\tilde{p})}{\partial y^2} & \frac{\partial^2 F(\tilde{p})}{\partial z \partial y} \\ \frac{\partial^2 F(\tilde{p})}{\partial x \partial z} & \frac{\partial^2 F(\tilde{p})}{\partial y \partial z} & \frac{\partial^2 F(\tilde{p})}{\partial z^2} \end{pmatrix} \begin{pmatrix} x - p_0 \\ y - p_1 \\ z - p_2 \end{pmatrix}.$$

The 3×3 matrix appearing in this expression is called the Hessian of F at $\tilde{p}$ and has rank $0 < r < 3$ as $\tilde{p}$ is a double point. As the Hessian does not have full rank, the above equation for the tangent lines factorises into a product of two linear polynomials.

For a plane cubic C, there are two types of singular double points:

(1) A *node (or ordinary double point)* is a double point with two distinct tangent lines (which is a self intersection of the curve, so that both branches of the curve have distinct tangent lines at the intersection point).
(2) A *cusp* is a double point with a single tangent line of multiplicity two (which is not a self intersection point of the curve).

Example 7.21. Let $F_1(x, y, z) = xz^2 + y^3 + y^2x$ and $F_2(x, y, z) = xz^2 + y^3$. The corresponding cubics are irreducible and have a singular point at $p = [1 : 0 : 0]$. The point p is a double point which is a node of the first cubic corresponding to $(F_1 = 0)$ as the tangent lines are given by

$$0 = y^2 + z^2 = (y - \sqrt{-1}z)(y + \sqrt{-1}z).$$

The point p is a double point of the second cubic corresponding to $(F_2 = 0)$, which is a cusp as the tangent lines are given by

$$0 = z^2.$$

Exercise 7.22. Fix a non-zero homogeneous polynomial

$$F(x, y, z) = \sum_{i=0}^{3} \sum_{j=0}^{3-i} a_{ij} x^{3-i-j} y^i z^j$$

of degree 3 and let C be the plane cubic curve defined by $F = 0$. For $p = [1 : 0 : 0] \in \mathbb{P}^2$, show the following statements hold.

i) $p \in C$ if and only if $a_{00} = 0$.
ii) p is a singular point of F if and only if $a_{00} = a_{10} = a_{01} = 0$.
iii) p is a triple point of F if and only if $a_{00} = a_{10} = a_{01} = a_{11} = a_{20} = a_{02} = 0$.
iv) If $p = [1 : 0 : 0]$ is a double point of F, then its tangent lines are defined by

$$a_{20}y^2 + a_{11}yz + a_{02}z^2 = 0.$$

For non-singular plane cubics, we have a classification following Bézout's Theorem in terms of Legendre cubics or Weierstrass cubics. It is important for the following classification, that we remember that the characteristic of k is assumed to be not equal to 2 or 3.

Proposition 7.23. *Let $C \subset \mathbb{P}^2$ be an irreducible plane cubic curve.*

(1) If C is non-singular it is projectively equivalent to a Legendre cubic *of the form*

$$y^2 z = x(x - z)(x - \lambda z)$$

for some $\lambda \in k - \{0, 1\}$.

(2) C is projectively equivalent to a Weierstrass cubic *of the form*

$$y^2 z = x^3 + axz^2 + bz^3$$

for scalars a and b.

Proof. i) Let C be a non-singular plane cubic defined by $P(x, y, z) = 0$. The Hessian $\mathcal{H}_P$ of P is the degree 3 polynomial which is the determinant of the 3×3 matrix of second order derivatives of P. By Bézout's theorem, $\mathcal{H}_P$ and P have at least one common solution, which gives a point $p \in C$ known as an inflection point. By a change of coordinates, we can assume $p = [0 : 1 : 0]$ and the tangent line T_pC is defined by $z = 0$. Hence P, $\partial P/\partial x$, $\partial P/\partial y$ and $\mathcal{H}_P$ all vanish at p, but $\partial P/\partial z$ is non-zero at p. It follows from the Euler relations that

$$\mathcal{H}_P(p) = -4 \left(\frac{\partial P}{\partial z}(p) \right)^2 \frac{\partial^2 P}{\partial x^2}(p)$$

and so also $\partial^2 P/\partial x^2(p) = 0$. Hence, P does not involve the monomials y^3, xy^2 and x^2y. Therefore,

$$P(x,y,z) = Q(x,z) + yz(\alpha x + \beta y + \gamma z)$$

where Q is homogeneous of degree 3 and $\beta \neq 0$. After a change of coordinates in the y variable, we may assume that

$$P(x,y,z) = R(x,z) + y^2 z$$

for R a degree 3 homogeneous polynomial in x and z. Since C is non-singular, z does not divide R; that is, the coefficient of x^3 in R is non-zero. We can factorise this homogeneous polynomial in two variables as:

$$R(x,z) = u(x - az)(x - bz)(x - cz)$$

where $u \neq 0$ and a, b, c are distinct as C is non-singular. Let $\lambda = (b - c)/(b - a)$; then one further change of coordinates reduces the equation to a Legendre cubic. As the characteristic of k is not equal to 3, any Legendre cubic can be transformed into a Weierstrass cubic by a change of coordinates.

ii) It suffices to consider irreducible singular plane conics. By a change of coordinates, we can assume that $[0 : 0 : 1]$ is a singular point and the equation of our cubic has the form

$$zQ(x,y) + R(x,y) = 0$$

where Q is homogeneous of degree 2 and R is homogeneous of degree 3. After a linear change of variables in x, y, the degree 2 polynomial Q in two variables is either $Q(x,y) = y^2$ or $Q(x,y) = xy$. T he first case corresponds to a cuspidal cubic and the second case corresponds to a nodal cubic; we merely sketch the argument below and refer to [4] §10.3 for further details, where a classification for fields of characteristic 2 and 3 is also given.

Consider the first case: $Q(x,y) = y^2$. Then our conic has the form

$$y^2 z + ax^3 + bx^2 y + cxy^2 + dy^3 = 0$$

where $a \neq 0$, as the conic is irreducible. By a linear change in the z-coordinate, we can assume $c = d = 0$ and by scaling x, we may assume $a = 1$. A final change of coordinates which fixes the singular point $[0 : 0 : 1]$ and moves the unique non-singular inflection point to $[0 : 1 : 0]$, with tangent line $z = 0$, reduces the equation to $zy^2 = x^3$, which is the Weierstrass cusp.

Consider the second case: $Q(x,y) = xy$. Then our conic has the form

$$xyz + ax^3 + bx^2 y + cxy^2 + dy^3 = 0.$$

By the change of coordinates in z, we can assume $b = c = 0$. Since C is irreducible, both a and d must be non-zero and so we can scale them to both be 1. After one more change of coordinates, we obtain a nodal Weierstrass form: $y^2 z = x^2(x + y)$. □

Remark 7.24. The constant λ occurring in the Legendre cubic is not unique: it depends on which two roots of the cubic equation are sent to 0 and 1. Hence, there are 6 possible choices of λ for each non-singular cubic: λ, $1 - \lambda$, $1/\lambda$, $1/(1-\lambda)$, $\lambda/(\lambda - 1)$ and $(\lambda - 1)/\lambda$. Similarly, in the Weierstrass cubic, the constants a and b are not unique: as a change of coordinates $y' = \eta^3 y$ and $x' = \eta^2 x$ gives a new Weierstrass cubic with $a' = \eta^4 a$ and $b' = \eta^6 b$.

Weierstrass cubics arise in the study of elliptic curves, which are classified up to isomorphism using the j-invariant. *Elliptic curves* are the non-singular Weierstrass cubics (those for which $4a^3 + 27b^2 \neq 0$). Two elliptic curves are isomorphic if and only if they have the same *j-invariant*, where

$$j = 1738 \frac{4a^3}{4a^3 + 27b^2}.$$

In terms of the Legendre cubic, we can write the j-invariant in terms of λ as

$$j = \frac{256(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}.$$

For further details on elliptic curves and the j-invariant, see [14] IV §4.

This classification of plane cubics does not tell us anything about which ones are (semi)stable. We will use the Hilbert–Mumford criterion to give a complete description of the (semi)stable locus. Any 1-PS of SL_3 is conjugate to a 1-PS of the form

$$\lambda(t) = \begin{pmatrix} t^{r_0} & & \\ & t^{r_1} & \\ & & t^{r_2} \end{pmatrix}$$

where r_i are integers such that $\sum_{i=0}^{2} r_i = 0$ and $r_0 \geq r_1 \geq r_2$. It is easy to calculate that

$$\mu(F,\lambda) = -\min\{-(3-i-j)r_0 - ir_1 - jr_2 : a_{ij} \neq 0\} = \max\{(3-i-j)r_0 + ir_1 + jr_2 : a_{ij} \neq 0\}.$$

Lemma 7.25. *A plane cubic curve C is semistable if and only if it has no triple point and no double point with a unique tangent. A plane cubic curve C is stable if and only if it is non-singular.*

Proof. Let C be defined by the vanishing of the non-zero degree 3 homogeneous polynomial

$$F(x,y,z) = \sum_{i=0}^{3}\sum_{j=0}^{3-i} a_{ij} x^{3-i-j} y^i z^j.$$

If F (or really the class of F in $Y_{3,n}$) is not semistable, then by the Hilbert–Mumford criterion there is a 1-PS λ of SL_3 such that $\mu(F,\lambda) < 0$. For some $g \in \mathrm{SL}_3$, the 1-PS $\lambda' := g\lambda g^{-1}$ is of the form $\lambda'(t) = \mathrm{diag}(t^{r_0}, t^{r_1}, t^{r_2})$ for integers $r_0 \geq r_1 \geq r_2$ which satisfy $\sum r_i = 0$. Then

$$\mu(g\cdot F, \lambda') = \mu(F,\lambda) < 0.$$

Let us write $F' := g \cdot F = \sum_{i,j} a'_{ij} x^{3-i-j} y^i z^j$; then

$$\lambda'(t) \cdot F'(x,y,z) = \sum_{i,j} t^{-r_0(3-i-j)-r_1 i - r_2 j} a'_{ij} x^{3-i-j} y^i z^j.$$

Since $\mu(F',\lambda') < 0$, we conclude that

$$-\min\{-r_0(3-i-j) - r_1 i - r_2 j : a'_{ij} \neq 0\} = \max\{r_0(3-i-j) + r_1 i + r_2 j : a'_{ij} \neq 0\} < 0;$$

that is, all weights of F' must be positive. The inequalities $r_0 \geq r_1 \geq r_2$ imply that the monomials with non-positive weights are: x^3, x^2y (which have strictly negative weights), and xy^2, x^2z, and xyz. Hence, $\mu(F,\lambda) < 0$ implies $a'_{00} = a'_{10} = a'_{20} = a'_{11} = a'_{01} = 0$ and so $p = [1:0:0]$ is a singular point of F' by Exercise 7.22. Then $g^{-1}\cdot p$ is a singular point of $F = g^{-1} \cdot F'$. Moreover, if $a'_{02} = 0$ also then $[1:0:0]$ is a triple point of F' and if $a_{02} \neq 0$ then $[1:0:0]$ is a double point with a single tangent.

Suppose that $F = \sum a_{ij} x_0^{3-i-j} x_1^i x_2^j$ has a double point with a unique tangent or triple point, then we can assume without loss of generality (by using the action of SL_3) that this point is $p = [1:0:0]$ and that $a_{00} = a_{10} = a_{01} = a_{20} = a_{11} = 0$. Then if $\lambda(t) = \mathrm{diag}(t^3, t^{-1}, t^{-2})$, we see $\mu(F,\lambda) < 0$. Therefore F is semistable if and only if it has no triple point or double point with a unique tangent.

For the second statement, if p is a singular point of C defined by $F = 0$, then using the SL_3-action, we can assume $p = [1:0:0]$ and so $a_{00} = a_{10} = a_{01} = 0$. For $\lambda(t) = \mathrm{diag}(t^2, t^{-1}, t^{-1})$, we see $\mu(F,\lambda) \leq 0$ by direct calculation; that is, F is not stable.

It remains to show that if F is not stable then F is not smooth. Without loss of generality, using the Hilbert–Mumford criterion and the action of SL_3 we can assume that $\mu(F,\lambda) \leq 0$ for $\lambda(t) = \mathrm{diag}(t^{r_0}, t^{r_1}, t^{r_2})$ where $r_0 \geq r_1 \geq r_2$ and $\sum r_i = 0$. In this case, we must have $a_{00} = a_{10} = 0$, as x^3 and x^2y have strictly negative weights. If also $a_{01} = 0$, then $p = [1:0:0]$ is a singular point as required. If $a_{01} \neq 0$, then

$$0 \geq \mu(F,\lambda) \geq (2r_0 + r_2). \tag{1}$$

The inequalities between the r_i imply that we must have equality in (1) and so $r_1 = r_0$ and $r_2 = -2r_0$. Then

$$\mu(F,\lambda) = \max\{(3-3j)r_0 : a_{ij} = 0\} \leq 0$$

and $r_0 > 0$; thus $a_{20} = a_{30} = 0$. In this case, F is reducible, as z divides F, and any reducible plane cubic has a singular point. □

There are three strictly semistable orbits:

(1) nodal irreducible cubics,
(2) cubics which are a union of a conic and a non-tangential line, and
(3) cubics which are the union of three non-concurrent lines.

The lowest dimensional strictly semistable orbit, which is the orbit of three non-concurrent lines (this has a two dimensional stabiliser group and so the orbit has dimension $6 = \dim \mathrm{SL}_3 - 2$), is closed in the semistable locus. One can show that the closure of the orbit of nodal irreducible cubics (which is 8 dimensional) contains both other strictly semistable orbits. In particular, the compactification of the geometric quotient $Y_{3,2}^s \to Y_{3,2}^s/\mathrm{SL}_3$ of smooth cubics is given by adding a single point corresponding to these three strictly semistable orbits.

The geometric quotient of the stable locus classifies isomorphism classes of non-singular plane cubics, and so via the theory of elliptic curves and the j-invariant, is isomorphic to $\mathbb{A}^1$ (see [14] IV Theorem 4.1). Hence its compactification, which is a good quotient of $Y_{3,2}^{ss}$, is $\mathbb{P}^1$.

The ring of invariants $R(Y_{3,2})^{\mathrm{SL}_2}$ is known to be freely generated by two invariants S and T by a classical result of Aaronhold (1850). In terms of the Weierstrass normal form, we have

$$S = \frac{a}{27} \quad T = \frac{4b}{27},$$

which both vanish on the cuspidal Weierstrass cubic (where $a = b = 0$), and $S \neq 0$ for the nodal Weierstrass cubic, which is strictly semi-stable.

Finally, we list the unstable orbits: cuspidal cubics, cubics which are the union of a conic and a tangent line, cubics which are the union of three lines with a common intersection, cubics which are the union of a double line with a distinct line and cubics which are given by a triple line.

8. Moduli of vector bundles on a curve

In this section, we describe the construction of the moduli space of (semi)stable vector bundles on a smooth projective curve X (always assumed to be connected) using geometric invariant theory.

The outline of the construction follows the general method described in §2.6. First of all, we fix the available discrete invariants, namely the rank n and degree d. This gives a moduli problem $\mathcal{M}(n,d)$, which is unbounded by Example 2.22. We can overcome this unboundedness problem by restricting to moduli of semistable vector bundles and get a new moduli problem $\mathcal{M}^{ss}(n,d)$. This moduli problem has a family with the local universal property over a scheme R. Moreover, we show there is a reductive group G acting on R such that two points lie in the same orbits if and only if they correspond to isomorphic bundles. Then the moduli space is constructed as a GIT quotient of the G-action on R. In fact, the notion of semistability for vector bundles was introduced by David Mumford following his study of semistability in geometric invariant theory, and we will see both concepts are closely related.

The construction of the moduli space of stable vector bundles on a curve was given by Seshadri [37], and later Newstead in [30, 31]. In these notes, we will essentially follow the construction due to Simpson [39] which generalises the curve case to a higher dimensional projective scheme. An in-depth treatment of the general construction following Simpson can be found in the book of Huybrechts and Lehn [16]. However, we will exploit some features of the curve case to simplify the situation; for example, we directly show that the family of semistable vector bundles with fixed invariants over a smooth projective curve is bounded, without using the Le Potier-Simpson estimates which are used to show boundedness in higher dimensions.

Convention: Throughout this section, X denotes a connected smooth projective curve. By 'sheaf' on a scheme Y, we always mean a coherent sheaf of $\mathcal{O}_Y$-modules.

8.1. **An overview of sheaf cohomology.** We briefly recall the definition of the cohomology groups of a sheaf $\mathcal{F}$ over X. By definition, the sheaf cohomology groups $H^i(X,\mathcal{F})$ are obtained by taking the right derived functors of the left exact global sections functor $\Gamma(X,-)$. Therefore,

$$H^0(X,\mathcal{F}) \cong \Gamma(X,\mathcal{F}).$$

As X is projective, $H^i(X,\mathcal{F})$ are finite dimensional k-vector spaces and, as X has dimension 1, we have $H^i(X,\mathcal{F}) = 0$ for $i > 1$. The cohomology groups can be calculated using Cech cohomology. The first Cech cohomology group is the group of 1-cochains modulo the group of 1-coboundaries. More precisely, given a cover $\mathcal{U} = \{U_i\}$ of X, we let $U_{ij} = U_i \cap U_j$ and $U_{ijk} = U_i \cap U_j \cap U_k$ denote the double and triple intersections; then we define

$$H^1(\mathcal{U},\mathcal{F}) := Z^1(\mathcal{U},\mathcal{F})/B^1(\mathcal{U},\mathcal{F})$$

where

$$Z^1(\mathcal{U},\mathcal{F}) := \mathrm{Ker}\ \delta_1 = \{(f_{ij}) \in \bigoplus_{i,j} H^0(\mathcal{U}_{ij},\mathcal{F}) : \forall i,j,k, f_{ij} - f_{jk} + f_{ki} = 0 \in \mathcal{F}(\mathcal{U}_{ijk})\}$$

$$B^1(\mathcal{U},\mathcal{F}) := \mathrm{Image}\ \delta_0 = \{(h_i - h_j) \text{ for } (h_i) \in \bigoplus_i \mathcal{F}(U_i)\}$$

are the group of 1-cochains and 1-coboundaries respectively. If $\mathcal{V}$ is a refinement of $\mathcal{U}$, then there is an induced homomorphism $H^1(\mathcal{U},\mathcal{F}) \to H^1(\mathcal{V},\mathcal{F})$ and the first cohomology group $H^1(X,\mathcal{F})$ is the direct limit of the groups $H^1(\mathcal{U},\mathcal{F})$ over all covers $\mathcal{U}$ of X. In fact, these definitions of Cech cohomology groups make sense for any scheme X and any coherent sheaf $\mathcal{F}$; however, higher dimensional X, will in general have non-zero higher degree cohomology groups.

