Freie Universität Berlin SoSe24
Institut für Mathematik A. Constantinescu
Stand: 30. Mai 2024

Zahlentheorie II – Homework 7

Submission: individually or in pairs,
on Whiteboard as `Names_ZT2_H7.pdf` **by 12:00 on Thursday**, the 6th. of June 2024.

Full written proofs are required in support of your answers.

Problem 1. **2 points**

Construct a field L together with a subgroup $G \leq \mathrm{Aut}(L)$ such that L/L^G is not a Galois extension.

Problem 2. **2 points**

Let K be a field, $f \in K[x]$ an irreducible separable polynomial, and L a splitting field of f over K, so that L/K is a finite Galois extension. If L/K is abelian, show that $L = K(\alpha)$ for every zero $\alpha \in L$ of f.

Problem 3. **2 points**

Determine $\mathrm{Gal}(L/K)$ as well as the normal intermediate extensions of L/K in the following cases.

1. $K = \mathbb{Q}$, $L =$ the splitting field of $x^3 - 2$ over $\mathbb{Q}$.
2. $K = \mathbb{Q}$, $L = \mathbb{Q}(i, \sqrt[4]{2})$.

Problem 4. **2 points**

Let $p \in \mathbb{N}$ be a prime, $r \in \mathbb{N}_{>0}$, and $\xi \in \mathbb{C}$ some primitive root of unity of order p^r. Show that $\mathbb{Q} \subset \mathbb{Q}(\xi)$ is a cyclic Galois extension, except for the case in which $p = 2$ and $r \geqslant 3$. Show that in this latter case the Galois group is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^{r-2}\mathbb{Z}$.

Total: 8 points

Extra Problem 5.

Let $K \subseteq \mathbb{R}$ be a field and $f \in K[x]$ be an irreducible polynomial in $K[x]$ with $\deg f = p$, where p is a prime, and with f having precisely $p - 2$ real roots. Let L be the splitting field of f. Show that $\mathrm{Gal}(L/K) \simeq S_p$, the symmetric group on p elements.

Extra Problem 6.

Let K be a filed and $L = K(x)$, the field of rational functions in one variable x over K. Let σ be the K-homomorphism of L given by $x \mapsto x + 1$. Determine the subfield

$$L^{\sigma} = \{a \in L \ : \ \sigma(a) = a\}.$$

Elementary Proble 7.

Find all $x, y, z, t \in \mathbb{N}$ such that $3^x - 3^y + 3^z - 3^t = 2025$.