Complex Analysis SS15
Fachbereich Mathematik und Informatik
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Complex analysis
Summer semester 2015

Exercise Sheet 1

Hand-in date: 5pm, Friday 17th April.

Exercise 1. Write the following complex numbers in the form $x+iy$ with $x, y \in \mathbb{R}$ and calculate their modulus and arguments:

a) $\frac{1+i}{1-i}$,

b) $\frac{3+4i}{1-2i}$,

c) $\left(\frac{1+i\sqrt{3}}{2}\right)^n$ for $n \in \mathbb{N}$,

d) $\exp(2\pi ik/n)$ for $n \neq 0, k \in \mathbb{Z}$.

Exercise 2.

(1) Let P(z) be a polynomial with real coefficients. Prove that $\overline{P(z)} = P(\overline{z})$ for all complex numbers z and deduce that all roots come in complex conjugate pairs (that is, w is a zero of P if and only if $\overline{w}$ is a zero of P.

(2) Show that the nth roots of unity other than 1 are the roots of the 'cyclotomic' equation $z^{n-1} + z^{n-2} + \cdots + z + 1 = 0$. [*Hint*: What is $z^n - 1$ divided by $z - 1$?]

(3) Let a, b, c be complex numbers and consider the equation $az^2 + bz + c = 0$. Show that this equation has a solution by giving a formula for the solutions. In particular, show that every complex number has a square root.

Exercise 3. Sketch the following subsets of the complex plane $\mathbb{C}$.

a) $\{z \in \mathbb{C} : |z - i| < 1\}$,

b) $\{z \in \mathbb{C} : |z| < 1 \text{ and } \mathrm{Im} z > 0\}$,

c) $\{z \in \mathbb{C} : \overline{z} = \frac{1}{z}\}$,

d) $\{z : \mathrm{Im}(z(1-i)) = 0\}$,

e) $\{z \in \mathbb{C} : |z - 1| = |z + 1|\}$,

f) $\{z \in \mathbb{C} : |z - 2| \geq |z - 3|\}$.

Exercise 4. Find the radius of convergence of the following power series.

a) $\sum_{n=1}^{\infty} \frac{(-i)^n z^n}{n!}$ b) $\sum_{n=0}^{\infty} (n + 2^n) z^n$ c) $\sum_{n=1}^{\infty} \frac{n! z^n}{n^n}$

d) $\displaystyle\sum_{n=0}^{\infty} n^m c_n z^n$ for $m \in \mathbb{N}$ (in terms of the radius of convergence R of $\displaystyle\sum_{n=0}^{\infty} c_n z^n$).