The above definition does not seem useful for computational purposes, but it is because of the following vanishing theorem of Serre.

Theorem 8.1 ([14] III Theorem 3.7)**.** *Let Y be an affine scheme and $\mathcal{F}$ be a coherent sheaf on Y; then for all $i > 0$, we have*

$$H^i(Y,\mathcal{F}) = 0.$$

Consequently, we can calculate cohomology of coherent sheaves on a separated scheme using an affine open cover.

Theorem 8.2 ([14] III Theorem 4.5)**.** *Let Y be a separated scheme and $\mathcal{U}$ be an open affine cover of Y. Then for any coherent sheaf $\mathcal{F}$ on Y and any $i \geq 0$, the natural homomorphism*

$$H^i(\mathcal{U},\mathcal{F}) \to H^i(Y,\mathcal{F})$$

is an isomorphism.

The assumption that Y is separated is used to ensure that the intersection of two open affine subsets is also affine (see [14] II Exercise 4.3). Hence, we can apply the above Serre vanishing theorem to all multi-intersections of the open affine subsets in the cover $\mathcal{U}$.

Exercise 8.3. Using the above theorem, calculate the sheaf cohomology groups

$$H^i(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(n))$$

by taking the standard affine cover of $\mathbb{P}^1$ consisting of two open sets isomorphic to $\mathbb{A}^1$.

One of the main reasons for introducing sheaf cohomology is that short exact sequences of coherent sheaves give long exact sequences in cohomology. The category of coherent sheaves on X is an abelian category, where a sequence of sheaves is exact if it is exact at every stalk. Furthermore, a short exact sequence of sheaves

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

induces a long exact sequence in sheaf cohomology

$$0 \to H^0(X,\mathcal{E}) \to H^0(X,\mathcal{F}) \to H^0(X,\mathcal{G}) \to H^1(X,\mathcal{E}) \to H^1(X,\mathcal{F}) \to H^1(X,\mathcal{G}) \to 0,$$

which terminates at this point as $\dim X = 1$.

Definition 8.4. For a coherent sheaf $\mathcal{F}$ on X, we let $h^i(X, \mathcal{F}) = \dim H^i(X, \mathcal{F})$ as a k-vector space. Then we define the *Euler characteristic* of $\mathcal{F}$ by

$$\chi(\mathcal{F}) = h^0(X, \mathcal{F}) - h^1(X, \mathcal{F}).$$

In particular, the Euler characteristic is additive on short exact sequences:

$$\chi(\mathcal{F}) = \chi(\mathcal{E}) + \chi(\mathcal{G}).$$

8.2. **Line bundles and divisors on curves.**

Example 8.5.

(1) For $x \in X$, we let $\mathcal{O}_X(-x)$ denote the sheaf of functions vanishing at x; that is, for $U \subset X$, we have

$$\mathcal{O}_X(-x)(U) = \{f \in \mathcal{O}_X(U) : f(x) = 0\}.$$

By construction, this is a subsheaf of $\mathcal{O}_X$ and, in fact, $\mathcal{O}_X(-x)$ is an invertible sheaf on X.

(2) For $x \in X$, we let k_x denote the skyscraper sheaf of x whose sections over $U \subset X$ are given by

$$k_x(U) := \begin{cases} k & \text{if } x \in U \\ 0 & \text{else.} \end{cases}$$

The skyscraper sheaf is not a locally free sheaf; it is a torsion sheaf which is supported on the point x. Since $H^0(X, k_x) = k_x(X) = k$ and $H^1(X, k_x) = 0$, we have $\chi(k_x) = 1$.

There is a short exact sequence of sheaves

$$0 \to \mathcal{O}_X(-x) \to \mathcal{O}_X \to k_x \to 0 \tag{2}$$

where for $U \subset X$, the homomorphism $\mathcal{O}_X(U) \to k_x(U)$ is given by evaluating a function $f \in \mathcal{O}_X(U)$ at x if $x \in U$. We can tensor this exact sequence by an invertible sheaf $\mathcal{L}$ to obtain

$$0 \to \mathcal{L}(-x) \to \mathcal{L} \to k_x \to 0$$

where $\mathcal{L}(-x)$ is also an invertible sheaf, whose sections over $U \subset X$ are the sections of $\mathcal{L}$ over U which vanish at x. Hence, we have the following formula

$$\chi(\mathcal{L}) = \chi(\mathcal{L}(-x)) + 1. \tag{3}$$

Definition 8.6. Let X be a smooth projective curve.

(i) A *Weil divisor* on X is a finite formal sum of points $D = \sum_{x \in X} m_x x$, for $m_x \in \mathbb{Z}$.

(ii) The *degree* of D is $\deg D = \sum m_x$.

(iii) We say D is *effective*, denoted $D \geq 0$, if $m_x \geq 0$ for all x.

(iv) For a rational function $f \in k(X)$, we define the associated *principal divisor*

$$\mathrm{div}(f) = \sum_{x \in X(k)} \mathrm{ord}_x(f) x,$$

where $\mathrm{ord}_x(f)$ is the order of vanishing of f at x (as $\mathcal{O}_{X,x}$ is a discrete valuation ring, we have a valuation $\mathrm{ord}_x : k(X)^* \to \mathbb{Z}$).

(v) We say two divisors are *linearly equivalent* if their difference is a principal divisor.

(vi) For a Weil divisor D, we define an invertible sheaf $\mathcal{O}_X(D)$ by

$$\mathcal{O}_X(D)(U) := \{0\} \cup \{f \in k(X)^* : (\mathrm{div} f + D)|_U \geq 0\}.$$

Remark 8.7.

(1) For $D = -x$, this definition of $\mathcal{O}_X(D)$ coincides with the definition of $\mathcal{O}_X(-x)$ above.

(2) As X is smooth, the notions of Weil and Cartier divisors coincide. The above construction $D \mapsto \mathcal{O}_X(D)$ determines a homomorphism from the group of Weil divisors modulo linear equivalence to the Picard group of isomorphism classes of line bundles, and this homomorphism is an isomorphism as X is smooth. In particular, any invertible sheaf $\mathcal{L}$ over X is isomorphic to an invertible sheaf $\mathcal{O}_X(D)$. For proofs of these statements, see [14] II §6.

For an effective divisor D, the dual line bundle $\mathcal{O}(-D)$ is isomorphic to the ideal sheaf of the (possibly non-reduced) subscheme $D \subset X$ given by this effective divisor (see [14] II Proposition 6.18) and we have a short exact sequence

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to k_D \to 0, \tag{4}$$

where k_D denotes the skyscraper sheaf supported on D; thus k_D is a torsion sheaf. This short exact sequence generalises the short exact sequence (2). In particular, any effective divisor admits a non-zero section $\mathcal{O}_X \to \mathcal{O}_X(D)$. In fact, a line bundle $\mathcal{O}_X(D)$ admits a non-zero section if and only if D is linearly equivalent to an effective divisor D by [14] II Proposition 7.7.

Definition 8.8. The Grothendieck group of X, denoted $K_0(X)$, is the free group generated by classes $[\mathcal{E}]$,for $\mathcal{E}$ a coherent sheaf on X, modulo the relations $[\mathcal{E}] - [\mathcal{F}] + [\mathcal{G}] = 0$ for short exact sequences $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$.

We claim that there is a homomorphism

$$(\det, \mathrm{rk}) : K_0(X) \to \mathrm{Pic}(X) \oplus \mathbb{Z} \tag{5}$$

which sends a locally free sheaf $\mathcal{E}$ to $(\det \mathcal{E} := \wedge^{\mathrm{rk}\,\mathcal{E}}\mathcal{E}, \mathrm{rk}\,\mathcal{E})$. To extend this to a homomorphism on $K_0(X)$, we need to define the map for coherent sheaves $\mathcal{F}$: for this, we can take a finite resolution of $\mathcal{F}$ by locally free sheaves, which exists because X is smooth, and use the relations defining $K_0(X)$. This map is surjective and in fact is an isomorphism (see [14] II, Exercise 6.11). Using this homomorphism we can define the degree of any coherent sheaf on X.

Definition 8.9. (The *degree* of a coherent sheaf).

(i) If D is a divisor, we define $\deg \mathcal{O}_X(D) := \deg D$.
(ii) If $\mathcal{F}$ is a torsion sheaf, we define $\deg \mathcal{F} = \sum_{x \in X} \mathrm{length}(\mathcal{F}_x)$.
(iii) If $\mathcal{E}$ is a locally free sheaf, $\deg \mathcal{E} = \deg(\det \mathcal{E})$.
(iv) If $\mathcal{F}$ is a coherent sheaf, we define $\deg \mathcal{E} := \deg(\det \mathcal{F})$, where $\det \mathcal{F}$ is the image of $\mathcal{F}$ in Pic(X) under the homomorphism (5).

In fact, the degree is uniquely determined by the first two properties and the fact that the degree is additive on short exact sequences (that is, if we have a short exact sequence $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$, then $\deg \mathcal{F} = \deg \mathcal{E} + \deg \mathcal{G}$); see [14] II, Exercise 6.12.

Example 8.10. The skyscraper sheaf k_x has degree 1.

8.3. Serre duality and the Riemann-Roch Theorem.

Proposition 8.11 (Riemann-Roch Theorem, version I)**.** *Let $\mathcal{L} = \mathcal{O}_X(D)$ be an invertible sheaf on a smooth projective curve X. Then*

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \deg D$$

Proof. We can write $D = x_1 + \cdots + x_n - y_1 - \cdots - y_m$ and then proceed by induction on $n + m \in \mathbb{Z}$. The base case where $D = 0$ is immediate. Now assume that the equality has been proved for D; then we can deduce the statement for $D+x$ (and $D-x$) from the equality (3). □

Definition 8.12. For a smooth projective curve X, the sheaf of differentials $\omega_X := \Omega^1_X$ on X is called the *canonical sheaf.* The *genus* of X is $g(X) := h^0(X, \omega_X)$.

The canonical bundle is a locally free sheaf of rank $1 = \dim X$; see [14] II Theorem 8.15.

Theorem 8.13 (Serre duality for a curve)**.** *Let X be a smooth projective curve and $\mathcal{E}$ be a locally free sheaf over X. There exists a natural perfect pairing*

$$H^0(X, \mathcal{E}^\vee \otimes \omega_X) \times H^1(X, \mathcal{E}) \to k.$$

Hence, $H^0(X, \mathcal{E}^\vee \otimes \omega_X) \cong H^1(X, \mathcal{E})^\vee$ and $h^0(X, \mathcal{E}^\vee \otimes \omega_X) = h^1(X, \mathcal{E})$.

Remark 8.14.

(1) Once one chooses an isomorphism $H^1(X,\omega_X) \simeq k$, the pairing can be described as the composition
$$H^0(X,\mathcal{E}^\vee \otimes \omega_X) \times H^1(X,\mathcal{E}) \to H^1(X,\mathcal{E}^\vee \otimes \mathcal{E} \otimes \omega_X) \to H^1(X,\omega_X) \simeq k$$
where the first map is a cup-product and the map $\mathcal{E}^\vee \otimes \mathcal{E} \to \mathcal{O}_X$ is the trace.

(2) In fact, Serre duality can be generalised to any projective scheme (see [14] III Theorem 7.6 for the proof) where ω_X is replaced by a dualising sheaf. If Y is a smooth projective variety of dimension n, then the dualising sheaf is the canonical sheaf $\omega_Y = \wedge^n \Omega_Y$, which is the nth exterior power of the sheaf of differentials, and the first cohomology group is replaced by the nth cohomology group.

An important consequence of Serre duality on curves is the Riemann–Roch Theorem.

Theorem 8.15 (Riemann–Roch theorem, version II)**.** *Let X be a smooth projective curve of genus g and let $\mathcal{L}$ be a degree d invertible sheaf on X. Then*
$$h^0(X,\mathcal{L}) - h^0(X,\mathcal{L}^\vee \otimes \omega_X) = d + 1 - g.$$

Proof. First, we use Serre duality to calculate the Euler characteristic of the structure sheaf
$$\chi(\mathcal{O}_X) := h^0(X,\mathcal{O}_X) - h^1(X,\mathcal{O}_X) = 1 - h^0(X,\omega_X) = 1 - g.$$
Then by Serre duality and the baby version of Riemann–Roch it follows that
$$h^0(X,\mathcal{L}) - h^0(X,\mathcal{L}^\vee \otimes \omega_X) = \chi(\mathcal{L}) = d + \chi(\mathcal{O}(X)) = d + 1 - g$$
as required. □

There is a Riemann–Roch formula for locally free sheaves due to Weil. The proof is given by induction on the rank of the locally free sheaf with the above version giving the base case. To go from a given locally free sheaf $\mathcal{E}$ to a locally free sheaf of lower rank $\mathcal{E}'$ one uses a short exact sequence
$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{E}' \to 0,$$
where $\mathcal{L}$ is an invertible subsheaf of $\mathcal{E}$ of maximal degree (this forces the quotient $\mathcal{E}'$ to be locally free; see Exercise 8.23 for the existence of such a short exact sequence).

Corollary 8.16 (Riemann–Roch for locally free sheaves on a curve)**.** *Let X be a smooth projective curve of genus g and $\mathcal{F}$ be a locally free sheaf of rank n and degree d over X. Then*
$$\chi(\mathcal{F}) = d + n(1-g).$$

Example 8.17. On a curve X of genus g, the canonical bundle has degree $2g-2$ by the Riemann–Roch Theorem:
$$h^0(X,\omega_X) - h^1(X,\mathcal{O}_X) = g - 1 = \deg \omega_X + 1 - g.$$
Therefore, on $\mathbb{P}^1$, we have $\omega_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-2)$.

8.4. **Vector bundles and locally free sheaves.** We will often use the equivalence between the category of algebraic vector bundles on X and the category of locally free sheaves. We recall that this equivalence is given by associating to an algebraic vector bundle $F \to X$ the sheaf $\mathcal{F}$ of sections of F. Under this equivalence, the trivial line bundle $X \times \mathbb{A}^1$ on X corresponds to the structure sheaf $\mathcal{O}_X$.

We will use the notation $\mathcal{E}$ to mean a sheaf or locally free sheaf and E to mean a vector bundle. We also denote the stalk of $\mathcal{E}$ at x by $\mathcal{E}_x$ and the fibre of E at x by E_x.

For a smooth projective curve X, the local rings $\mathcal{O}_{X,x}$ are DVRs, which are PIDs. Using this one can show the following.

Exercise 8.18. Prove the following statements for a smooth projective curve X.

a) Any torsion free sheaf on X is locally free.
b) A subsheaf of a locally free sheaf over X is locally free.
c) A non-zero homomorphism $f : \mathcal{L} \to \mathcal{E}$ of locally free sheaves over X with $\operatorname{rk} \mathcal{L} = 1$ is injective.

One should be careful when going between vector bundles and locally free sheaves, as this correspondence does not preserve subobjects. More precisely, if $\mathcal{F}$ is a locally free sheaf with associated vector bundle F and $\mathcal{E} \subset \mathcal{F}$ is a subsheaf, then the map on stalks $\mathcal{E}_x \to \mathcal{F}_x$ is injective for all $x \in X$. However, the map on fibres of the associated vector bundles $E_x \to F_x$ is not necessarily injective, as E_x is obtained by tensoring $\mathcal{E}_x$ with the residue field $k(x) \cong k$, which is not exact in general.

Example 8.19. For an effective divisor D, we have that $\mathcal{O}_X(-D) \hookrightarrow \mathcal{O}_X$ is a locally free subsheaf but this does not induce a vector subbundle of the trivial line bundle, as a line bundle has no non-trivial vector subbundles.

However, if we have a subsheaf $\mathcal{E}$ of a locally free sheaf $\mathcal{F}$ for which the quotient $\mathcal{G} := \mathcal{F}/\mathcal{E}$ is torsion free (and so locally free, as X is a curve), then the associated vector bundle E is a vector subbundle of F, because if we tensor the short exact sequence

$$0 \to \mathcal{E}_x \to \mathcal{F}_x \to \mathcal{G}_x \to 0$$

with the residue field, then we get a long exact sequence

$$\cdots \to \mathrm{Tor}^1_{\mathcal{O}_{X,x}}(k, \mathcal{G}_x) \to E_x \to F_x \to G_x \to 0,$$

where $\mathrm{Tor}^1_{\mathcal{O}_{X,x}}(k, \mathcal{G}_x) = 0$ as $\mathcal{G}_x$ is flat.

