Zahlentheorie II – Homework 5

Submission: individually or in pairs,
on Whiteboard as `Names_ZT2_H5.pdf` **by 12:00 on Thursday**, the 23rd. of May 2024.

Full written proofs are required in support of your answers.

Problem 1. **2 points**

Let K be a field and $f \in K[x]$ a polynomial with $\deg f = n > 0$. Let L be a splitting field of f over K. Show that:

1. $[L : K]$ divides $n!$.
2. If $[L : K] = n!$, then f is irreducible.

Problem 2. **2 points**

For a prime number p consider the function field $L = \mathbb{F}_p(X, Y)$ in two variables over $\mathbb{F}_p$, as well as the Frobenius homomorphism $\sigma : L \longrightarrow L$, $\sigma(a) = a^p$. Let $K = \sigma(L)$. Determine the degrees $[L : K]$ and $[L : K]_s$ and show that the field extension L/K is not simple.

Problem 3. **2 points**

Let $K \subset L$ be an algebraic extension. Show that if K is perfect, then L is perfect.

Porblem 4. **2 points**

Let K be a field of positive characteristic p. Prove that K is perfect if and only if the Frobenius homomorphism is surjective.

Total: 8 points

Extra Problem 5.

Determine a splitting field $\mathbb{F}$ over $\mathbb{Q}$ of the family $\{x^4 + 1, x^5 + 2\}$ and compute $[\mathbb{F} : \mathbb{Q}]$.

Extra Problem 6.

Let $K \subset L$ be a finitely generated field extension with $\operatorname{char} K = p > 0$. Show that if L is a perfect field, then L/K is algebraic and K is perfect.

The following "Elementary problems" are not related to this weeks lecture.

They are here only for your entertainment. "Elementary" does not mean easy. It means that the problem can be solved with basic instruments.

Elementary Problem 7.

Let $a, b, c, d, e \in \mathbb{Z}$. Show that if $9 \mid a^3 + b^3 + c^3 + d^3 + e^3$, then $3 \mid abcde$.

Elementary Problem 8.

Show that if x, y are positive integers such that $x(5x - 4) = y^2$, then x is a perfect square and find all the positive integer solutions of the equation

$$x(5x - 4) = y^2.$$