Complex Analysis SS15
Fachbereich Mathematik und Informatik
J-Prof. Victoria Hoskins
Teaching Assistant: Alejandra Rincón

Complex analysis
Summer semester 2015

Exercise Sheet 9

Hand-in date: 12:00, Friday 12th June.

Exercise 1. Let $0 \leq r < R \leq \infty$. Prove that the Laurent series of an analytic function f on an annulus $\mathcal{A} = \{z \in \mathbb{C} : r < |z - a| < R\}$ around $a \in \mathbb{C}$ is unique. [Hint: consider a Laurent series of the zero function in $\mathcal{A}$ and extend this to an entire function which tends to zero as $z \to \infty$, then deduce the principal and analytic parts are both zero.].

Exercise 2. Calculate the Laurent series of the following functions at the given points and state the punctured disk on which the Laurent series converges.

a)
$$\exp(1/z) \quad \text{at} \quad 0.$$

b)
$$\frac{1}{z^2(z-1)} \quad \text{at} \quad 0, 1.$$

c)
$$\frac{1 + z^2 + z^4}{z(z^2+1)} \quad \text{at} \quad 0.$$

d)
$$\frac{\exp(1/z^2)}{z-1} \quad \text{at} \quad 0.$$

Exercise 3. Let $f : D \to \overline{\mathbb{C}}$ be a meromorphic function on a domain D.

a) Prove that f is locally the quotient of two analytic functions (that is, for any $a \in D$, there exists an open neighbourhood U of a in D such that $f|_U = g/h$ for $g, h \in \mathcal{O}(U)$).

b) If the pole set $P(f)$ is finite, then prove that there exists $g, h \in \mathcal{O}(D)$ such that $h|_{P(f)} = 0$ and $f = g/h$ on $D - P(f)$.