Freie Universität Berlin SoSe24
Institut für Mathematik A. Constantinescu
Stand: 25. April 2024

Zahlentheorie II – Homework 2

Submission: individually or in pairs,
on Whiteboard as `Names_ZT2_H2.pdf` **by 12:00 on Thursday**, the 2nd. May 2024.

Full written proofs are required in support of your answers.

Prboblem 1. **2 points**

Find all positive integers n such that $f_n = \sum_{i=0}^{n}(-1)^i x^{n-i}$ is irreducible in $\mathbb{Z}[x]$.

Problem 2. **2 points**

Let K be a field of characteristic $p > 0$.

1. Show that if K is finite, then the Frobenius endomorphism $\sigma : K \longrightarrow K$ is an automorphism.
2. Does the assertion extend to the case that K is not necessarily finite?

Problem 3. **2 Points**

Show that a field extension L/K is algebraic if and only if every intermediate subring R, that is every R satisfying $K \subset R \subset L$, is a field.

Problem 4. **2 points**

Let $\alpha \in \mathbb{C}$ satisfy $\alpha^3 + 2\alpha - 1 = 0$. So α is algebraic over $\mathbb{Q}$. Determine in each case the minimal polynomial of α as well as that of $\alpha^2 + \alpha$ over $\mathbb{Q}$.

Total: 8 points

Extra Problem 5.

Let L/K be a field extension. Show that two elements $\alpha, \beta \in L$ are algebraic over K if and only if $\alpha + \beta$ and $\alpha \cdot \beta$ are algebraic over K.

Extra Problem 6.

Show that there are infinitely many primes such that $p \equiv \pm 3 \bmod 8$.