Freie Universität Berlin
Institut für Mathematik
Stand: 13. Juni 2024

SoSe24
A. Constantinescu

Zahlentheorie II – Homework 9

Submission: individually or in pairs,
on Whiteboard as `Names_ZT2_H9.pdf` **by 12:00 on Thursday**, the 20th. of June 2024.

Full written proofs are required in support of your answers.

Problem 1. **2 points**

Let $K = \mathbb{Q}$ and $L = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}, \dots)$ be obtained by adjoining the square root of every prime. Prove that

$$\mathrm{Gal}(L/K) \simeq \prod_{p \text{ prime}} \mathbb{Z}/2\mathbb{Z}.$$

Problem 2. **2 points**

Show that the only automorphism of the field of p-adic numbers $\mathbb{Q}_p$ is the identity.

(**Hint:** Show that it must be continuous with respect to p-adic topology.)

Problem 3. **2 points**

Consider the standard representation of p-adic numbers:

$$x = \sum_{i=-m}^{\infty} a_i p^i \in \mathbb{Q}_p \qquad a_i \in \{0, \dots, p-1\} \;\; \forall\, i.$$

Show that $x \in \mathbb{Q}$ precisely when this representation is periodic.

Problem 4. **2 points**

Let $\mathbb{F}$ be a finite field and $\overline{\mathbb{F}}$ an algebraic closure of it.

1. Determine the subfield of $\overline{\mathbb{F}}$ which is fixed by $\mathbb{Z}_\ell$, the group of ℓ−adic integers viewed as a subgroup of $\mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F})$.
2. Determine all intermediate subfields of $\overline{\mathbb{F}}/\mathbb{F}$.

Total: 8 points

Extra Problem 5.

Figure out how to add and multiply p-adic numbers in their standard presentation.

Extra Problem 6.

1. Is 2 a square in $\mathbb{Q}_5$? What about -1 in $\mathbb{Q}_5$? Or 2 in $\mathbb{Q}_7$?
2. When they are squares, can you compute some p-adic digits of one of their square roots?
3. Show that $\mathbb{Z}_p$ contains all the $(p-1)$st roots of unity.

Hint: Hensel's Lemma. You probably saw it in Algebra 1 or at a previous point in your life.

Extra Problem* 7.

Show that when $p \neq 2$ there are 4 elements in $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{Q}_p$ such that for every nonzero $x \in \mathbb{Q}_p$ precisely one of the elements $\alpha_1 x, \alpha_2 x, \alpha_3 x, \alpha_4 x$ is a square.
(In $\mathbb{R}$ there are two such elements: ± 1. But any choice of a positive and a negative also works.)

When $p = 2$, prove that there are 8 such numbers in $\mathbb{Q}_2$.