Definition 8.20. Let $\mathcal{E}$ be a subsheaf of a locally free sheaf $\mathcal{F}$ and let E and F denote the corresponding vector bundles. Then the *vector subbundle of F generically generated by E* is a vector subbundle $\overline{E}$ of F which is the vector bundle associated the locally free sheaf

$$\overline{\mathcal{E}} := \pi^{-1}(\mathcal{T}(\mathcal{F}/\mathcal{E}))$$

where $\pi : \mathcal{F} \to \mathcal{F}/\mathcal{E}$ and $\mathcal{T}(\mathcal{F}/\mathcal{E})$ denotes the torsion subsheaf of $\mathcal{F}/\mathcal{E}$ (i.e. $(\mathcal{F}/\mathcal{E})/\mathcal{T}(\mathcal{F}/\mathcal{E})$ is torsion free).

Indeed, as $\mathcal{F}/\overline{\mathcal{E}}$ is torsion free (and so locally free), the vector bundle homomorphism associated to $\overline{\mathcal{E}} \to \mathcal{F}$ is injective; that is, $\overline{E}$ is a vector subbundle of F. Furthermore, we have that

$$\mathrm{rk}\, \overline{\mathcal{E}} = \mathrm{rk}\, \mathcal{E} \qquad \text{and} \qquad \deg \overline{\mathcal{E}} \geq \mathcal{E}.$$

Example 8.21. Let D be an effective divisor and consider the subsheaf $\mathcal{L}' := \mathcal{O}_X(-D)$ of $\mathcal{L} := \mathcal{O}_X$; then the vector subbundle of L generically generated by L' is $\overline{L'} = L$.

The category of locally free sheaves is not an abelian category and also the category of vector bundles is not abelian. Given a homomorphism of locally free sheaves $f : \mathcal{E} \to \mathcal{G}$, the quotient $\mathcal{E}/\mathrm{ker} f$ may not be locally free (and similarly for the image). Similarly, the kernel (and the image) of a morphism of vector bundles may not be a vector bundle; essentially because the rank can jump. Instead, we can define the vector subbundle that is generically generated by the kernel (and the same for the image) sheaf theoretically.

Definition 8.22. Let $f : E \to F$ be a morphism of vector bundles; then we can define

(1) the vector subbundle K of E generically generated by the kernel $\mathrm{Ker} f$, which satisfies

$$\mathrm{rk}\, K = \mathrm{rk}\, \mathrm{Ker} f \qquad \deg K \geq \deg \mathrm{Ker} f;$$

(2) the vector subbundle I of F generically generated by the image $\mathrm{Image} f$, which satisfies

$$\mathrm{rk}\, I = \mathrm{rk}\, \mathrm{Image} f \qquad \deg I \geq \deg \mathrm{Image} f.$$

Exercise 8.23. Let $\mathcal{E}$ be a locally free sheaf of rank r over X. In this exercise, we will prove that there exists a short exact sequence of locally free sheaves over X

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0$$

such that $\mathcal{L}$ is an invertible sheaf and $\mathcal{F}$ has rank $r - 1$.

a) Show that for any effective divisor D with $r \deg D > h^1(\mathcal{E})$, the vector bundle $\mathcal{E}(D)$ admits a section by considering the long exact sequence in cohomology associated to the short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to k_D \to 0$$

tensored by $\mathcal{E}$ (here k_D denotes the skyscraper sheaf with support D). Deduce that $\mathcal{E}$ has an invertible subsheaf.

b) For an invertible sheaf $\mathcal{L}$ with $\deg \mathcal{L} > 2g-2$, prove that $h^1(\mathcal{L}) = 0$ using Serre duality.

c) Show that the degree of an invertible subsheaf $\mathcal{L}$ of $\mathcal{E}$ is bounded above, using the Riemann–Roch formula for invertible sheaves and part b).

d) Let $\mathcal{L}$ to be an invertible subsheaf of $\mathcal{E}$ of maximal degree; then verify that the quotient $\mathcal{F}$ of $\mathcal{L} \subset \mathcal{E}$ is locally free.

Exercise 8.24. In this exercise, we will prove for locally free sheaves $\mathcal{E}$ and $\mathcal{F}$ over X that

$$\deg(\mathcal{E} \otimes \mathcal{F}) = \operatorname{rk} \mathcal{E} \deg \mathcal{F} + \operatorname{rk} \mathcal{F} \deg \mathcal{E}$$

by induction on the rank of $\mathcal{E}$.

a) Prove the base case where $\mathcal{E} = \mathcal{O}_X(D)$ by splitting into two cases. If D is effective, use the short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \mathcal{O}_D \to 0$$

and the Riemann–Roch Theorem to prove the result. If D is not effective, write D as $D_1 - D_2$ for effective divisors D_i and modify $\mathcal{F}$ by twisting by a line bundle.

b) For the inductive step, use Exercise 8.23.

Example 8.25. Let $\mathcal{E}$ be a locally free sheaf of rank r and degree d over a genus g smooth projective curve X; then for any line bundle $\mathcal{L}$, we have that

$$\chi(\mathcal{E} \otimes \mathcal{L}^{\otimes m}) = d + rm \deg \mathcal{L} + r(1-g)$$

is a degree 1 polynomial in m.

8.5. **Semistability.** In order to construct moduli spaces of algebraic vector bundles over a smooth projective curve, Mumford introduced a notion of semistability for algebraic vector bundles. One advantage to restricting to semistable bundles of fixed rank and degree is that the moduli problem is then bounded (without adding the semistability hypothesis, the moduli problem is unbounded; see Example 2.22). A second advantage, which explains the term semistable, is that the notion of semistability for vector bundles corresponds to the notion of semistability coming from an associated GIT problem (which we will describe later on).

Definition 8.26. The slope of a non-zero vector bundle E on X is the ratio

$$\mu(E) := \frac{\deg E}{\operatorname{rk} E}.$$

Remark 8.27. Since the degree and rank are both additive on short exact sequences of vector bundles

$$0 \to E \to F \to G \to 0,$$

it follows that

(1) If two out of the three bundles have the same slope μ, the third also has slope μ;
(2) $\mu(E) < \mu(F)$ (resp. $\mu(E) > \mu(F)$) if and only if $\mu(F) < \mu(G)$ (resp. $\mu(F) > \mu(G)$).

Definition 8.28. A vector bundle E is stable (resp. semistable) if every proper non-zero vector subbundle $S \subset E$ satisfies

$$\mu(S) < \mu(E) \qquad \text{(resp. } \mu(S) \leq \mu(E) \text{ for semistability).}$$

A vector bundle E is polystable if it is a direct sum of stable bundles of the same slope.

Remark 8.29. If we fix a rank n and degree d such that n and d are coprime, then the notion of semistability for vector bundles with invariants (n, d) coincides with the notion of stability.

Lemma 8.30. *Let L be a line bundle and E a vector bundle over X; then*

i) L is stable.
ii) If E is stable (resp. semistable), then $E \otimes L$ is stable (resp. semistable).

Proof. Exercise. □

Lemma 8.31. *Let $f : E \to F$ be a non-zero homomorphism of vector bundles over X; then*

i) If E and F are semistable, $\mu(E) \leq \mu(F)$.
ii) If E and F are stable of the same slope, then f is an isomorphism.
iii) Every stable vector bundle E is simple i.e. $\operatorname{End} E = k$.

Proof. Exercise. □

If E is a vector bundle which is not semistable, then there exists a subbundle $E' \subset E$ with larger slope that E, by taking the sum of all vector subbundles of E with maximal slope, one obtains a unique maximal destabilising vector subbundle of E, which is semistable by construction. By iterating this process, one obtains a unique maximal destabilising filtration of E known as the Harder–Narasimhan filtration of E [13].

Definition 8.32. Let E be a vector bundle; then E has a Harder–Narasimhan filtration

$$0 = E^{(0)} \subsetneq E^{(1)} \subsetneq \cdots \subsetneq E^{(s)} = E$$

where $E_i := \mathcal{E}^{(i)}/E^{(i-1)}$ are semistable with slopes

$$\mu(E_1) > \mu(E_2) > \cdots > \mu(E_s).$$

As we have already mentioned, the moduli problem of vector bundles on X with fixed rank n and degree d is unbounded. Therefore, we restrict to the moduli functors $\mathcal{M}^{(s)s}(n,d)$ of (semi)stable locally free sheaves. Let us refine our notion of families to families of semistable vector bundles.

Definition 8.33. A family over a scheme S of (semi)stable vector bundles on X with invariants (n,d) is a coherent sheaf $\mathcal{E}$ over $X \times S$ which is flat over S and such that for each $s \in S$, the sheaf $\mathcal{E}_s$ is a (semi)stable vector bundle on X with invariants (n,d).

We say two families $\mathcal{E}$ and $\mathcal{F}$ over S are equivalent if there exists an invertible sheaf $\mathcal{L}$ over S and an isomorphism

$$\mathcal{E} \cong \mathcal{F} \otimes \pi_S^* \mathcal{L}$$

where $\pi_S : X \times S \to S$ denotes the projection.

Lemma 8.34. *If there exists a semistable vector bundle over X with invariants (n,d) which is not polystable, then the moduli problem of semistable vector bundles $\mathcal{M}^{ss}(n,d)$ does not admit a coarse moduli space.*

Proof. If there exists a semistable sheaf $\mathcal{F}$ on X which is not polystable, then there is a non-split short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0,$$

where $\mathcal{F}'$ and $\mathcal{F}''$ are semistable vector bundles with the same slope as $\mathcal{F}$. The above short exact sequence corresponds to a non-zero point in $\operatorname{Ext}(\mathcal{F}'', \mathcal{F}')$ and if we take the affine line through this extension class then we obtain a family of extensions over $\mathbb{A}^1 = \operatorname{Spec} k[t]$. More precisely, let $\mathcal{E}$ be the coherent sheaf over $X \times \mathbb{A}^1$ given by this extension class. Then $\mathcal{E}$ is flat over $\mathbb{A}^1$ and

$$\mathcal{E}_t \cong \mathcal{F} \quad \text{for } t \neq 0, \quad \text{and} \quad \mathcal{E}_0 \cong \mathcal{F}'' \oplus \mathcal{F}'.$$

Since $\mathcal{E}$ is a family of semistable locally free sheaves with the fixed invariants (n,d) which exhibits the jump phenomenon, there is no coarse moduli space by Lemma 2.27. □

However, this cannot happen if the notions of semistability and stability coincide, which happens when n and d are coprime.

8.6. **Boundedness of semistable vector bundles.** To construct a moduli space of vector bundles on X using GIT, we would like to find a scheme R that parametrises a family $\mathcal{F}$ of semistable vector bundles on X of fixed rank n and degree d such that any vector bundle of the given invariants is isomorphic to $\mathcal{F}_p$ for some $p \in R$. In this section, we prove an important boundedness result for the family of semistable vector bundles on X of fixed rank and degree which will enable us to construct such a scheme. In fact, we will show that we can construct a scheme R which parametrises a family with the local universal property.

First, we note that we can assume, without loss of generality, that the degree of our vector bundle is sufficiently large: for, if we take a line bundle $\mathcal{L}$ of degree e, then tensoring with $\mathcal{L}$ preserves (semi)stability and so induces an isomorphism between the moduli functor of (semi)stable vector bundles with rank and degree (n, d) and those with rank and degree $(n, d + ne)$

$$- \otimes \mathcal{L} : \mathcal{M}^{ss}(n, d) \cong \mathcal{M}^{ss}(n, d + ne).$$

Hence, we can assume that $d > n(2g - 1)$ where g is the genus of X. This assumption will be used to prove the boundedness result for semistable vector bundles. However, first we need to recall the definition of a sheaf being generated by global sections.

Definition 8.35. A sheaf $\mathcal{F}$ is generated by its global sections if the natural evaluation map

$$\mathrm{ev}_{\mathcal{F}} : H^0(X, \mathcal{F}) \otimes \mathcal{O}_X \to \mathcal{F}$$

is a surjection.

Lemma 8.36. *Let $\mathcal{F}$ be a locally free sheaf over X of rank n and degree $d > n(2g - 1)$. If the associated vector bundle F is semistable, then the following statements hold:*

i) $H^1(X, \mathcal{F}) = 0$;
ii) $\mathcal{F}$ is generated by its global sections.

Proof. For i), we argue by contradiction using Serre duality: if $H^1(X, \mathcal{F}) \neq 0$, then dually there would be a non-zero homomorphism $f : \mathcal{F} \to \omega_X$. We let $K \subset F$ be the vector subbundle generically generated by the kernel of f which is a vector subbundle of rank $n - 1$ with

$$\deg K \geq \deg \ker f \geq \deg \mathcal{F} - \deg \omega_X = d - (2g - 2).$$

In this case, by semistability of F, we have

$$\frac{d - (2g - 2)}{n - 1} \leq \mu(K) \leq \mu(F) = \frac{d}{n};$$

this gives $d \leq n(2g - 2)$, which contradicts our assumption on the degree of F.

For ii), we let F_x denote the fibre of the vector bundle at a point $x \in X$. If we consider the fibre F_x as a torsion sheaf over X, then we have a short exact sequence

$$0 \to \mathcal{F}(-x) := \mathcal{O}_X(-x) \otimes \mathcal{F} \to \mathcal{F} \to F_x = \mathcal{F} \otimes k_x \to 0$$

which gives rise to an associated long exact sequence in cohomology

$$0 \to H^0(X, \mathcal{F}(-x)) \to H^0(X, \mathcal{F}) \to H^0(F_x) \to H^1(X, \mathcal{F}(-x)) \to \cdots .$$

Then we need to prove that, for each $x \in X$, the map $H^0(X, \mathcal{F}) \to H^0(X, F_x) = F_x$ is surjective. We prove this map is surjective by showing that $H^1(X, \mathcal{F}(-x)) = 0$ using the same argument as in part i) above, where we use the fact that twisting by a line bundle does not change semistability: $\mathcal{F}(-x) := \mathcal{O}_X(-x) \otimes \mathcal{F}$ is semistable with degree $d - n > n(2g - 2)$ and thus $H^1(X, \mathcal{O}_X(-x) \otimes \mathcal{F}) = 0$. □

As mentioned above, these two properties are important for showing boundedness. In fact, we will see that a strictly larger family of vector bundles of fixed rank and degree are bounded; namely those that are generated by their global sections and have vanishing 1st cohomology. Given a locally free sheaf $\mathcal{F}$ of rank n and degree d that is generated by its global sections, we can consider the evaluation map

$$\mathrm{ev}_{\mathcal{F}} : H^0(X, \mathcal{F}) \otimes \mathcal{O}_X \to \mathcal{F}$$

which is, by assumption, surjective. If also $H^1(X, \mathcal{F}) = 0$, then by the Riemann–Roch formula

$$\chi(\mathcal{F}) = d + n(1-g) = \dim H^0(X, \mathcal{F}) - \dim H^1(X, \mathcal{F}) = \dim H^0(X, \mathcal{F});$$

that is, the dimension of the 0th cohomology is fixed and equal to $N := d + n(1-g)$. Therefore, we can choose an isomorphism $H^0(X, \mathcal{F}) \cong k^N$ and combine this with the evaluation map for $\mathcal{F}$, to produce a surjection

$$q_{\mathcal{F}} : \mathcal{O}_X^N = k^N \otimes \mathcal{O}_X \to \mathcal{F}$$

from a fixed trivial vector bundle. Such surjective homomorphisms from a fixed coherent sheaf are parametrised by a projective scheme known as a Quot scheme, which generalises the Grassmannians.

8.7. **The Quot scheme.** The Quot scheme is a fine moduli space which generalises the Grassmannian in the sense that it parametrises quotients of a fixed sheaf. In this section, we will define the moduli problem that the Quot scheme represents and give an overview of the construction of the Quot scheme following [33].

Let Y be a projective scheme and $\mathcal{F}$ be a fixed coherent sheaf on X. Then one can consider the moduli problem of classifying quotients of $\mathcal{F}$. More precisely, we consider surjective sheaf homomorphisms $q : \mathcal{F} \to \mathcal{E}$ up to the equivalence relation

$$(q : \mathcal{F} \to \mathcal{E}) \sim (q' : \mathcal{F} \to \mathcal{E}') \iff \ker q = \ker q'.$$

Equivalently, there is a sheaf isomorphism $\phi : \mathcal{E} \to \mathcal{E}'$ such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{q} & \mathcal{E} \\ {\scriptstyle \mathrm{Id}}\downarrow & & \downarrow{\scriptstyle \phi} \\ \mathcal{F} & \xrightarrow[q']{} & \mathcal{E}'. \end{array}$$

This gives the naive moduli problem and the following definition of families gives the extended moduli problem.

Definition 8.37. Let $\mathcal{F}$ be a coherent sheaf over Y. Then for any scheme S, we let $\mathcal{F}_S := \pi_Y^* \mathcal{F}$ denote the pullback of $\mathcal{F}$ to $Y \times S$ via the projection $\pi_Y : Y \times S \to Y$. A *family of quotients of $\mathcal{F}$ over a scheme S* is a surjective $\mathcal{O}_{Y \times S}$-linear homomorphism of sheaves over $Y \times S$

$$q_S : \mathcal{F}_S \to \mathcal{E},$$

such that $\mathcal{E}$ is flat over S. Two families $q_S : \mathcal{F}_S \to \mathcal{E}$ and $q'_S : \mathcal{F}_S \to \mathcal{E}'$ are equivalent if $\ker q_S = \ker q'_S$. It is easy to check that we can pullback families, as flatness is preserved by base change; therefore, we let

$$\mathcal{Q}uot_Y(\mathcal{F}) : \mathrm{Sch} \to \mathrm{Set}$$

denote the associated moduli functor.

Remark 8.38.

(1) With these definitions, it is clear that we can think of the Quot scheme as instead parametrising coherent subsheaves of $\mathcal{F}$ up to equality rather than quotients of $\mathcal{F}$ up to the above equivalence. Indeed this perspective can also be taken with the Grassmannian (and even projective space). For us, the quotient perspective will be the most useful.

