Complex Analysis SS15
Fachbereich Mathematik und Informatik
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Complex analysis
Summer semester 2015

Exercise Sheet 6

Hand-in date: 12:00, Friday 22nd May.

Exercise 1. For a non-zero real number c, compute the integral

$$\int_{-\infty}^{\infty} \exp(-x^2 - 2icx)\, dx$$

using the formula $\int_{-\infty}^{\infty} \exp(-x^2)\, dx = \sqrt{\pi}$.
[Hint: consider the integral of $\exp(-z^2)$ along the boundary of a rectangle with vertices $\pm R$ and $\pm R + ic$].

For $\lambda \in \mathbb{R}$, compute

$$\int_{-\infty}^{\infty} \exp(-x^2) \cos(\lambda x)\, dx.$$

Exercise 2. Compute the following complex line integrals.

a) $\int_{\partial D_1(-1)} \frac{1}{(z+1)(z-1)^3}\, dz$

b) $\int_{\partial D_0(3)} \frac{\cos(\pi z)}{z^2-1} dz$

c) $\int_{\partial D_{1/2}(0)} \frac{\exp(1-z)}{z^3(1-z)}\, dz$

d) $\int_{\partial D_1(1)} \left(\frac{z}{z-1}\right)^n dz \quad$ for $n \geq 1$

Exercise 3.

a) Let $f : \mathbb{C} \to \mathbb{C}$ be a function that is holomorphic in a neighbourhood of a closed disc $\overline{D} \subset \mathbb{C}$. Show that the function

$$z \mapsto \int_{\partial D} \frac{f(w)}{w-z}\, dw$$

determines a holomorphic function on $\mathbb{C} - \overline{D}$. Which function is this?

b) Show that the function

$$z \mapsto \frac{1}{2\pi i} \int_{\partial D_1(0)} \frac{dw}{w(w-z)}$$

defines holomorphic functions $f_1 : D_1(0) \to \mathbb{C}$ and $f_2 : \mathbb{C} - \overline{D_1(0)} \to \mathbb{C}$, and determine these functions. For which boundary points $z \in \partial D_1(0)$, does it hold that either

$$\lim_{z \to w} f_1(z) = \frac{1}{w} \text{ or } \lim_{z \to w} f_2(z) = \frac{1}{w}?$$