Freie Universität Berlin SoSe24
Institut für Mathematik A. Constantinescu
Stand: 11. Juli 2024

Zahlentheorie II – Homework 13

Submission: individually or in pairs,
on Whiteboard as `Names_ZT2_H13.pdf` **by 9:00 on Thursday**, the 18th. of July 2024.

Full written proofs are required in support of your answers.

Problem 1. **2 points**

Let K be a field, $\overline{K}$ an algebraic closure of it, and $n \in \mathbb{Z}_{>0}$ such that $\operatorname{char} K \nmid n$ and $U_n \leq K^\times$.

For $c, c' \in K^\times$ consider L and L' the splitting fields of the polynomials $x^n - c$ and $x^n - c'$ over K, respectively. Show that

$$L = L' \Leftrightarrow \exists\ r \in \mathbb{N} \text{ such that } \gcd(r, n) = 1 \text{ and } c^r \cdot c' \in (K^\times)^n.$$

Problem 2. **2 points**

Let K be a field, $\overline{K}$ an algebraic closure of it, and $n \in \mathbb{Z}_{>0}$ such that $\operatorname{char} K \nmid n$ and $U_n \leq K^\times$.

Show for every finite Galois extension L/K that there is a canonical isomorphism of groups

$$\frac{L^n \cap K^\times}{(K^\times)^n} \xrightarrow{\ \sim\ } \operatorname{Hom}(\operatorname{Gal}(L/K), U_n).$$

Problem 3. **2 points**

Let K'/K be a field extension, such that $K = (K')^G$ for some $G \leq \operatorname{Aut}(K')$. Let V' be a K'-vector space together with a G-action $G \times V' \longrightarrow V'$, written as

$$(\sigma, v) \mapsto f_\sigma(v).$$

Let $V := (V')^G$ be the set of vectors in V' which are fixed by G.

1. Show [†] that $V \subseteq_K V'$, that is V is a K-subspace of V'.
2. Show that the the inclusion map $\iota : V \hookrightarrow V'$ induces an injective map $\iota' : K' \otimes_K V \longrightarrow V'$ and find an example for which this map is not surjective.

Problem 4. **2 poits**

Let $K \subset K' \subset K''$ be a chain of proper extensions of finite fields. View K'' as a K'-vector space and denote it by V'. Give examples of a K'-subspace $U' \subseteq_{K'} V'$ and of an endomorphism $\varphi \in \operatorname{End}_{K'}(V')$ which are not defined over K.

(Start with explicit finite fields and then try to find a general method to produce conrete examples.)

Total: 8 points

[†] Try to prove it by elementary methods and, most importantly, do not use Proposition 4 from the lecture.