(2) For the moduli problem of the Grassmannian, we fix the dimension of the quotient vector spaces. Similarly for the quotient moduli problem, we can fix invariants, as for two quotient sheaves to be equivalent, they must be isomorphic. Thus we can refine the above moduli functor by fixing the invariants of our quotient sheaves.

Definition 8.39. For a coherent sheaf $\mathcal{E}$ over a projective scheme Y equipped with a fixed ample invertible sheaf $\mathcal{L}$, the Hilbert polynomial of $\mathcal{E}$ with respect to $\mathcal{L}$ is a polynomial $P(\mathcal{E}, \mathcal{L}) \in \mathbb{Q}[t]$ such that for $l \in \mathbb{N}$ sufficiently large,

$$P(\mathcal{E}, \mathcal{L}, l) = \chi(\mathcal{E} \otimes \mathcal{L}^{\otimes l}) := \sum_{i \geq 0} (-1)^i \dim H^i(Y, \mathcal{E} \otimes \mathcal{L}^{\otimes l}).$$

Serre's vanishing theorem states that for l sufficiently large (depending on $\mathcal{E}$), all the higher cohomology groups of $\mathcal{E} \otimes \mathcal{L}^{\otimes l}$ vanish (see [14] III Theorem 5.2). Hence, for l sufficiently large, $P(\mathcal{E}, \mathcal{L}, l) = \dim H^0(Y, \mathcal{E} \otimes \mathcal{L}^{\otimes l})$.

The proof that there is such a polynomial is given by reducing to the case of $\mathbb{P}^n$ (as L is ample, we can use a power of L to embed X into a projective space) and then the proof proceeds by induction on the dimension d of the support of the sheaf (where the inductive step is given by restricting to a hypersurface and the base case $d = 0$ is trivial as the Hilbert polynomial is constant); for a proof, see [16] Lemma 1.2.1. However, for a smooth projective curve X, we can explicitly write down the Hilbert polynomial of a locally free sheaf over X using the Riemann–Roch Theorem.

Example 8.40. On a smooth projective genus g curve X, we fix a degree 1 line bundle $\mathcal{L} = \mathcal{O}_X(x) =: \mathcal{O}_X(1)$. For a vector bundle $\mathcal{E}$ over X of rank n and degree d, the twist $\mathcal{E}(m) := \mathcal{E} \otimes \mathcal{O}_X(m)$ has rank n and degree $d + mn$. The Riemann–Roch formula gives

$$\chi(\mathcal{E}(m)) = d + mn + n(1 - g).$$

Thus $\mathcal{E}$ has Hilbert polynomial $P(t) = nt + d + n(1 - g)$ of degree 1 with leading coefficient given by the rank n.

Definition 8.41. For a fixed ample line bundle L on Y, we have a decomposition

$$\mathcal{Q}uot_Y(\mathcal{F}) = \bigsqcup_{P \in \mathbb{Q}[t]} \mathcal{Q}uot_Y^{P,L}(\mathcal{F})$$

into Hilbert polynomials P taken with respect to L.

If $Y = X$ is a curve, then we have a decomposition of the Quot moduli functor by ranks and degrees of the quotient sheaf:

$$\mathcal{Q}uot_X(\mathcal{F}) = \bigsqcup_{(n,d)} \mathcal{Q}uot_X^{n,d}(\mathcal{F}).$$

Example 8.42. The grassmannian moduli functor is a special example of the Quot moduli functor:

$$\mathcal{G}r(d, n) = \mathcal{Q}uot^{n-d}_{\operatorname{Spec} k}(k^n).$$

Theorem 8.43 (Grothendieck). *Let Y be a projective scheme and $\mathcal{L}$ an ample invertible sheaf on Y. Then for any coherent sheaf $\mathcal{F}$ over Y and any polynomial P, the functor $\mathcal{Q}uot_Y^{P,\mathcal{L}}(\mathcal{F})$ is represented by a projective scheme $\mathrm{Quot}_Y^{P,\mathcal{L}}(\mathcal{F})$.*

The idea of the construction is very beautiful but also technical; therefore, we will just give an outline of a proof. We split the proof up into the 4 following steps.

Step 1. Reduce to the case where $Y = \mathbb{P}^n$, $L = \mathcal{O}_{\mathbb{P}^n}(1)$ and $\mathcal{F}$ is a trivial vector bundle $\mathcal{O}_{\mathbb{P}^n}^N$.

Step 2. For m sufficiently large, construct an injective natural transformation of moduli functors

$$\mathcal{Q}uot_{\mathbb{P}^n}^{P,\mathcal{O}(1)}(\mathcal{O}_{\mathbb{P}^n}^N) \hookrightarrow \mathcal{G}r(V, P(m))$$

to the Grassmannian moduli functor of $P(m)$-dimensional quotients of $V := k^N \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(m))$.

Step 3. Prove that $\mathcal{Q}uot_{\mathbb{P}^n}^{P,\mathcal{O}(1)}(\mathcal{O}_{\mathbb{P}^n}^N)$ is represented by a locally closed subscheme of $\mathrm{Gr}(V, P(m))$.

Step 4. Prove that the Quot scheme is proper using the valuative criterion for properness.

Before we explain the proof of each step, we need the following definition.

Definition 8.44. A natural transformation of presheaves $\eta : \mathcal{M}' \to \mathcal{M}$ is a closed (resp. open, resp. locally closed) immersion if η_S is injective for every scheme S and moreover, for any natural transformation $\gamma : h_S \to \mathcal{M}$ from the functor of points of a scheme S, there is a closed (resp. open, resp. locally closed) subscheme $S' \subset S$ such that

$$h_{S'} \cong \mathcal{M}' \times_{\mathcal{M}} h_S$$

where the fibre product is given by

$$(\mathcal{M}' \times_{\mathcal{M}} h_S)(T) = \left\{(f : T \to S \in h_S(T), F \in \mathcal{M}'(T)) : \gamma_T(f) = \eta_T(F) \in \mathcal{M}(T)\right\}.$$

Sketch of Step 1. First, we can assume that L is very ample by replacing L by a sufficiently large positive power of L; this will only change the Hilbert polynomial. Then L defines a projective embedding $i : Y \hookrightarrow \mathbb{P}^n$ such that L is the pullback of $\mathcal{O}_{\mathbb{P}^n}(1)$. Since i is a closed immersion, $i_*\mathcal{F}$ is coherent and, moreover i_* is exact; therefore, we can push-forward quotient sheaves on Y to $\mathbb{P}^n$. Hence, one obtains a natural transformation

$$\mathcal{Q}uot_Y(\mathcal{F}) \to \mathcal{Q}uot_{\mathbb{P}^n}(i_*\mathcal{F}),$$

which is injective as $i^*i_* = \mathrm{Id}$. We claim that this natural transformation is a closed immersion in the sense of the above definition. More precisely, we claim for any scheme S and natural transformation $h_S \to \mathcal{Q}uot_{\mathbb{P}^n}(i_*\mathcal{F})$ there exists a closed subscheme $S' \subset S$ with the following property: a morphism $f : T \to S$ determines a family in $\mathcal{Q}uot_Y(\mathcal{F})(T)$ if and only if the morphism f factors via S'. To define the closed subscheme associated to a map $\eta : h_S \to \mathcal{Q}uot_{\mathbb{P}^n}(i_*\mathcal{F})$, we let $(i_*\mathcal{F})_S \twoheadrightarrow \mathcal{E}$ denote the family over S of quotients associated to $\eta_S(\mathrm{id}_S)$ and apply $(\mathrm{id}_S \times i)^*$ to obtain a homomorphism of sheaves over $Y \times S$

$$(\mathcal{F})_S \cong i^*(i_*\mathcal{F})_S \to i^*\mathcal{E};$$

then we take $S' \subset S$ to be the closed subscheme on which this homomorphism is surjective (the fact that this is closed follows from a semi-continuity argument). Hence, we may assume that $(Y, L) = (\mathbb{P}^n, \mathcal{O}(1))$.

We can tensor any quotient sheaf by a power of $\mathcal{O}(1)$ and this induces a natural transformation between

$$\mathcal{Q}uot_{\mathbb{P}^n}(\mathcal{F}) \cong \mathcal{Q}uot_{\mathbb{P}^n}(\mathcal{F} \otimes \mathcal{O}(r))$$

(under this natural transformation the Hilbert polynomial undergoes a explicit transformation corresponding to this tensorisation). Hence, by replacing $\mathcal{F}$ with $\mathcal{F}(r) := \mathcal{F} \otimes \mathcal{O}(r)$, we can assume without loss of generality that $\mathcal{F}$ has trivial higher cohomology groups and is globally generated; that is, the evaluation map

$$\mathcal{O}_{\mathbb{P}^n}^{\oplus N} \cong H^0(\mathbb{P}^n, \mathcal{F}) \otimes \mathcal{O}_{\mathbb{P}^n} \to \mathcal{F}$$

is surjective and $N = P(\mathcal{F}, 0)$. By composition, this surjection induces a natural transformation

$$\mathcal{Q}uot_{\mathbb{P}^n}(\mathcal{F}) \to \mathcal{Q}_{\mathbb{P}^n}(\mathcal{O}_{\mathbb{P}^n}^N)$$

which can also be shown to be a closed embedding. In conclusion, we obtain a natural transformation

$$\mathcal{Q}uot_Y^{P',L}(\mathcal{F}) \to \mathcal{Q}uot_{\mathbb{P}^n}^{P,\mathcal{O}(1)}(\mathcal{O}_{\mathbb{P}^n}^N)$$

which is a closed embedding of moduli functors.

Sketch of Step 2. By a result of Mumford and Castelnuovo concerning 'Castelnuovo–Mumford regularity' of subsheaves of $\mathcal{O}_{\mathbb{P}^n}^N$, there exists $M \in \mathbb{N}$ (depending on P, n and N) such that for all $m \geq M$, the following holds for any short exact sequence of sheaves

$$0 \to \mathcal{K} \to \mathcal{O}_{\mathbb{P}^n}^N \to \mathcal{F} \to 0$$

such that $\mathcal{F}$ has Hilbert polynomial P:

(1) the sheaf cohomology groups H^i of $\mathcal{K}(m)$, $\mathcal{F}(m)$ vanish for $i > 0$,
(2) $\mathcal{K}(m)$ and $\mathcal{F}(m)$ are globally generated.

The proof of this result is by induction on n, where one restricts to a hyperplane $H \cong \mathbb{P}^{n-1}$ in $\mathbb{P}^n$ to do the inductive step; for a full proof, see [33] Theorem 2.3. Now if we fix $m \geq M$, we claim there is a natural transformation

$$\eta : \mathcal{Q}uot_{\mathbb{P}^n}^{P,\mathcal{O}(1)}(\mathcal{O}_{\mathbb{P}^n}^N) \hookrightarrow \mathcal{G}r(k^N \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(m)), P(m)).$$

First, let us define this for families over $S = \operatorname{Spec} k$: for a quotient $q : \mathcal{O}_{\mathbb{P}^n}^{\oplus N} \twoheadrightarrow \mathcal{F}$ with kernel $\mathcal{K}$, we have an associated long exact sequence in cohomology

$$0 \to H^0(\mathcal{K}(m)) \to H^0(\mathcal{O}_{\mathbb{P}^n}^N(m)) \to H^0(\mathcal{F}(m)) \to H^1(\mathcal{K}(m)) = 0.$$

Hence, we define

$$\eta_{\operatorname{Spec} k}(q : \mathcal{O}_{\mathbb{P}^n}^N \twoheadrightarrow \mathcal{F}) = \left(H^0(q(m)) : W \twoheadrightarrow H^0(\mathcal{F}(m))\right),$$

where $W := H^0(\mathcal{O}_{\mathbb{P}^n}^N(m)) = k^N \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(m))$ and we have $\dim H^0(\mathcal{F}(m)) = P(m)$, as all higher cohomology of $\mathcal{F}(m)$ vanishes.

To define η_S for a family of quotients $q_S : \mathcal{O}_{\mathbb{P}^n \times S}^N \to \mathcal{E}$ over an arbitrary scheme S, we essentially do the above process in a family. More precisely, we let $\pi_S : X \times S \to S$ be the projection and push forward $q_S(m)$ by π_S to S to obtain a surjective homomorphism of sheaves over S

$$\mathcal{O}_S \otimes W \cong \pi_{S*}(\mathcal{O}_{\mathbb{P}^n \times S}^N(m)) \to \pi_{S*}(\mathcal{E}(m)).$$

By our assumptions on m and the semi-continuity theorem, $\pi_{S*}(\mathcal{E}(m))$ is locally free of rank $P(m)$ (as the higher rank direct images of $\mathcal{E}(m)$ vanish so the claim follows by EGA III 7.9.9). Hence, we have a family of $P(m)$-dimensional quotients of W over S, which defines the desired S-point in the Grassmannian.

Let $\mathrm{Gr} = \mathrm{Gr}(W, P(m))$. We claim this natural transformation η is an injection. Let us explain how to reconstruct q_S from the morphism $f_{q_s} : S \to \mathrm{Gr}$ corresponding to the surjection

$$\pi_{S*}(q_s(m)) : \mathcal{O}_S \otimes W \to \pi_{S*}(\mathcal{E}(m)).$$

Over the Grassmannian, we have a universal inclusion (and a corresponding surjection)

$$\mathcal{K}_{\mathrm{Gr}} \hookrightarrow \mathcal{O}_{\mathrm{Gr}} \otimes W,$$

whose pullback to S via the morphism $S \to \mathrm{Gr}$ is the homomorphism

$$\pi_S^* \pi_{S*}(\mathcal{K}_S(m)) \to V \otimes \mathcal{O}_S = \pi_S^* \pi_{S*}(\mathcal{O}_{\mathbb{P}^n \times S}^N(m)),$$

where $\mathcal{K}_S := \ker q_S$. We claim that the homomorphism

$$f : \pi_S^* \pi_{S*}(\mathcal{K}_S(m)) \to \pi_S^* \pi_{S*}(\mathcal{O}_{\mathbb{P}^n \times S}^N(m)) \to \mathcal{O}_{\mathbb{P}^n \times S}^N(m)$$

has cokernel $q_S(m)$. To prove the claim, consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_S^*\pi_{S*}(\mathcal{K}_S(m)) & \longrightarrow & \pi_S^*\pi_{S*}(\mathcal{O}_{\mathbb{P}^n \times S}^N(m)) & \longrightarrow & \pi_S^*\pi_{S*}(\mathcal{E}(m)) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{K}_S(m) & \longrightarrow & \mathcal{O}_{\mathbb{P}^n \times S}^N(m) & \longrightarrow & \mathcal{E}(m) & \longrightarrow & 0 \end{array}$$

whose rows are exact and whose columns are surjective by our assumption on m. Finally, we can recover q_S from $q_S(m)$ by twisting by $\mathcal{O}(-m)$.

Sketch of Step 3. For any morphism $f : T \to \mathrm{Gr} = \mathrm{Gr}(W, P(m))$, we let $\mathcal{K}_{T,f}$ denote the pullback of the universal subsheaf $\mathcal{K}_{\mathrm{Gr}}$ on the Grassmannian to T via f. Then consider the induced composition

$$h_{T,f} : \pi_T^* \mathcal{K}_{T,f} \to \pi_T^*(W \otimes \mathcal{O}_T) \cong \pi_T^* \pi_{T*}(\mathcal{O}_{\mathbb{P}^n \times T}^N(m)) \to \mathcal{O}_{T \times \mathbb{P}^n}^N(m)$$

where $\pi_T : \mathbb{P}^n \times T \to T$ denotes the projection. Let $q_{T,f}(m) : \mathcal{O}_{\mathbb{P}^n \times T}^N(m) \to \mathcal{F}_{T,f}(m)$ denote the cokernel of $h_{T,f}$; then $\mathcal{F}_{T,f}$ is a coherent sheaf over $\mathbb{P}^n \times T$.

We claim that the natural transformation defined in Step 2 is a locally closed immersion. To prove the claim, we need to show for any morphism $h : S \to \mathrm{Gr}$, there is a unique locally closed subscheme $S' \hookrightarrow S$ with the property that a morphism $f : T \to S$ factors via $S' \hookrightarrow S$ if and only if the sheaf $\mathcal{F}_{T,h \circ f}$ is flat over T and has Hilbert polynomial P at each $t \in T$. This locally closed subscheme $S' \subset S$ is constructed as the stratum with Hilbert polynomial P in the flattening stratification for the sheaf $\mathcal{F}_{T,f'}$ over $T \times \mathbb{P}^n$. For details, see [33] Theorem 4.3.

Let $\mathrm{Quot}^P_{\mathbb{P}^n}(\mathcal{O}^N_{\mathbb{P}^n})$ be the locally closed subscheme of Gr associated to the identity morphism on Gr (which corresponds to the universal family on the Grassmannian); then it follows from the above arguments that this scheme represents the functor $\mathcal{Q}uot^P_{\mathbb{P}^n}(\mathcal{O}^N_{\mathbb{P}^n})$.

The Grassmannian $\mathrm{Gr} = \mathrm{Gr}(W, P(m))$ has its Pücker embedding into projective space

$$\mathrm{Gr}(W, P(m)) \hookrightarrow \mathbb{P}(\wedge^{P(m)} W^\vee).$$

Therefore, we have a locally closed embedding

$$\mathrm{Quot}_{\mathbb{P}^n}(\mathcal{O}^{\oplus N}_{\mathbb{P}^n}, P) \hookrightarrow \mathbb{P}(\wedge^{P(m)} W^\vee). \tag{6}$$

In particular, the Quot scheme is quasi-projective; hence, separated and of finite type.

