Freie Universität Berlin
Institut für Mathematik
Stand: 4. Juli 2024

SoSe24
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Zahlentheorie II – Homework 12

Submission: individually or in pairs,
on Whiteboard as `Names_ZT2_H12.pdf` **by 12:00 on Thursday**, the 11th. of July 2024.

Full written proofs are required in support of your answers.

Problem 1. **2 points**

Consider a finite Galois extension L/K with Galois group $G = \mathrm{Gal}(L/K)$. Equip the group $(L, +)$ with the canonical action of G and show that

$$H^1(G, L) = 0.$$

Hint: Adapt the proof of [Bosch.4.8/Theorem 2] to the additive point of view.

Problem 2. **2 points**

Use the Hilbert 90 Theorem [Bosch.4.8/Theorem 1] to show that for two rational numbers $a, b \in \mathbb{Q}$ the relation $a^2 + b^2 = 1$ is equivalent to the existence of integers $m, n \in \mathbb{Z}$, not both zero, satisfying

$$a = \frac{m^2 - n^2}{m^2 + n^2}, \qquad b = \frac{2mn}{m^2 + n^2}.$$

Problem 3. **2 points**

Let G be a group and cosider $m \in \mathbb{Z}_{\geqslant 0}$ with $m\mathbb{Z} = \{a \in \mathbb{Z} \ : \ g^a = 1 \ \ \forall\, g \in G\} \subseteq \mathbb{Z}$. If $m > 0$, then we call m **the exponent of** G. If $m = 0$, then we say that G has infinite exponent.

1. Characterize all subgroups of $\mathbb{Z}/n\mathbb{Z}$ and show that for each $d \,|\, n$ there exists preciesely one subgroup of order d.
2. Show that if G has exponent 2, then G is Abelian.
3. **Bonus:** Find a group of exponent 3 that is not Abelian.

Problem 4. **2 points**

Show that $L_2 = \mathbb{Q}(i, \sqrt{2}, \sqrt{3}, \sqrt{5}, \dots)$ is the largest among the Abelian Galois extensions of $\mathbb{Q}$ in $\mathbb{C}$, with exponent of the Galois Group equal to 2, and determine $\mathrm{Gal}(L_2/\mathbb{Q})$.
(Compare with Homework 9).

Total: 8 points