Freie Universität Berlin SoSe24
Institut für Mathematik A. Constantinescu
Stand: 28. Juni 2024

Zahlentheorie II – Homework 11

Submission: individually or in pairs,
on Whiteboard as `Names_ZT2_H11.pdf` **by 12:00 on Thursday**, the 4th. of July 2024.

Full written proofs are required in support of your answers.

Problem 1. **2 points**

Let $\widehat{G} = \{\chi : G \longrightarrow \mathbb{C}^\times\}$ denote the set of complex characters on the group G. Together with the multiplication defined for all $\chi, \eta \in \widehat{G}$ by

$$(\chi \cdot \eta)(g) := \chi(g) \cdot \eta(g) \qquad \forall\, g \in G,$$

$\widehat{G}$ forms an Abelian group.

1. Show that if $\#G < \infty$ and G is Abelian, then $G \simeq \widehat{G}$.
2. Show that if $\#G < \infty$ and G is Abelian, then for every nontrivial* character $\chi \in \widehat{G}$ we have
$$\sum_{g \in G} \chi(g) = 0.$$

Problem 2. **2 points**

Show for monic polynomials $f, g \in R[X]$ over a ring R that

$$\Delta_{f \cdot g} = \Delta_f \cdot \Delta_g \cdot (\mathrm{res}(f, g))^2.$$

Problem 3. **2 points**

Let K be a field and $L = K(a)$ be a simple algebraic field extension with $f_a \in K[X]$ the minimal polynomial of a over K. Show that

$$f(x) = N_{L/K}(x - a) \qquad \text{for any element } x \in K.$$

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* i.e. it does not map everything to 1.

Problem 4. **2 points**

Let L/K be a finite Galois extension and $x_1, \ldots, x_n$ a K-basis of L. Show for a subgroup $H \leq \mathrm{Gal}(L/K)$ that its corresponding fixed field satisfies

$$L^H = K(\mathrm{tr}_{L/K}(x_1), \ldots, \mathrm{tr}_{L/K}(x_1)).$$

Total: 8 points

Extra Porblem 5.

Let R be a ring and $f \in R[X]$ a monic polynomial. Show that for any $c \in R$ the subsitution $X \mapsto X + c$ does not change the discriminant. That is, that $\Delta_f = \Delta_g$ with $g(X) = f(X + c)$.

What does the substitution $X \mapsto aX + c$, for $a, c \in R$ with $a \neq 0$, do to the discriminant?

Extra Problem 6.

Let L/K be a finite extension of degree m. Show that if $n \in \mathbb{N}_{>0}$ is coprime with m, then every element $a \in K$ which admits an nth root in L already admits an nth root in K.

Extra Problem 7.

Check out Proposition 12 from Bosch's §4.5.

1. Try to prove it on your own.
2. In some explicit instances in which a cyclotomic polynomial is not irreducible in $\mathbb{F}_q[X]$, find its irreducible factors.