Sketch of Step 4. We will prove the valuative criterion for the Quot scheme using its moduli functor. The Quot scheme $\mathrm{Quot}^P_{\mathbb{P}^n}(\mathcal{O}^{\oplus N}_{\mathbb{P}^n})$ is proper over $\mathrm{Spec}\, k$ if and only if for every discrete valuation ring R over k with quotient field K, the restriction map

$$\mathcal{Q}uot^P_{\mathbb{P}^n}(\mathcal{O}^N_{\mathbb{P}^n})(\mathrm{Spec}\, R) \to \mathcal{Q}uot^P_{\mathbb{P}^n}(\mathcal{O}^N_{\mathbb{P}^n})(\mathrm{Spec}\, K)$$

is bijective. Since the Quot scheme is separated, we already know that this map is injective. Let $j : \mathbb{P}^n_K \hookrightarrow \mathbb{P}^n_R$ denote the open immersion. Any quotient sheaf $q_K : \mathcal{O}^N_{\mathbb{P}^n_K} \to \mathcal{F}_K$ can be lifted to a quotient sheaf $q_R : \mathcal{O}^N_{\mathbb{P}^n_R} \to \mathcal{F}_R$ where $\mathcal{F}_R$ is the image of the homomorphism

$$q_R : \mathcal{O}^{\oplus N}_{\mathbb{P}^n_R} \to j_*(\mathcal{O}^{\oplus N}_{\mathbb{P}^n_K}) \to j_*\mathcal{F}_K.$$

The sheaf $\mathcal{F}_R$ is torsion free as it as a subsheaf of $j_*\mathcal{F}_K$, which is torsion free as j^* is exact, $j^*j_*\mathcal{F}_K = \mathcal{F}_K$ and $\mathcal{F}_K$ is torsion free (as it is flat over K). Hence, $\mathcal{F}_R$ is flat over R, as over a DVR flat is equivalent to torsion free and so this gives a well defined R-valued point of the Quot scheme. It remains to check that the image of q_R under the restriction map is q_K. As j^* is left exact, the map $j^*\mathcal{F}_R \to j^*j_*\mathcal{F}_K = \mathcal{F}_K$ is injective and the following commutative diagram

$$\begin{array}{ccc} \mathcal{O}^{\oplus N}_{\mathbb{P}^n_K} & \xrightarrow{j^*q_R} & j^*\mathcal{F}_R \\ & \searrow^{q_K} & \downarrow \\ & & \mathcal{F}_K \end{array}$$

implies that the vertical homomorphism must also be surjective; thus $j^*\mathcal{F}_R \cong j^*j_*\mathcal{F}_K = \mathcal{F}_K$ as required. Hence, the Quot scheme is proper over $\mathrm{Spec}\, k$.

Since $\mathrm{Quot}_{\mathbb{P}^n}(\mathcal{O}^{\oplus N}_{\mathbb{P}^n}; P)$ is proper over $\mathrm{Spec}\, k$, the embedding (6) is a closed embedding.

Remark 8.45.

(1) As the Quot scheme $Q := \mathrm{Quot}^{P,\mathcal{L}}_Y(\mathcal{F})$ is a fine moduli space, the identity morphism on Q corresponds to a universal quotient homomorphism

$$\pi_Y^*\mathcal{F} \twoheadrightarrow \mathcal{U}$$

over $Q \times Y$, where $\pi_Y : Q \times Y \to Y$ denotes the projection to Y.

(2) The Quot scheme can also be defined in the relative setting, where we replace our field k by a general base scheme S and look at quotients of a fixed coherent sheaf on a scheme $X \to S$; the construction in the relative case is carried out in [33].

The Hilbert schemes are special examples of Quot schemes, which also play an important role in the construction of many moduli spaces.

Definition 8.46. A *Hilbert scheme* is a Quot scheme of the form $\mathrm{Quot}^P_Y(\mathcal{O}_Y)$ and represents the moduli functor that sends a scheme S to the set of closed subschemes $Z \subset Y \times S$ that are proper and flat over S with the given Hilbert polynomial.

Exercise 8.47. For a natural number $d \geq 1$, prove that the Hilbert scheme

$$\mathrm{Quot}^d_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1})$$

is isomorphic to $\mathbb{P}^d$ by showing they both have the same functor of points in the following way.

a) Show that any family $Z \subset \mathbb{P}^1$ over $\mathrm{Spec}\, k$ in this Hilbert scheme is a degree d hypersurface in $\mathbb{P}^1$.
b) Let S be a scheme and $\pi_S : \mathbb{P}^1_S := \mathbb{P}^1 \times S \to S$ denote the projection. Show that any family $Z \subset \mathbb{P}^1_S$ over S in this Hilbert scheme is a Cartier divisor in $\mathbb{P}^1_S$ and so there is a line subbundle of $\pi_{S*}(\mathcal{O}_{\mathbb{P}^1_S}(d))$ which determines a morphism $f_Z : S \to \mathbb{P}^d$. In particular, this gives a natural transformation

$$\mathcal{Q}uot^d_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}) \to h_{\mathbb{P}^d}.$$

c) Construct the inverse to the above natural transformation using the tautological family of degree d hypersurfaces in $\mathbb{P}^1$ parametrised by $\mathbb{P}^d$.

8.8. **GIT set up for construction of the moduli space.** Throughout this section, we fix a connected smooth projective curve X and we assume the genus of X is greater than or equal to 2 to avoid special cases in low genus. We fix a rank n and a degree $d > n(2g-1)$ (recall that tensoring with a line bundle does not alter semistability and so we can pick the degree to be arbitrarily large; in fact, eventually we will choose d to be even larger). It follows from Lemma 8.36 that any locally free semistable sheaf $\mathcal{E}$ of rank r and degree d is globally generated with $H^1(X,\mathcal{E}) = 0$. By the Riemann–Roch Theorem,

$$\dim H^0(X,\mathcal{E}) = d + n(1-g) =: N.$$

If we choose an identification $H^0(X,\mathcal{E}) \cong k^N$, then the evaluation map

$$H^0(X,\mathcal{E}) \otimes \mathcal{O}_X \to \mathcal{E},$$

which is surjective as $\mathcal{E}$ is globally generated, determines a quotient sheaf $q : \mathcal{O}_X^N \twoheadrightarrow \mathcal{E}$.

Let $Q := \mathrm{Quot}_X^{n,d}(\mathcal{O}_X^N)$ be the Quot scheme of quotient sheaves of the trivial rank N vector bundle $\mathcal{O}_X^N$ of rank n and degree d. Let $R^{(s)s} \subset Q$ denote the open subscheme consisting of quotients $q : \mathcal{O}_X^N \to \mathcal{F}$ such that $\mathcal{F}$ is a (semi)stable locally free sheaf and $H^0(q)$ is an isomorphism. For a proof that these conditions are open see [16] Proposition 2.3.1.

The Quot scheme Q parametrises a universal quotient

$$q_Q : \mathcal{O}^N_{Q \times X} \twoheadrightarrow \mathcal{U}$$

and we let $q^{(s)s} : \mathcal{O}^N_{R^{(s)s} \times X} \to \mathcal{U}^{(s)s} := \mathcal{U}|_{R^{(s)s}}$ denote the restriction to $R^{(s)s}$.

Lemma 8.48. *The universal quotient sheaf $\mathcal{U}^{(s)s}$ over $R^{(s)s} \times X$ is a family over $R^{(s)s}$ of (semi)stable locally free sheaves over X with invariants (n,d) with the local universal property.*

Proof. Let $\mathcal{F}$ be a family over a scheme S of (semi)stable locally free sheaves over X with fixed invariants (n,d). Then for each $s \in S$, the locally free semistable sheaf $\mathcal{F}_s$ is globally generated with vanishing first cohomology by our assumption on d. Therefore, by the semi-continuity Theorem, $\pi_{S*}\mathcal{F}$ is a locally free sheaf over S of rank $N = d + n(1-g)$. For each $s \in S$, we need to show there is an open neighbourhood $U \subset S$ of $s \in S$ and a morphism $f : S \to R^{(s)s}$ such that $\mathcal{F}|_U \sim f^*\mathcal{U}^{(s)s}$. Pick an open neighbourhood $U \ni s$ on which $\pi_{S*}\mathcal{F}$ is trivial; that is, we have an isomorphism

$$\Phi : \mathcal{O}_U^N \cong (\pi_{S*}\mathcal{F})|_U.$$

Then the surjective homomorphism of sheaves over $U \times X$

$$q_U : \mathcal{O}^N_{U \times X} \xrightarrow{\pi_U^*\Phi} \pi_U^*\pi_{U*}(\mathcal{F}|_U) \longrightarrow \mathcal{F}|_U$$

determines a morphism $f : U \to Q$ to the quot scheme such that there is a commutative diagram

$$\begin{array}{ccc} \mathcal{O}^N_{U\times X} & \xrightarrow{q_u} & \mathcal{F}|_U \\ \downarrow \mathrm{Id} & & \downarrow \cong \\ \mathcal{O}^N_{U\times X} & \xrightarrow{f^*q_Q} & f^*\mathcal{U} \end{array}$$

In particular $\mathcal{F}|_U \cong f^*\mathcal{U}$ and, as $\mathcal{F}$ is a family of (semi)stable vector bundles, the morphism $f : U \to Q$ factors via $R^{(s)s}$. □

These families $\mathcal{U}^{(s)s}$ over $R^{(s)s}$ are not universal families as the morphisms described above are not unique: if we take $S = \operatorname{Spec} k$ and $\mathcal{E}$ to be a (semi)stable locally free sheaf, then different choices of isomorphism $H^0(X,\mathcal{E}) \cong k^N$ give rise to different points in $R^{(s)s}$.

Any two choices of the above isomorphism are related by an element in the general linear group GL_N and so it is natural to mod out by the action of this group.

Lemma 8.49. *There is an action of* GL_N *on* $Q := \mathrm{Quot}^{n,d}_X(\mathcal{O}^N_X)$ *such that the orbits in* $R^{(s)s}$ *are in bijective correspondence with the isomorphism classes of (semi)stable locally free sheaves on* X *with invariants* (n, d).

Proof. We claim there is a (left) action

$$\sigma : \mathrm{GL}_N \times Q \to Q$$

which on k-points is given by

$$g \cdot (\mathcal{O}^N_X \xrightarrow{q} \mathcal{E}) = (\mathcal{O}^N_X \xrightarrow{g^{-1}} \mathcal{O}^N_X \xrightarrow{q} \mathcal{E}).$$

To construct the action morphism it suffices to give a family over $\mathrm{GL}_N \times Q$ of quotients of $\mathcal{O}^N_X$ with invariants (n, d). The inverse map on the group $i^{-1} : \mathrm{GL}_N \to \mathrm{GL}_N$ determines a universal inversion which is a sheaf isomorphism

$$\tau : k^N \otimes \mathcal{O}_{\mathrm{GL}_N} \to k^N \otimes \mathcal{O}_{\mathrm{GL}_N}. \tag{7}$$

Let $q_Q : k^N \otimes \mathcal{O}_{Q\times X} \to \mathcal{U}$ denote the universal quotient homomorphism on $Q \times X$. Then the action $\sigma : \mathrm{GL}_N \times Q \to Q$ is determined by the following family of quotient maps over $\mathrm{GL}_N \times Q$

$$k^N \otimes \mathcal{O}_{\mathrm{GL}_N\times Q\times X} \xrightarrow{p^*_{\mathrm{GL}_N}\tau} k^N \otimes \mathcal{O}_{\mathrm{GL}_N\times Q\times X} \xrightarrow{(p_{Q\times X})^*q_Q} p^*_{Q\times X}\mathcal{U}$$

where $p_{\mathrm{GL}_N} : \mathrm{GL}_N \times Q \times X \to \mathrm{GL}_N$ and $p_{Q\times X} : \mathrm{GL}_N \times Q \times X \to Q \times X$ denote the projection morphisms.

From the definition of $R^{(s)s}$, we see these subschemes are preserved by the action. Consider quotient sheaves $q_\mathcal{E} : \mathcal{O}^N_X \twoheadrightarrow \mathcal{E}$ and $q_\mathcal{E} : \mathcal{O}^N_X \twoheadrightarrow \mathcal{F}$ in $R^{(s)s}$. If $g\cdot q_E \sim q_F$, then there is an isomorphism $\mathcal{E} \cong \mathcal{F}$ which fits into a commutative square, and so $\mathcal{E}$ and $\mathcal{F}$ are isomorphic. Conversely, if $\mathcal{E} \cong \mathcal{F}$, then there is an induced isomorphism $\phi : H^0(\mathcal{E}) \cong H^0(\mathcal{F})$. The composition

$$k^N \xrightarrow{H^0(q_\mathcal{E})} H^0(\mathcal{E}) \xrightarrow{\phi} H^0(\mathcal{F}) \xrightarrow{H^0(q_\mathcal{F})^{-1}} k^N$$

is an isomorphism which determines a point $g \in \mathrm{GL}_N$ such that $g \cdot q_E \sim q_F$. □

Remark 8.50. In particular, any coarse moduli space for (semi)stable vector bundles is constructed as a categorical quotient of the GL_N-action on $R^{(s)s}$. Furthermore, if there is an orbit space quotient of the GL_N-action on $R^{(s)s}$, then this is a coarse moduli space. In fact, as the diagonal $\mathbb{G}_m \subset \mathrm{GL}_N$ acts trivially on the Quot scheme, we do not lose anything by instead working with the SL_N-action.

Finally, we would like to linearise the action to construct a categorical quotient via GIT. There is a natural family of invertible sheaves on the Quot scheme arising from Grothendieck's embedding of the Quot scheme into the Grassmannians: for sufficiently large m, we have a closed immersion

$$Q = \mathrm{Quot}_X^{n,d}(\mathcal{O}_X^N) \hookrightarrow \mathrm{Gr}(H^0(\mathcal{O}_X^N(m)), M) \hookrightarrow \mathbb{P} := \mathbb{P}(\wedge^M H^0(\mathcal{O}_X^N(m))^\vee)$$

where $M = mr+d+r(1-g)$. We let $\mathcal{L}_m$ denote the pull back of $\mathcal{O}_\mathbb{P}(1)$ to the Quot scheme via this closed immersion. There is a natural linear action of SL_N on $H^0(\mathcal{O}_X^N(m)) = (k^N \otimes H^0(\mathcal{O}_X(m))$, which induces a linear action of SL_N on $\mathbb{P}(\wedge^M H^0(\mathcal{O}_X^N(m))^\vee)$; hence, $\mathcal{L}_m$ admits a linearisation of the SL_N-action.

We can define the linearisation $\mathcal{L}_m$ using the universal quotient sheaf $\mathcal{U}$ on $Q \times X$: we have

$$\mathcal{L}_m = \det(\pi_{Q*}(\mathcal{U} \otimes \pi_X^*\mathcal{O}_X(m)))$$

where $\pi_X : Q \times X \to X$ and $\pi_Q : Q \times X \to Q$ are the projection morphisms. Furthermore, the universal quotient sheaf $\mathcal{U}$ admits a SL_N-linearisation: we have equivalent families of quotients sheaves over $\mathrm{SL}_N \times Q$ given by

$$k^N \otimes \mathcal{O}_{\mathrm{SL}_N \times Q \times X} \xrightarrow{(\sigma\times \mathrm{id}_X)^* q_Q} (\sigma \times \mathrm{id}_X)^*\mathcal{U}$$

and

$$k^N \otimes \mathcal{O}_{\mathrm{SL}_N \times Q \times X} \xrightarrow{p^*_{\mathrm{SL}_N}\tau} k^N \otimes \mathcal{O}_{\mathrm{SL}_N \times Q \times X} \xrightarrow{p^*_{Q\times X} q_Q} p^*_{Q\times X}\mathcal{U},$$

where $q_Q : k^N \otimes \mathcal{O}_{Q\times X} \to \mathcal{U}$ denotes the universal quotient homomorphism, $\sigma : \mathrm{SL}_N \times Q \to Q$ denotes the group action, $p_{Q\times X}$ and p_{SL_N} denote the projections from $\mathrm{SL}_N \times Q \times X$ to the relevant factor and τ is the isomorphism given in (7). Hence, there is an isomorphism

$$\Phi : (\sigma \times \mathrm{id}_X)^*\mathcal{U} \to (p_{Q\times X})^*\mathcal{U}$$

satisfying the cocycle condition, which gives a linearisation of the SL_N-action on $\mathcal{U}$. For m sufficiently large, $\mathcal{L}_m$ is ample and admits an SL_N-linearisation, as the construction of $\mathcal{L}_m$ commutes with base change for m sufficiently large. Hence, at $q : \mathcal{O}_X^N \to \mathcal{F}$ in Q, the fibre of the of the associated line bundle L_m is naturally isomorphic to an alternating tensor product of exterior powers of the cohomology groups of $\mathcal{F}(m)$:

$$L_{m,q} \cong \det H^*(X, \mathcal{F}(m)) = \bigotimes_{i\geq 0} \det H^i(X, \mathcal{F}(m))^{\otimes(-1)^i}.$$

In fact, by the Castelnuovo–Mumford regularity result explained in the second step of the construction of the quot scheme, for m sufficiently large, we have $H^i(X, \mathcal{F}(m)) = 0$ for all $i > 0$ for all points $q : \mathcal{O}_X^N \to \mathcal{F}$ in Q. Therefore, for m sufficiently large, the fibre at q is

$$L_{m,q} \cong \det H^0(X, \mathcal{F}(m)).$$

8.9. **Analysis of semistability.** Let SL_N act on $Q := \mathrm{Quot}_X^{n,d}(\mathcal{O}_X^N)$ as above. In this section, we will determine the GIT (semi)stable points in Q with respect to the SL_N-linearisation $\mathcal{L}_m$. In fact, we will prove that $Q^{ss}(\mathcal{L}_m) = R^{ss}$ and $Q^s(\mathcal{L}_m) = R^s$.

