Freie Universität Berlin
Institut für Mathematik
Stand: 9. Mai 2024

SoSe24
A. Constantinescu

Zahlentheorie II – Homework 4

Submission: individually or in pairs,
on Whiteboard as `Names_ZT2_H4.pdf` **by 12:00 on Thursday**, the 16th. of May 2024.

Full written proofs are required in support of your answers.

Problem 1. **2 points**

Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$. Determine all homomorphisms $\mathbb{Q}(\sqrt[4]{2}, i) \longrightarrow \overline{\mathbb{Q}}$ as well as their images.

Problem 2. **2 points**

Let L/K be a finite field extension of degree $[L : K] = n$. Assume there is an element $\alpha \in L$ together with isomorphisms $\sigma_i : L \longrightarrow L$, $i = 1, \dots, n$ satisfying $\sigma_i|_K = \mathrm{id}_K$ and $\sigma_i(\alpha) \neq \sigma_j(\alpha)$ for $i \neq j$. Show that $L = K(\alpha)$.

Porblem 3. **2 points**

Let $\mathbb{F}$ be a finite field and let $f \in \mathbb{F}[x]$ be an irreducible polynomial. Show that $L = \mathbb{F}[x]/(f)$ is a splitting field for f over $\mathbb{F}$.

Problem 4. **2 points**

Show that every field extension of degree two is normal.

Total: 8 points

Extra Problem 5.

Which of the following field extensions is normal?

1. $\mathbb{F}_2 \subset \mathbb{F}_2[x]/(x^3 + x + 1)$.
2. $\mathbb{F}_2(x^2) \subset \mathbb{F}_2(x)$.
3. $\mathbb{Q} \subset \mathbb{Q}(\sqrt{2 + \sqrt{2}})$.

Extra Problem 6.

Let $\mathbb{F}$ be a finite field.

1. Show that for every positive integer d, the polynomial ring $\mathbb{F}[x]$ contains an irreducible polynomial of degree d.
2. Show that for every irreducible polynomial $f \in \mathbb{F}[x]$, there exists an $n \in \mathbb{N}_{>1}$ such that $f \mid x^{p^n} - x$, where $p = \text{char } \mathbb{F}$.
3. Describe the roots of $x^4 + x^3 + 1$ in $F_2[x]/(x^4 + x^3 + 1)$.

The following "Elementary problems" are not related to this weeks lecture.

They are here only for your entertainment. "Elementary" does not mean easy. It means that the problem can be solved with basic instruments.

Elementary Problem 7.

Prove that any number which is not divisible by 2, by 3, or by 5 has an infinity of multiples of the form $111\ldots1$.

Comment: Clearly there are infinitely many multiples of 3 or of 9 which are of the form $111\ldots1$. I think this hypothesis is there to allow a uniform proof, without case distinctions. (I found this problem in a 1938 issue of a Romanian mathematics journal aimed at high school students.)

Elementary Problem 8.

Show that for every polynomial $f \in \mathbb{Z}[x]$, there exists an integer a, such that $f(a)$ is not a prime number.