Complex Analysis SS15
Fachbereich Mathematik und Informatik
J-Prof. Victoria Hoskins
Teaching Assistant: Alejandra Rincón

Complex analysis
Summer semester 2015

Exercise Sheet 4

Hand-in date: 12:00, Friday 8th May.

Exercise 1. At which points are the following functions holomorphic (respectively analytic)?

a) $f : \mathbb{C} \to \mathbb{C}$ given by $f(x + iy) = x^2 + iy^2$.

b) $g : \mathbb{C} - \{\beta_1, \ldots, \beta_n\} \to \mathbb{C}$ given by

$$f(z) = \frac{(z - \alpha_1) \cdots (z - \alpha_m)}{(z - \beta_1) \cdots (z - \beta_m)},$$

for complex numbers α_j, β_k.

c) $h : \mathbb{C} \to \mathbb{C}$ given by $h(z) := \sum_{k=0}^{\infty} \frac{z^{2n}}{(2n)!}$.

Exercise 2. Let $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$ and consider the function $f : \mathbb{H} \to \mathbb{C}$ given by

$$f(z) = \frac{z - i}{z + i}.$$

a) Prove that f is holomorphic and injective with image equal to $\mathbb{U} := D_1(0)$.

b) What are the images of the horizontal lines $H_r := \{z \in \mathbb{H} : \mathrm{Im}\, z = r\}$, for $r > 0$, and the vertical rays $V_x := \{z \in \mathbb{H} : \mathrm{Re}\, z = x\}$, for $x \in \mathbb{R}$, under f?

c) Give a formula for the inverse $f^{-1} : \mathbb{U} \to \mathbb{H}$ of f and prove that it is also holomorphic.

Exercise 3. (The Uniqueness Theorem for power series). Let $S \subset \mathbb{C}$ be a subset and suppose that the origin is an accumulation point of S. The Uniqueness Theorem states that if the power series $\sum_{k=0}^{\infty} a_k z^k$ and $\sum_{k=0}^{\infty} b_k z^k$ both converge and agree on S, then $a_k = b_k$ for all k. We will prove this theorem in the following steps:

a) Show that there is a non-zero sequence of points in S which converges to the origin.

b) Let $f(z) := \sum_{k=0}^{\infty} c_k z^k$, for $c_k := a_k - b_k$. Using the above sequence, show that $c_0 = 0$.

c) Prove by induction that $c_k = 0$ for all k, by considering the functions $f(z)/z^k$.