We will use the Hilbert–Mumford criterion for our stability analysis. Let $q : \mathcal{O}_X^N \twoheadrightarrow \mathcal{F}$ denote a closed point in the Quot scheme Q and let $\lambda : \mathbb{G}_m \to \mathrm{SL}_N$ be a 1-parameter subgroup. We recall that the action is given by

$$\lambda(t) \cdot q : \mathcal{O}_X^N \xrightarrow{\lambda^{-1}(t)} \mathcal{O}_X^N \xrightarrow{q} \mathcal{F}.$$

First of all, we would like to calculate the limit as $t \to 0$. For this, we need some notation. The action of λ^{-1} on $V := k^N$ determines a weight space decomposition

$$V = \oplus_{r\in\mathbb{Z}} V_r$$

where $V_r := \{v \in V : \lambda^{-1}(t)v = t^r v\}$ are zero except for finitely many weights r and, as λ is a 1-PS of the special linear group, we have

$$\sum_{r\in\mathbb{Z}} r \dim V_r = 0. \tag{8}$$

There is an induced ascending filtration of $V = k^N$ given by $V^{\leq r} := \oplus_{s \leq r} V_s$ and an induced ascending filtration of $\mathcal{F}$ given by

$$\mathcal{F}^{\leq r} := q(V^{\leq r} \otimes \mathcal{O}_X)$$

and q induces surjections $q_r : V_r \otimes \mathcal{O}_X \to \mathcal{F}_r := \mathcal{F}_{\leq r}/\mathcal{F}_{\leq r-1}$ which fit into a commutative diagram

$$\begin{array}{ccccc} V^{\leq r-1} \otimes \mathcal{O}_X & \longrightarrow & V^{\leq r} \otimes \mathcal{O}_X & \longrightarrow & V_r \otimes \mathcal{O}_X \\ \downarrow & & \downarrow & & \downarrow q_r \\ \mathcal{F}^{\leq r-1} & \longrightarrow & \mathcal{F}^{\leq r} & \twoheadrightarrow & \mathcal{F}_r. \end{array}$$

Lemma 8.51. *Let $q : \mathcal{O}_X^N \twoheadrightarrow \mathcal{F}$ be a k-point in Q and $\lambda : \mathbb{G}_m \to \mathrm{SL}_N$ be a 1-PS as above; then*

$$\lim_{t \to 0} \lambda(t) \cdot q = \bigoplus_{r \in \mathbb{N}} q_r.$$

Proof. As the quot scheme is projective, there is a unique limit. Therefore, it suffices to construct a family of quotient sheaves of $\mathcal{O}_X^N$ over $\mathbb{A}^1 = \operatorname{Spec} k[t]$

$$\Phi : \mathcal{O}_{X \times \mathbb{A}^1}^N \twoheadrightarrow \mathcal{E}$$

such that $\Phi_t = \lambda(t) \cdot q$ for all $t \neq 0$ and $\Phi_0 = \oplus_r q_r$.

We will use the equivalence between quasi-coherent sheaves on $\mathbb{A}^1$ and $k[t]$-modules. Consider the $k[t]$-module

$$\mathcal{V} := \bigoplus_r V^{\leq r} \otimes_k t^r k$$

with action given by $t \cdot (v^{\leq r} \otimes t^r) = v^{\leq r} \otimes t^{r+1} \in V^{\leq r+1} \otimes r$, which works as the filtration is increasing. Since the filtration on V is zero for sufficiently small r and stabilises to V for sufficiently large r: there is an integer R such that $V^{\leq r} = 0$ for all $r \leq R$ and so $\mathcal{V} \subset V \otimes_k t^R k[t]$; hence, $\mathcal{V}$ is coherent. The 1-PS λ^{-1} determines a sheaf homomorphism over $\mathbb{A}^1$

$$\gamma : V \otimes_k k[t] \to \mathcal{V} := \bigoplus_r V^{\leq n} \otimes_k t^r k$$

given by $v \otimes t^s = \sum_r v_r \otimes t^s \mapsto \sum_r v_r \otimes t^{r+s}$, where $v_r \in V_r$ and so, as s is non-negative, $v_r \in V^{\leq r+s}$. By construction, $\gamma|_{V_r} = t^r \cdot \mathrm{Id}_{V_r}$. We leave it to the reader to write down an inverse which shows that γ is an isomorphism.

Over $\operatorname{Spec} k$, the k-module $k[t]$ determines a quasi-coherent (but not coherent) sheaf, we let $\mathcal{O}_X \otimes_k k[t]$ denote the pullback of this quasi-coherent sheaf to X. Then to describe coherent sheaves on $X \times \mathbb{A}^1$, we will use the equivalence between the category of quasi-coherent sheaves on $X \times \mathbb{A}^1$ and the category of $\mathcal{O}_X \otimes_k k[t]$-modules. Using the filtration $\mathcal{F}^{\leq r}$ we construct a quasi-coherent sheaf $\mathcal{E}$ over $X \times \mathbb{A}^1$ as follows. Let

$$\mathcal{E} := \bigoplus_n \mathcal{F}^{\leq r} \otimes_k t^r k \subset \mathcal{F} \otimes_k t^R k[t]$$

for R as above. The action of t is identical to the action of t on $\mathcal{V}$ given above. Furthermore, we have the above inclusion as the filtration is zero for r sufficiently small and stabilises to $\mathcal{F}$ for r sufficiently large; in particular $\mathcal{E}$ is a coherent sheaf on $X \times \mathbb{A}^1$.

The homomorphism q induces a surjective homomorphism of coherent sheaves over $X \times \mathbb{A}^1$

$$q_{\mathbb{A}^1} : \bigoplus_n V^{\leq r} \otimes_k t^r k \to \mathcal{E} := \bigoplus_n \mathcal{F}^{\leq r} \otimes_k t^r k$$

and we define our family of quotient sheaves over $X \times \mathbb{A}^1$ to be $\Phi := q_{\mathbb{A}^1} \circ \pi_{\mathbb{A}^1}^* \gamma$, where $\pi_{\mathbb{A}^1} : X \times \mathbb{A}^1 \to \mathbb{A}^1$ is the projection.

If we restrict Φ to $\mathbb{A}^1 - \{0\}$, then this corresponds to inverting the variable t. In this case, we have an commutative diagram

$$\begin{array}{ccc} \mathcal{O}_X^N \otimes_k k[t,t^{-1}] & \xrightarrow{\pi^*_{\mathbb{A}^1-\{0\}}(\gamma)} & \mathcal{O}_X^N \otimes_k k[t,t^{-1}] \\ \downarrow{\scriptstyle \Phi|_{\mathbb{A}^1-\{0\}}} & & \downarrow{\scriptstyle q\otimes \mathrm{id}} \\ \mathcal{E}\otimes_k k[t,t^{-1}] & \xrightarrow{\cong} & \mathcal{F}\otimes_k k[t,t^{-1}] \end{array}$$

where γ gives the action of λ^{-1}; hence $[\Phi_t] = [\lambda(t)\cdot q]$ for all $t \neq 0$. Let $i : 0 \hookrightarrow \mathbb{A}^1$ denote the closed immersion; then the composition i_*i^* kills the action of t. We have

$$i_*i^*(\mathcal{E}) = \mathcal{E}/t\cdot\mathcal{E} = (\bigoplus_{r\geq R}\mathcal{F}^{\leq r}\otimes_k t^r k)/(\bigoplus_{r\geq R}\mathcal{F}^{\leq r}\otimes_k t^{r+1}k) = \bigoplus_r \mathcal{F}_r\otimes_k t^r k,$$

with trivial action by t. Hence, the restriction of $\mathcal{E}$ to the special fibre $0\in\mathbb{A}^1$ is $\mathcal{E}_0 = \oplus_r\mathcal{F}_r$ and this completes the proof of the lemma. □

Lemma 8.52. *Let $\lambda : \mathbb{G}_m \to \mathrm{SL}_N$ be a 1-PS and $q : \mathcal{O}_X^N \twoheadrightarrow \mathcal{F}$ be a k-point in Q. Then using the notation introduced above for the weight decomposition of λ^{-1} acting on $V = k^N$, we have*

$$\mu^{\mathcal{L}_m}(q,\lambda) = -\sum_{r\in\mathbb{Z}} rP(\mathcal{F}_r,m) = \sum_{r\in\mathbb{Z}}\left(P(\mathcal{F}^{\leq r},m) - \frac{\dim V^{\leq r}}{N}P(\mathcal{F},m)\right).$$

Proof. By definition, this Hilbert–Mumford weight is negative the weight of the action of $\lambda(\mathbb{G}_m)$ on the fibre of the line bundle L_m over the fixed point $q' := \bigoplus_{r\in\mathbb{N}} q_r = \lim_{t\to 0}\lambda(t)\cdot q$. The fibre over this fixed point is

$$L_{m,q'} = \bigotimes_{r\in\mathbb{Z}}\det H^*(X,\mathcal{F}_r(m)),$$

where $H^*(X,F_r(m))$ denotes the complex defining the cohomology groups $H^i(X,F_r(m))$ for $i=1,2$ and the determinant of this complex is the 1-dimensional vector space

$$\bigotimes_{i\geq 0}\det H^i(X,\mathcal{F}_r(m))^{\otimes(-1)^i} = \wedge^{h^0(X,\mathcal{F}_r(m))}H^0(X,\mathcal{F}_r(m))\otimes\wedge^{h^1(X,\mathcal{F}_r(m))}H^1(X,\mathcal{F}_r(m))^\vee.$$

The virtual dimension of $H^*(X,\mathcal{F}_r(m))$ is given by the alternating sums of the dimensions of the cohomology groups of $\mathcal{F}_r(m)$ and thus is equal to $P(\mathcal{F}_r,m)$. Since λ acts with weight r on $\mathcal{F}_r$, it also acts with weight r on $H^i(X,\mathcal{F}_r(m))$. Therefore, the weight of λ acting on $\det H^*(X,\mathcal{F}_r(m))$ is $rP(\mathcal{F}_r,m)$. The first equality then follows from this and the definition of the Hilbert–Mumford weight.

For the second equality, we recall that as λ is a 1-PS of SL_N, we have a relation (8) and by definition, we have $\dim V_r = \dim V^{\leq r} - \dim V^{\leq r-1}$. Furthermore, as $\mathcal{F}_r := \mathcal{F}^{\leq r}/\mathcal{F}^{\leq r-1}$, we have

$$P(\mathcal{F}_r) = P(\mathcal{F}^{\leq r}) - P(\mathcal{F}^{\leq r-1}).$$

The second equality then follows from these observations. □

Remark 8.53. The second expression for the Hilbert–Mumford weight is of greater use to us, as it is expressed in terms of subsheaves of $\mathcal{F}$. The number of distinct weights for the λ^{-1}-action on $V = k^N$, tells us the number of jumps in the filtration of $\mathcal{F}$.

If we suppose there are only two weights $r_1 < r_2$ for λ, then we get a filtration of $\mathcal{F}$ by a single subsheaf $0 \subsetneq \mathcal{F}' \subsetneq \mathcal{F}$:

$$0 = \cdots = 0 = \mathcal{F}^{\leq r_1 - 1} \subsetneq \mathcal{F}' := \mathcal{F}^{\leq r_1} = \cdots = \mathcal{F}^{\leq r_2-1} \subsetneq \mathcal{F}^{\leq r_2} = \mathcal{F} = \cdots\mathcal{F}.$$

Let $V' := V^{\leq r_1}$; then we have

$$\mu^{\mathcal{L}_m}(q,\lambda) = (r_2 - r_1)\left(P(\mathcal{F}',m) - \frac{\dim V'}{\dim V}P(\mathcal{F},m)\right),$$

where $r_2 - r_1 > 0$.

Proposition 8.54. *Let $q : \mathcal{O}_X^N \twoheadrightarrow \mathcal{F}$ be a k-point in Q. Then $q \in Q^{(s)s}(\mathcal{L}_m)$ if and only if for all subspaces $0 \neq V' \subsetneq V = K^N$ we have an inequality*

$$\frac{\dim V'}{P(\mathcal{F}', m)} \ (\leq) \ \frac{\dim V}{P(\mathcal{F}, m)} \tag{9}$$

where $\mathcal{F}' := q(V' \otimes \mathcal{O}_X) \subset \mathcal{F}$.

Proof. Suppose the inequality (9) holds for all subspaces V'. We will show q is (semi)stable using the Hilbert–Mumford criterion. For any 1-PS $\lambda : \mathbb{G}_m \to \mathrm{GL}_N$, there are finitely many weights $r_1 < r_2 < \cdots < r_s$ for the λ^{-1}-action on $V = K^N$, which give rise to subspaces $V^{(i)} = V^{\leq r_i} \subset V$ and subsheaves $\mathcal{F}^{(i)} := q(V^{(i)} \otimes \mathcal{O}_X) \subset \mathcal{F}$. Furthermore, we have $\mathcal{F}^{\leq n} = \mathcal{F}^{(i)}$ for $r_i \leq n < r_{i+1}$. Therefore, by Lemma 8.52, we have

$$\mu^{\mathcal{L}_m}(q, \lambda) = \sum_{i=1}^{s-1} (r_{i+1} - r_i) \left(P(\mathcal{F}^{(i)}, m) - \frac{\dim V^{(i)}}{\dim V} P(\mathcal{F}, m) \right) \ (\geq) 0.$$

Conversely, if there exists a subspace $0 \subsetneq V' \subsetneq k^N$ for which the inequality (9) does not hold (or holds with equality respectively), then we can construct a 1-PS λ with two weights $r_1 > r_2$ such that $V^{(1)} = V'$ and $V^{(2)} = V$. Then

$$\mu^{\mathcal{L}_m}(q, \lambda) = (r_2 - r_1) \left(P(\mathcal{F}', m) - \frac{\dim V'}{\dim V} P(\mathcal{F}, m) \right) < 0 \quad (\text{resp. } \mu^{\mathcal{L}_m}(q, \lambda) = 0);$$

that is q is unstable for the SL_N-action with respect to $\mathcal{L}_m$. □

Remark 8.55. For m sufficiently large $P(\mathcal{F}', m)$ and $P(\mathcal{F}, m)$ are both positive; thus, we can multiply by the denominators in the inequality (9) to obtain an equivalent inequality

$$(\dim V' \operatorname{rk} \mathcal{F})m + (\dim V')(\deg \mathcal{F} + \operatorname{rk} \mathcal{F}(1-g)) (\leq) (\dim V \operatorname{rk} \mathcal{F}')m + (\dim V)(\deg \mathcal{F}' + \operatorname{rk} \mathcal{F}'(1-g)).$$

An inequality of polynomials in a variable m holds for all m sufficiently large if and only if there is an inequality of their leading terms. If $\operatorname{rk} \mathcal{F}' \neq 0$, then the leading term of the polynomial $P(\mathcal{F}')$ is $\operatorname{rk} \mathcal{F}'$ and if $\operatorname{rk} \mathcal{F}' = 0$, then the Hilbert polynomial of $\mathcal{F}'$ is constant. Therefore, there exists M (depending on $\mathcal{F}$ and $\mathcal{F}'$) such that for $m \geq M$

$$\operatorname{rk} \mathcal{F}' > 0 \quad \text{and} \quad \frac{\dim V'}{\operatorname{rk} \mathcal{F}'} (\leq) \frac{\dim V}{\operatorname{rk} \mathcal{F}} > 0 \quad \iff \quad \frac{\dim V'}{P(\mathcal{F}', m)} (\leq) \frac{\dim V}{P(\mathcal{F}, m)}.$$

In fact, M only depends on $P(\mathcal{F})$ and $P(\mathcal{F}')$. Moreover, as the subspaces $0 \neq V' \subsetneq V = k^N$ form a bounded family (they are parametrised by a product of Grassmannians) and the quotient sheaves $q : \mathcal{O}_X^N \to \mathcal{F}$ form a bounded family (they are parametrised by the Quot scheme Q), the family of sheaves $\mathcal{F}' = q(V' \otimes \mathcal{F})$ are also bounded. Therefore, there are only finitely many possibilities for $P(\mathcal{F}')$. Hence, there exists M such that for $m \geq M$ the following holds: for any $q : \mathcal{O}_X^N \to \mathcal{F}$ in Q and $0 \neq V' \subsetneq V = k^N$, we have

$$\operatorname{rk} \mathcal{F}' > 0 \quad \text{and} \quad \frac{\dim V'}{\operatorname{rk} \mathcal{F}'} (\leq) \frac{\dim V}{\operatorname{rk} \mathcal{F}} > 0 \quad \iff \quad \frac{\dim V'}{P(\mathcal{F}', m)} (\leq) \frac{\dim V}{P(\mathcal{F}, m)}$$

where $\mathcal{F}' = q(V' \otimes \mathcal{O}_X)$.

Remark 8.56. Let $q : \mathcal{O}_X^N \twoheadrightarrow \mathcal{F} \in Q(k)$. Then we note

(1) If $0 \subsetneq V' \subset V = k^N$ and $\mathcal{F}' := q(V' \otimes \mathcal{O}_X)$, then $V' \subset H^0(q)^{-1}(H^0(\mathcal{F}'))$,
(2) If $\mathcal{G} \subset \mathcal{F}$ and $V' = H^0(q)^{-1}(H^0(\mathcal{G}))$, then $q(V' \otimes \mathcal{O}_X) \subset \mathcal{G}$.

Using these two remarks, we obtain a corollary to Proposition 8.54.

Corollary 8.57. *There exists M such that for $m \geq M$ and for a k-point $q : \mathcal{O}_X^N \twoheadrightarrow \mathcal{F}$ in Q, the following statements are equivalent:*

(1) q is GIT (semi)stable for SL_N-acting on Q with respect to $\mathcal{L}_m$;
(2) for all subsheaves $\mathcal{F}' \subset \mathcal{F}$ with $V' := H^0(q)^{-1}(H^0(\mathcal{F}')) \neq 0$, we have $\operatorname{rk} \mathcal{F}' > 0$ and

$$\frac{\dim V'}{\operatorname{rk} \mathcal{F}'} (\leq) \frac{\dim V}{\operatorname{rk} \mathcal{F}}.$$

In the remaining part of this section, we prove some additional results concerning semistability of vector bundles, which we will eventually relate to GIT semistability.

