Freie Universität Berlin
Institut für Mathematik
Stand: 26. Juni 2024

SoSe24
A. Constantinescu

Zahlentheorie II – Homework 10

Submission: individually or in pairs,
on Whiteboard as `Names_ZT2_H10.pdf` **by 12:00 on Thursday**, the 20th. of June 2024.

Full written proofs are required in support of your answers.

Problem 1. **2 points**

Let G be a finite group.

1. Show that there exists a Galois extension L/K such that $\mathrm{Gal}(L/K) \simeq G$.
2. Show that if K is fixed there may not exist an extension $K \subseteq L$ with $\mathrm{Gal}(L/K) \simeq G$.

Problem 2. **2 points**

Consider a subfield $L \subset \mathbb{C}$ such that $L/\mathbb{Q}$ is a cyclic Galois extension of degree 4. Show that $L/\mathbb{Q}$ admits a unique nontrivial intermediate field E, and that E is contained in $\mathbb{R}$.

`You may solve Extra Problem 5 instead of this one.`

Problem 3. **2 points**

Determine a splitting field and the Galois group of the polynomial $x^4 - 5$ over $\mathbb{Q}$, resp. $\mathbb{Q}(i)$, as well as all intermediate fields of the arising extensions.

Problem 4. **2 points**

Write the symmetric polynomial $T_1^2T_2^2T_3 + T_1^2T_2T_3^2 + T_1T_2^2T_3^2 \in \mathbb{Q}[T_1, T_2, T_3]$ as a polynomial expression $g(s_1, s_2, s_3)$ in the elementary symmetric polynomials.

Total: 8 points

Extra Problem 5. **2 points (instead of Pb.2)**

Let K be a field of characteristic $\neq 2$ and $f \in K[x]$ a separable irreducible polynomial with zeros $\alpha_1, \ldots, \alpha_n$ in a splitting field L of f over K. Assume that the Galois group of f is cyclic of even order and show:

1. The discriminant $\Delta = \prod_{i<j}(\alpha_i - \alpha_j)^2$ does not admit a square root in K.

2. There is a unique intermediate field E of L/K which satisfies $[E : K] = 2$, namely $E = K(\Delta)$.

Extra Problem 6.

Consider $L = \mathbb{Q}(\frac{i+1}{\sqrt{2}})$ as a subfield of $\mathbb{C}$. Show that L/Q is Galois and determine the corresponding Galois group, as well as all intermediate fields of $L/\mathbb{Q}$.

Extra Problem 7.

Determine the Galois Groups of the following polynomials.

1. $x^3 + 6x^2 + 11x + 7 \in \mathbb{Q}[x]$.
2. $x^4 + 2x^2 - 2 \in \mathbb{Q}[x]$.
3. $x^4 - x^2 - 3 \in \mathbb{F}_5[x]$.
4. $x^4 + 7x^2 - 3 \in \mathbb{F}_{13}[x]$.