Freie Universität Berlin SoSe24
Institut für Mathematik A. Constantinescu
Stand: 24. Mai 2024

Zahlentheorie II – Homework 6

Submission: individually or in pairs,
on Whiteboard as `Names_ZT2_H6.pdf` **by 12:00 on Thursday**, the 30th. of May 2024.

Full written proofs are required in support of your answers.

Problem 1. **2 points**

Let K be a field of characteristic $p > 0$. Show that:

1. For every $n \in \mathbb{N}$ there exists a field $K^{p^{-n}}$ extending K with the following two properties:
   - If $a \in K^{p^{-n}}$, then $a^{p^n} \in K$.
   - For every $b \in K$, there exists an $a \in K^{p^{-n}}$ such that $b = a^{p^n}$
2. $K^{p^{-n}}$ is unique up to canonical isomorphism and there are canonical embeddings

$$K \subset K^{p^{-1}} \subset K^{p^{-2}} \subset \dots$$

3. The field[‡] $K^{p^{-\infty}} = \bigcup_{i=1}^{\infty} K^{p^{-i}}$ is a perfect field.

Problem 2. **2 points**

Let L/K be an algebraic field extension with the property that every irreducible polynomial in $K[x]$ admits at least one zero in L. Show that L is an algebraic closure of K.

(This means that in the Proof of § 3.4 Theorem 4, we could have stopped at L_1.)

Problem 3. **2 points**

Show that $\mathbb{F}_{p^\infty} := \bigcup_{n=0}^{\infty} \mathbb{F}_{p^{n!}}$ is an algebraic closure of $\mathbb{F}_p$.

Problem 4. **2 points**

Let $n \in \mathbb{Z}_{>0}$ and $q = p^n$. Let $f \in \mathbb{F}_q[x]$ be an irreducible polynomial of degree $n > 0$ and $\alpha \in \overline{\mathbb{F}}_p$ be a root of f. Show that the roots of f in $\mathbb{F}_q(\alpha)$ are

$$\alpha, \alpha^q, \dots, \alpha^{q^{n-1}}.$$

Total: 8 points

[‡] This field is called the **purely inseparable closure** of K.

Extra Problem 5.

Let $\overline{\mathbb{F}}_p$ be an algebraic closure of $\mathbb{F}_p$. Show that besides the powers of the Frobenius homomorphism, there exist other automorphisms of $\overline{\mathbb{F}}_p$.

Extra Problem 6.

Let $\mathbb{F}$ be a finite field of characteristic p and let $f \in \mathbb{F}[x]$ be an irreducible polynomial. Show that there exists an $n \in \mathbb{N}_{>0}$ such that f divides $x^{p^n} - x$.

* 7.

Prove that there is an positive integer n such that the decimal expansion of 2^n begins with your phone number.

(I found this on Facebook[†]. I am not sure how elementary this is.)

[†] https://en.wikipedia.org/wiki/Facebook