Freie Universität Berlin SoSe24
Institut für Mathematik A. Constantinescu
Stand: 6. Juni 2024

Zahlentheorie II – Homework 8

Submission: individually or in pairs,
on Whiteboard as `Names_ZT2_H8.pdf` **by 12:00 on Thursday**, the 13th. of June 2024.

Full written proofs are required in support of your answers.

Problem 1. **2 points**

Let K be a field of characteristic $p > 0$ and $f = x^p - x + a \in \mathbb{K}[x]$ be an irreducible polynomial. Show that if $\alpha \in \overline{K}$ a root of f in some algebraic closure of K, then the field extension $K \subset K(\alpha)$ is Galois and compute $\mathrm{Gal}(K(\alpha)/K)$.

Problem 2. **2 points**

Can there exist a Galois extension L/K such that $\mathrm{Gal}(L/K) \simeq \mathbb{Z}$?

Problem 3. **2 points**

Let $K \subset E \subset L$ be field extensions with both L/K and E/K Galois. Show that:

1. The restriction homomorphism $\varphi : \mathrm{Gal}(L/K) \longrightarrow \mathrm{Gal}(E/K)$ is continuous.
2. A subset $U \subset \mathrm{Gal}(E/K)$ is open if and only if $\varphi^{-1}(U)$ open in $\mathrm{Gal}(L/K)$.

Problem 4. **2 points**

Let X be a set and $(X_i)_{i\in I}$ be a system of subsets of X. We define on I the partial order:

$$i \leqslant j \Leftrightarrow X_j \subseteq X_i,$$

and we define for all $i \leqslant j$ the map $f_{ij} : X_j \longrightarrow X_i$ as the inclusion map. Show that (X_i, f_{ij}) is a projective system (or a "diagram indexed by the poset I", viewed as the opposite small category) and that

$$\bigcap_{i\in I} X_i = \varprojlim_{i\in I} X_i.$$

Total: 8 points

Extra Problem 5.

Find all field automorphisms of $\mathbb{R}$, the field of real numbers.

Extra Problem*** 6.

Describe $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. How many elements can you find in this group?