Lemma 8.58. *Let n and d be fixed such that $d > n^2(2g-2)$. Then a locally free sheaf $\mathcal{F}$ of rank n and degree d is (semi)stable if for all $\mathcal{F}' \subset \mathcal{F}$ we have*

$$\frac{h^0(X,\mathcal{F}')}{\operatorname{rk}\mathcal{F}'}(\leq)\frac{\chi(\mathcal{F})}{\operatorname{rk}\mathcal{F}}. \tag{10}$$

Proof. Suppose $\mathcal{F}$ is not semistable; then there exists a subsheaf $\mathcal{F}' \subset \mathcal{F}$ with $\mu(\mathcal{F}') > \mu(\mathcal{F})$. In fact, we can assume $\mathcal{F}'$ is semistable (if not, there is a vector subbundle F'' of F' with larger slope, and so we can replace $\mathcal{F}'$ with $\mathcal{F}''$). Then

$$\deg \mathcal{F}' > \frac{d}{n}\operatorname{rk}\mathcal{F}' > \frac{d}{n} > n(2g-2) > \operatorname{rk}\mathcal{F}'(2g-2).$$

Then it follow from Lemma 8.36 that $H^1(X,\mathcal{F}') = 0$. However, in this case

$$\frac{h^0(X,\mathcal{F}')}{\operatorname{rk}\mathcal{F}'} = \mu(\mathcal{F}') + (1-g) > \mu(\mathcal{F}) + (1-g) = \frac{\chi(\mathcal{F})}{\operatorname{rk}\mathcal{F}}$$

which contradicts (10). Furthermore, if the inequality (10) holds with a strict inequality and $\mathcal{F}$ is not stable, then we can apply the above argument to any subsheaf $\mathcal{F}' \subset \mathcal{F}$ with the same slope as $\mathcal{F}$ and get a contradiction. □

The converse to this lemma also holds for d sufficiently large, as we will demonstrate in Proposition 8.61; however, first we need some preliminary results.

Lemma 8.59. *(Le Potier bounds) For any semi-stable locally free sheaf $\mathcal{F}$ of rank n and slope μ, we have*

$$\frac{h^0(X,\mathcal{F})}{n} \leq [\mu+1]_+ := \max(\mu+1, 0)$$

Proof. If $\mu < 0$, then $H^0(X,\mathcal{F}) = 0$. For $\mu \geq 0$, we proceed by induction on the degree d of $\mathcal{F}$. If we assume the lemma holds for all degrees less than d, then we can consider the short exact sequence

$$0 \to \mathcal{F}(-x) \to \mathcal{F} \to F_x \to 0$$

where $x \in X$. By considering the associated long exact sequence, we see that

$$h^0(X,\mathcal{F}) \leq h^0(X,\mathcal{F}(-x)) + n.$$

Since $\mu(\mathcal{F}) = \mu(\mathcal{F}(-x)) + 1$, the result follows by applying the inductive hypothesis to $\mathcal{F}(-x)$. □

We recall that any vector bundle has a unique maximal destabilising sequence of vector subbundles, known as its Harder–Narasimhan filtration (cf. Definition 8.32).

Corollary 8.60. *Let $\mathcal{F}$ be a locally free sheaf of rank n and slope μ with Harder–Narasimhan filtration*

$$0 = \mathcal{F}^{(0)} \subsetneq \mathcal{F}^{(1)} \subsetneq \cdots \subsetneq \mathcal{F}^{(s)} = \mathcal{F}$$

i.e. $\mathcal{F}_i = \mathcal{F}^{(i)}/\mathcal{F}^{(i-1)}$ are semistable and $\mu_{\max}(\mathcal{F}) = \mu(\mathcal{F}_1) > \cdots > \mu(\mathcal{F}_s) = \mu_{\min}(\mathcal{F})$; then

$$\frac{h^0(X,\mathcal{F})}{n} \leq \sum_{i=1}^{s} \frac{\operatorname{rk}\mathcal{F}_i}{n}[\mu(\mathcal{F}_i)+1]_+ \leq \left(1-\frac{1}{n}\right)[\mu+1]_+ + \frac{1}{r}[\mu_{\min}(\mathcal{F})+1]_+.$$

Proposition 8.61. *Let n and d be fixed such that $d > gn^2 + n(2g-2)$. Let $\mathcal{F}$ be a semistable locally free sheaf over X with rank r and degree d. Then for all non-zero subsheaves $0 \neq \mathcal{F}' \subsetneq \mathcal{F}$, we have*

$$\frac{h^0(X,\mathcal{F}')}{\operatorname{rk}\mathcal{F}'} \leq \frac{\chi(\mathcal{F})}{\operatorname{rk}\mathcal{F}}$$

and if equality holds, then $h^1(X,\mathcal{F}') = 0$ and $\mu(\mathcal{F}') = \mu(\mathcal{F})$.

Proof. Let $\mu = d/n$ denote the slope and pick a constant C such that $2g-2 < C < \mu - gn$ (this is possible, as $\mu - gn > 2g-2$ by our choice of d). We will prove the following statements for subsheaves $\mathcal{F}' \subset \mathcal{F}$.

(1) If $\mu_{\min}(\mathcal{F}') \leq C$, then
$$\frac{h^0(X,\mathcal{F}')}{\operatorname{rk}\mathcal{F}'} < \frac{\chi(\mathcal{F})}{\operatorname{rk}\mathcal{F}}.$$
(2) If $\mu_{\min}(\mathcal{F}') > C$, then $h^1(X,\mathcal{F}) = 0$ and
$$\frac{h^0(X,\mathcal{F}')}{\operatorname{rk}\mathcal{F}'} \leq \frac{\chi(\mathcal{F})}{\operatorname{rk}\mathcal{F}}$$
and if equality holds, then $\mu(\mathcal{F}') = \mu(\mathcal{F})$.

We can apply Corollary 8.60 to a subsheaf $\mathcal{F}' \subset \mathcal{F}$ to obtain the bound
$$\frac{h^0(X,\mathcal{F}')}{\operatorname{rk}\mathcal{F}'} \leq \left(1 - \frac{1}{n}\right)[\mu+1]_+ + \frac{1}{n}[\mu_{\min}(\mathcal{F}')+1]_+.$$
If $\mu_{\min}(\mathcal{F}') \leq C$, then
$$\frac{h^0(X,\mathcal{F}')}{\operatorname{rk}\mathcal{F}'} \leq \left(1 - \frac{1}{n}\right)(\mu+1) + \frac{1}{n}(C+1) < \mu + 1 + g = \frac{\chi(\mathcal{F})}{\operatorname{rk}\mathcal{F}}$$
by our choice of C, which proves (1).

For (2), suppose $\mu_{\min}(\mathcal{F}') > C$; then we claim that $H^1(X,\mathcal{F}') = 0$. To prove the claim, it suffices to show that $H^1(X,\mathcal{F}'_i) = 0$, where $\mathcal{F}'_i$ are the semistable subquotients appearing in the Harder–Narasimhan filtration of $\mathcal{F}'$. For each $\mathcal{F}'_i$, we have
$$\mu(\mathcal{F}'_i) \geq \mu_{\min}(\mathcal{F}') > C > 2g-2.$$
Hence, $\deg \mathcal{F}'_i > \operatorname{rk}\mathcal{F}'_i(2g-2)$ and, as $\mathcal{F}'_i$ is semistable, we conclude that $H^1(X,\mathcal{F}'_i) = 0$ by Lemma 8.36. Then by semistability of $\mathcal{F}$, we have $\mu(\mathcal{F}') \leq \mu(\mathcal{F})$; hence
$$\frac{h^0(X,\mathcal{F}')}{\operatorname{rk}\mathcal{F}'} = \mu(\mathcal{F}') + 1 - g \leq \mu(\mathcal{F}) + 1 - g = \frac{\chi(\mathcal{F})}{\operatorname{rk}\mathcal{F}}$$
with equality only if $\mu(\mathcal{F}') = \mu(\mathcal{F})$. □

Remark 8.62. Proposition 8.61 and Lemma 8.58 together say, for sufficiently large degree d, that (semi)stability of a locally free sheaf $\mathcal{F}$ over X is equivalent to
$$\frac{h^0(X,\mathcal{F}')}{\operatorname{rk}\mathcal{F}'} (\leq) \frac{h^0(X,\mathcal{F})}{\operatorname{rk}\mathcal{F}}$$
for all non-zero proper subsheaves $\mathcal{F}' \subset \mathcal{F}$. This result was first proved by Le Potier for curves (see [35] Propositions 7.1.1 and 7.1.3) and was later generalised to higher dimensions by Simpson [39].

We recall that we defined open subschemes $R^{(s)s} \subset Q := \operatorname{Quot}_X^{n,d}(\mathcal{O}^N)$ whose k-points are quotient sheaves $q : \mathcal{O}_X^N \to \mathcal{F}$ such that $\mathcal{F}$ is a locally free (semi)stable sheaf and $H^0(q)$ is an isomorphism. The following theorem shows that GIT semistability for SL_N acting on Q coincides with vector bundle semistability (provided d and m are sufficiently large).

Theorem 8.63. *Let n and d be fixed such that $d > \max(n^2(2g-2), gn^2 + n(2g-2))$. Then there exists a natural number $M > 0$ such that for all $m \geq M$, we have*
$$Q^{ss}(\mathcal{L}_m) = R^{ss} \qquad \text{and} \qquad Q^s(\mathcal{L}_m) = R^s.$$

Proof. We pick M as required by Corollary 8.57. Since these subschemes are all open subschemes of Q, it suffices to check these equalities of schemes on k-points.

First, let $q : \mathcal{O}_X^N \to \mathcal{F}$ be a k-point in R^{ss}; that is, $\mathcal{F}$ is a semistable locally free sheaf and $H^0(q) : V \to H^0(X,\mathcal{F})$ is an isomorphism. We will show that q is GIT semistable using Corollary 8.57. Let $\mathcal{F}' \subset \mathcal{F}$ be a subsheaf with $\operatorname{rk}\mathcal{F}' > 0$ and let $V' := H^0(q)^{-1}(H^0(X,\mathcal{F}')$. As $H^0(q)$ is a isomorphism, we have $\dim V' = h^0(X,\mathcal{F}')$. By Proposition 8.61, we have either

(1) $h^0(X,\mathcal{F}') < \operatorname{rk}\mathcal{F}'\chi(\mathcal{F})/\operatorname{rk}\mathcal{F}$, or

(2) $h^1(X,\mathcal{F}')=0$ and $\mu(\mathcal{F}')=\mu(\mathcal{F})$.

In the first case,

$$\frac{\dim V'}{\operatorname{rk}\mathcal{F}'}=\frac{h^0(X,\mathcal{F}')}{\operatorname{rk}\mathcal{F}'}<\frac{\chi(\mathcal{F})}{\operatorname{rk}\mathcal{F}}=\frac{\dim V}{\operatorname{rk}\mathcal{F}}$$

and in the second case, $\dim V'=h^0(X,\mathcal{F}')=P(\mathcal{F}')$, and we have

$$\frac{\dim V'}{\operatorname{rk}\mathcal{F}'}=\frac{\chi(\mathcal{F}')}{\operatorname{rk}\mathcal{F}'}=\frac{\chi(\mathcal{F})}{\operatorname{rk}\mathcal{F}}=\frac{\dim V}{\operatorname{rk}\mathcal{F}}.$$

Hence $q\in\overline{Q}^{ss}(\mathcal{L}_m)$ by Corollary 8.57. In fact, this argument shows that if, moreover, $\mathcal{F}$ is a stable locally free sheaf, then $q\in\overline{Q}^{s}(\mathcal{L}_m)$, because, in this case, condition (2) is not possible and so we always have a strict inequality. Hence, we have inclusions $R^{(s)s}(k)\subset Q^{(s)s}(\mathcal{L}_m)(k)$.

Suppose that $q:\mathcal{O}_X^N\to\mathcal{F}$ is a k-point in $Q^{(s)s}(\mathcal{L}_m)$; then for every subsheaf $\mathcal{F}'\subset\mathcal{F}$ such that $V':=H^0(q)^{-1}(H^0(X,\mathcal{F}'))$ is non-zero, we have $\operatorname{rk}\mathcal{F}'>0$ and an inequality

$$\frac{\dim V'}{\operatorname{rk}\mathcal{F}'}(\leq)\frac{\dim V}{\operatorname{rk}\mathcal{F}}$$

by Corollary 8.57.

We first observe that $H^0(q):V\to H^0(X,\mathcal{F})$ is injective, as otherwise let K be the kernel, then $\mathcal{F}'=q(K\otimes\mathcal{O}_X)=0$ has rank equal to zero, and so contradicts GIT semistability of q. In fact, we claim that GIT semistability also implies $H^1(X,\mathcal{F})=0$; thus, $\dim H^0(X,\mathcal{F})=\chi(\mathcal{F})=N=\dim V$ and so the injective map $H^0(q)$ is an isomorphism. If $H^1(X,\mathcal{F})\neq 0$, then by Serre duality, there is a non-zero homomorphism $\mathcal{F}\to\omega_X$ whose image $\mathcal{F}''\subset\omega_X$ is an invertible sheaf. We can equivalently phrase the GIT (semi)stability of q in terms of quotient sheaves $\mathcal{F}\twoheadrightarrow\mathcal{F}''$ as giving an inequality

$$\frac{\dim V}{n}\leq\frac{\dim V''}{\operatorname{rk}\mathcal{F}''}$$

where V'' denotes the image of the composition $V\to H^0(X,\mathcal{F})\to H^0(X,\mathcal{F}'')$. We note that $\dim V''\leq g$, as $V''\subset H^0(X,\mathcal{F}'')\subset H^0(X,\omega_X)$. Therefore, GIT semistability would imply

$$\frac{d}{n}+(1-g)\leq g,$$

which contradicts our choice of d. Thus $H^0(q)$ is an isomorphism.

We next claim that $\mathcal{F}$ is locally free. Since we are working over a curve, the claim is equivalent to showing that $\mathcal{F}$ is torsion free. If $\mathcal{F}'\subset\mathcal{F}$ is a torsion subsheaf (i.e. $\operatorname{rk}\mathcal{F}'=0$), then $H^0(X,\mathcal{F}')\neq 0$, as every torsion sheaf has a section, and so this would contradict GIT semistability.

Let $\mathcal{F}'\subset\mathcal{F}$ be a subsheaf and $V':=H^0(q)^{-1}(H^0(X,\mathcal{F}'))$; then by GIT (semi)stability

$$\frac{h^0(X,\mathcal{F}')}{\operatorname{rk}\mathcal{F}'}=\frac{\dim V'}{\operatorname{rk}\mathcal{F}'}(\leq)\frac{\dim V}{\operatorname{rk}\mathcal{F}}=\frac{\chi(\mathcal{F})}{\operatorname{rk}\mathcal{F}}.$$

Hence, $\mathcal{F}$ is (semi)stable by Lemma 8.58. Since also $H^0(q)$ is an isomorphism, we have shown that $q\in R^{(s)s}$. This completes the proof of the opposite inclusion $Q^{(s)s}(\mathcal{L}_m)(k)\subset R^{(s)s}(k)$. □

8.10. **Construction of the moduli space.** Let X be a connected smooth projective curve of genus $g\geq 2$. We fix a rank n and a degree d. In this section, we will give the construction of the moduli space of stable vector bundles on X.

We defined open subschemes $R^{(s)s}\subset Q:=\operatorname{Quot}_X^{n,d}(\mathcal{O}^N)$ (where $N:=d+n(1-g)$) whose k-points are quotients $q:\mathcal{O}_X^N\to\mathcal{F}$ such that $\mathcal{F}$ is (semi)stable and $H^0(q)$ is an isomorphism.

The construction of the moduli space of stable vector bundles is originally due to Seshadri [37]; however, we have not followed his construction (Seshadri uses a different linearisation which embeds the Quot scheme in a product of Grassmannians). Instead, we are following the construction due to Le Potier [35] and Simpson [39], which generalises more naturally to higher dimensions; see Remark 8.70 for some comments on the additional complications for higher dimensional base schemes.

Theorem 8.64. *There is a coarse moduli space $M^s(n,d)$ for moduli of stable vector bundles of rank n and degree d over X that has a natural projective completion $M^{ss}(n,d)$ whose k-points parametrise polystable vector bundles of rank n and degree d.*

Proof. We first construct these spaces for large d and then, by tensoring with invertible sheaves of negative degree, we obtain the moduli spaces for smaller degree d. Hence, we may assume that $d > \max(n^2(2g-2), gn^2 + n(2g-2))$ We linearise the SL_N-action on Q in the invertible sheaf $\mathcal{L}_m$, where m is taken sufficiently large as required for the statement of Theorem 8.63. Then $Q^{(s)s}(\mathcal{L}_m) = R^{(s)s}$ and there is a projective GIT quotient

$$\pi : R^{ss} = Q^{ss}(\mathcal{L}_m) \to Q//_{\mathcal{L}_m}\mathrm{SL}_N =: M^{ss}(n,d)$$

which is a categorical quotient of the SL_N-action on R^{ss} and π restrict to a geometric quotient

$$\pi^s : R^s = Q^s(\mathcal{L}_m) \to Q^s(\mathcal{L}_m)/\mathrm{SL}_N =: M^s(n,d).$$

Furthermore, $R^{(s)s}$ parametrises a family $\mathcal{U}^{(s)s}$ of (semi)stable vector bundles over X of rank n and degree d which has the local universal property and such that two k-points in $R^{(s)s}$ lie in the same orbit if and only if the corresponding vector bundles parametrised by these points are isomorphic; see Lemmas 8.48 and 8.49. By Proposition 3.35, a coarse moduli space is a categorical quotient of the SL_N-action on $R^{(s)s}$ if and only if it is an orbit space. Therefore, as π^s is a categorical quotient which is also an orbit space, $M^s(n,d)$ is a coarse moduli space for stable vector bundles on X of rank n and degree d.

Since the k-points of the GIT quotient parametrise closed orbits, to complete the proof it remains to show that the orbit of $q : \mathcal{O}_X^N \to \mathcal{F}$ in R^{ss} is closed if and only if $\mathcal{F}$ is polystable. If $\mathcal{F}$ is not polystable, then there is a non-split short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

where $\mathcal{F}'$ and $\mathcal{F}''$ are semistable with the same slope as $\mathcal{F}$. In this case, we can find a 1-PS λ such that $\lim_{t\to 0}\lambda(t)\cdot[q] = [\mathcal{O}_X^N \to \mathcal{F}'' \oplus \mathcal{F}']$, which shows that the orbit is not closed. In fact, by repeating this argument one case show that a quotient homomorphism for a semistable sheaf contains a quotient homomorphism for a polystable sheaf in its orbit closure. More precisely, one can define a Jordan–Holder filtration of $\mathcal{F}$ by stable vector bundles of the same slope as $\mathcal{F}$:

$$0 \subsetneq \mathcal{F}_{(1)} \subsetneq \mathcal{F}_{(2)} \subsetneq \cdots \subsetneq \mathcal{F}_{(s)} = \mathcal{F}$$

and then pick out a 1-PS λ which inducing this filtration so that the limit as $t \to 0$ is the associated graded object $\mathrm{gr}_{JH}(\mathcal{F}) := \oplus_i \mathcal{F}_{(i)}/\mathcal{F}_{(i-1)}$. We note that unlike the Harder–Narasimhan filtration, the Jordan–Holder filtration is not unique but the associated graded object is unique. Now suppose that $\mathcal{F}$ is polystable so we have $\mathcal{F} = \oplus \mathcal{F}_i^{\oplus n_i}$ for non-isomorphic stable vector bundles $\mathcal{F}_i$; then we want to show the orbit of q is closed: i.e. for every point $q' : \mathcal{O}_X^N \to \mathcal{F}'$ in the closure of the orbit of q, we have an isomorphism $\mathcal{F} \cong \mathcal{F}'$. Using Theorem 6.13, we can produce a 1-PS λ such that $\lim_{t\to 0}\lambda(t)\cdot q = q'$. This corresponds to a family $\mathcal{E}$ over $\mathbb{A}^1$ of semistable vector bundles such that

$$\mathcal{E}_t \cong \mathcal{F} \quad \text{for } t \neq 0, \quad \text{and} \quad \mathcal{E}_0 = \mathcal{F}'.$$

Since the stable bundles $\mathcal{F}_i$ are simple and any non-zero homomorphism between stable vector bundles of the same slope is an isomorphism, we see that $\dim \mathrm{Hom}(\mathcal{F}_i, \mathcal{F}) = n_i$. As $\mathcal{E}$ is flat over $\mathbb{A}^1$, this dimension function is upper semi-continuous; hence $\dim \mathrm{Hom}(\mathcal{F}_i, \mathcal{F}') =: n_i' \geq n_i$. As $\mathcal{F}_i$ is stable, the evaluation map $e_i : \mathcal{F}_i \otimes \mathrm{Hom}(\mathcal{F}_i, \mathcal{F}') \to \mathcal{F}'$ must be injective. Moreover sum $\sum \mathcal{F}_i^{n_i'} \subset \mathcal{F}'$ is a direct sum as $\mathcal{F}_i \ncong \mathcal{F}_j$ by assumption. By comparing the ranks, we must have $n_i = n_i'$ for all i and $\mathcal{F}' \cong \oplus \mathcal{F}_i^{\oplus n_i} = \mathcal{F}$. □

Proposition 8.65. *The moduli space $M^s(n,d)$ of stable vector bundles is a smooth quasi-projective variety of dimension $n^2(g-1)+1$.*

Proof. We claim that the open subscheme $R^s \subset Q$ is smooth and has dimension $n^2(g-1)+N^2$. To prove this claim, we use the following results concerning the local smoothness and Zariski tangent spaces of the quot scheme: for a k-point $q : \mathcal{O}_X^N \to \mathcal{F}$ of Q, we have

(1) $T_qQ \cong \mathrm{Hom}(\mathcal{K}, \mathcal{F})$, where $\mathcal{K} = \ker q$.
(2) If $\mathrm{Ext}^1(\mathcal{K}, \mathcal{F}) = 0$, then Q is smooth in a neighbourhood of q.

For a proof of these results, see [16] Propositions 2.2.7 and 2.2.8; in fact, the description of the tangent spaces should remind you of the description of the tangent spaces to the Grassmannian. To prove the claim, for $q \in R^s$, we apply $\mathrm{Hom}(-, \mathcal{F})$ to the short exact sequence

$$0 \to \mathcal{K} \to \mathcal{O}_X^N \to \mathcal{F} \to 0$$

to obtain a long exact sequence

$$\cdots \to \mathrm{Hom}(\mathcal{K}, \mathcal{F}) \to \mathrm{Ext}^1(\mathcal{F}, \mathcal{F}) \to \mathrm{Ext}^1(\mathcal{O}_X^N, \mathcal{F}) \to \mathrm{Ext}^1(\mathcal{K}, \mathcal{F}) \to 0.$$

Since $\mathrm{Ext}^1(\mathcal{O}_X^N, \mathcal{F}) = H^1(X, \mathcal{F})^N = 0$ (by our assumption on the degree of d), we see that Q is smooth in a neighbourhood of every point $q \in Q$. To calculate the dimension, we consider the following long exact sequence for $q \in R^s$:

$$0 \to \mathrm{Hom}(\mathcal{F}, \mathcal{F}) \to \mathrm{Hom}(\mathcal{O}_X^N, \mathcal{F}) \to \mathrm{Hom}(\mathcal{K}, \mathcal{F}) \to \mathrm{Ext}^1(\mathcal{F}, \mathcal{F}) \to 0$$

where $\hom(\mathcal{F}, \mathcal{F}) = 1$ as every stable bundle is simple, and $\hom(\mathcal{O}_X^N, \mathcal{F}) = N^2$ as our assumption on d implies $H^1(X, \mathcal{F}) = 0$, and $\mathrm{Ext}^1(\mathcal{F}, \mathcal{F}) = H^1(\mathcal{F}^\vee \otimes \mathcal{F}) = n^2(g-1)+1$ by the Riemann–Roch formula. Hence,

$$\dim R^{ss} = \dim T_qQ = \dim \mathrm{Hom}(\mathcal{K}, \mathcal{F}) = n^2(g-1) + 1 + N^2 - 1 = n^2(g-1) + N^2.$$

Since SL_N acts with only a finite global stabiliser on the smooth quasi-projective variety R^s and the quotient $R^s \to M^s(n, d)$ is geometric, it follows from a deep result concerning étale slices of GIT quotients known as Luna's slice theorem [21], that $M^s(n, d)$ is smooth. Furthermore, we have

$$\dim M^s(n, d) = \dim R^s - \dim \mathrm{SL}_N = n^2(g-1) + 1$$

which completes the proof. □

Remark 8.66. In fact, using deformation theory of vector bundles, one can identify the Zariski tangent space to $M^s(n, d)$ at the isomorphism class $[E]$ of a stable vector bundle E as follows

$$T_{[E]}M^s(n, d) \cong \mathrm{Ext}^1(E, E).$$

The obstruction to $M^s(n, d)$ being smooth is controlled by $\mathrm{Ext}^2(E, E)$, which vanishes as we are working over a curve. The same description holds in higher dimensions, except now this second Ext group could be non-zero and so in general the moduli space is not smooth; see [16] Corollary 4.52.

If the degree and rank are coprime, the notions of semistability and stability coincide; hence, in the coprime case, the moduli space of stable vector bundles of rank r and degree d on X is a smooth projective variety.

Finally, we ask whether this coarse moduli space is ever a fine moduli space. In fact, we see why it is necessary to allow a more general notion of equivalence of families of vector bundles with a twist by a line bundle:

Remark 8.67. Two families $\mathcal{E}$ and $\mathcal{F}$ parametrised by S determine the same morphism to $M^s(n, d)$ if $\mathcal{E} \cong \mathcal{F} \otimes \pi_S^*\mathcal{L}$ for a line bundle $\mathcal{L}$ on S where $\pi_S : S \times X \to S$ is the projection map and, in fact, this is an if and only if statement by [31] Lemma 5.10.

It is a result of Mumford and Newstead, for $n = 2$ [26], and Tjurin [43] in general that the moduli space of stable vector bundles is a fine moduli space for coprime rank and degree.

Theorem 8.68. *If $(n, d) = 1$, then $M^s(n, d) = M^{ss}(n, d)$ is a fine moduli space.*

The idea of the proof is to construct a universal family over this moduli space by descending the universal family $\mathcal{U}$ over $R^s \times X$ to the GIT quotient. For more details, we recommend the exposition given by Newstead [31], Theorem 5.12.

Remark 8.69. If $(n, d) \neq 1$, then Ramanan observes that a fine moduli space for stable sheaves does not exist [36].

Remark 8.70. In this remark, we briefly explain some of the additional complications that arise when studying moduli of vector bundles over a higher dimensional projective base Y.

(1) Instead of fixing just the rank and degree, one must fix higher Chern classes (or the Hilbert polynomial) of the sheaves.
(2) In higher dimensions, torsion free and locally free not longer agree; therefore, rather than working with locally free sheaves, we must enlarge our category to torsion free sheaves in order to get a projective completion of the moduli space of stable sheaves.
(3) As we have seen for curves, slope (semi)stability is equivalent to an inequality of reduced Hilbert polynomials, known as Gieseker (semi)stability
$$\mu(\mathcal{E}') \leq \mu(\mathcal{E}) \iff \frac{P(\mathcal{E}')}{\mathrm{rk}\mathcal{E}'} \leq \frac{P(\mathcal{E})}{\mathrm{rk}\mathcal{E}}.$$
However, in higher dimensions, slope (semi)stability and Gieseker (semi)stability do not coincide: we have
$$\text{slope stable} \implies \text{Gieseker stable} \implies \text{Gieseker semistable} \implies \text{slope semistable.}$$
In higher dimensions, one constructs a moduli space for Gieseker stable torsion free sheaves (or for Gieseker semistable pure sheaves).
(4) Since the Hilbert polynomial is taken with respect to a choice of ample line bundle on Y, the notion of Gieseker (semi)stability also depends on this choice. Over a curve, the Hilbert polynomial of a vector bundle only depends on the degree of the ample line bundle we take and consequently all ample line bundles determine the same notion of semistability. In particular, one can study how the moduli space changes as we vary this ample line bundle on Y.
(5) The Quot scheme is longer smooth, due to the existence of some non-vanishing second Ext groups. In particular, the moduli space of stable torsion free sheaves is no longer smooth in general.
(6) To construct the moduli spaces in higher dimensions, we do not take a GIT quotient of the whole Quot scheme, but rather the closure of R^{ss} in Q. The reason for this, is that there may be semistable points in the quot scheme which are not torsion free (or pure) sheaves;
(7) In higher dimensions, the Le Potier bounds become more difficult to prove; although there are essentially analogous statements.

For the interested reader, we recommend the excellent book of Huybrechts and Lehn [16].

References

1. A. Borel, *Linear algebraic groups*, 2nd ed., Graduate Texts in Mathematics, vol. 126, Springer, 1991.
2. M. Brion, *Introduction to actions of algebraic groups*, Les cours du CIRM **1** (2010), no. 1, 1–22.
3. B. Conrad, *Reductive group schemes*, Notes for SGA3 summer school, Luminy, 2011.
4. I. Dolgachev, *Lectures on invariant theory*, Cambridge University Press, 2003.
5. I. Dolgachev and Y. Hu, *Variation of geometric invariant theory quotients*, P. Math. de L'IHÉS **87** (1998), 5–51.
6. B. Doran and F. Kirwan, *Towards non-reductive geometric invariant theory*, Pure Appl. Math. Q. **3** (2007), 61–105.
7. I. M. Gelfand, M. Kapranov, and A. V. Zelevinsky, *Discriminants, resultants and multidimensional determinants*, Birkhäuser, 1994.
8. U. Görtz and T. Wedhorn, *Algebraic geometry I: schemes with examples and exercises*, Vieweg/Teubner, 2010.
9. A. Grothendieck, *Fondements de la g éométrie algébriques*, Extraits du Séminaire Bourbaki, 1957–62.
10. ______, *Techniques de construction et théorèmes d'existence en géométrie algébrique IV: Les schémas de Hilbert*, Soc. Math. France **221** (1995), 249–276.
11. A. Grothendieck and M. Dezamure, *Séminaire de Géométrie Algébrique du Bois Marie - 1962-64 - schémas en groupes - (SGA 3) - vol. 1-3*, Lecture notes in mathematics, Springer-Verlag, 1970.
12. W. J. Haboush, *Reductive groups are geometrically reductive*, Ann. of Math. **102** (1975), 67–83.
13. G. Harder and M. S. Narasimhan, *On the cohomology groups of moduli spaces of vector bundles on curves*, Math. Annalen **212** (1975), 215–248.
14. R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, Springer, 1977.
15. J. E. Humphreys, *Linear algebraic groups*, Graduate texts in Mathematics, Springer, 1981.
16. D. Huybrechts and M. Lehn, *The geometry of moduli spaces of sheaves*, Aspects of Mathematics, Vieweg, 1997.
17. N. Iwahori and M. Matsumo, *On some Bruhart decompositions and the structure of the Hecke map of p-adic Chevalley groups*, Publ. I.H.E.S. **25** (1965), 5–48.
18. M. Kashiwara and P. Schapira, *Categories and sheaves: An introduction to ind-objects and derived categories*, A series of comprehensive studies in mathematics, vol. 332, Springer-Verlag, 2006.
19. F. Kirwan, *Complex algebraic curves*, LMS Student Texts, no. 23, Cambridge University Press, 1992.
20. K. Kodaira and D. C. Spencer, *On deformations of complex analytic structures*, Ann. of Math. **67** (1958), 328–466.
21. D. Luna, *Slices étales*, Bull. Soc. Math. de France **33** (1973), 81–105.
22. J. Milne, *Algebraic groups*, e-book: www.jmilne.org/math/CourseNotes/iAG.pdf.
23. ______, *The basic theory of affine group schemes*, e-book: www.jmilne.org/math/CourseNotes/AGS.pdf.
24. S. Mukai, *An introduction to invariants and moduli*, Cambridge studies in advanced mathematics, Cambridge University Press, 2003.
25. D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, third ed., Springer, 1993.
26. D. Mumford and P. E. Newstead, *Periods of moduli spaces of bundles on curves*, Amer. J. Math. **90** (1968), 1200–1208.
27. M. Nagata, *On the 14th problem of Hilbert*, Amer. J. Math. **81** (1959), 766–772.
28. ______, *Complete reducibility of rational representations of a matrix group*, J. Math. Kyoto Univ. **87** (1961), no. 1, 87–99.
29. ______, *Invariants of a group in an affine ring*, J. Math. Kyoto Univ. **3** (1963), no. 3, 369–378.
30. P. E. Newstead, *Characteristic classes of stable bundles of rank 2 over an algebraic curve*, Trans. Am. Math. Soc. **169** (1972), pp. 337–345.
31. P. E. Newstead, *Introduction to moduli problems and orbit spaces*, T.I.F.R. Lecture Notes, Springer-Verlag, 1978.
32. ______, *Geometric invariant theory*, Moduli spaces and vector bundles, London Math. Soc. Lecture Notes Ser., vol. 359, Cambridge University Press, 2009, pp. 99–127.
33. N. Nitsure, *Construction of Hilbert and Quot schemes. (Fundamental algebraic geometry. Grothendieck's FGA explained)*, Math. Surveys Monogr. **123** (2005), 105–137.
34. A. L. Onishchik and E. B. Vinberg, *Lie groups and Lie algebras III: Structure of Lie groups and Lie algebras*, Encyclopedia of mathematical sciences, vol. 41, Springer-Verlag, 1990.
35. J. Le Potier, *Lectures on vector bundles*, Cambridge studies in advanced mathematics, vol. 54, Cambridge University Press, 1997.
36. S. Ramanan, *The moduli space of vector bundles over an algebraic curve*, Math. Ann. **200** (1973), 69–84.
37. C. S. Seshadri, *Spaces of unitary vector bundles on a compact Riemann surface*, Ann. of Math. **85** (1967), no. 1, 303–336.
38. T. Shioda, *On the graded ring of invariants of binary octavics*, Amer. J. Maths **89** (1967), 1022–1046.
39. C. T. Simpson, *Moduli of representations of the fundamental group of a smooth projective variety*, Inst. Hautes Etudes Sci. Publ. Math. **79** (1994), 47–129.
40. T. A. Springer, *Linear algebraic groups*, Birkhäuser, 1998.

41. M. Thaddeus, *Geometric invariant theory and flips*, J. Amer. Math. Soc. **9** (1996), no. 3, 691–723.
42. R. P. Thomas, *Notes on GIT and symplectic reduction for bundles and varieties*, Surveys in differential geometry **10** (2006), 221–273.
43. A. N. Tjurin, *Analogue of Torelli's theorem for two-dimensional bundles over algebraic curves of arbitrary genus*, Izv. Akad. Nauk SSSR, Ser. Mat. **33** (1969), 1149–1170.