Freie Universität Berlin
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WiSe19/20
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Algebra 1 - First examination

12.2.2020

Problem Sheet

Problem 1 — 20 points

a. (4 P) Give the definition of a maximal ideal, and formulate an equivalent characterization of maximal ideals in terms of the quotient ring.

b. (6 P) Let R be a ring and $\mathfrak{m} \subseteq R$ a maximal ideal. Prove that if $1 + \mathfrak{m}$ consists of units, then the ring R is local.

c. (5 P) Is $\mathbb{Q}[x, y]/(x^{20}, y^{20})$ a local ring?

d. (5 P) Is the ideal $(x^3 - y^5) \subseteq \mathbb{C}[x, y]$ prime? Is it maximal?

Problem 2 — 20 points

a. (4 P) Formulate the universal property of the tensor product.

b. (3 P) Let R be a ring, U a multiplicatively closed set and M an R-module. Prove that every element of $U^{-1}R \otimes_R M$ is of the form $\frac{1}{u} \otimes m$ with $u \in U$ and $m \in M$.

c. (4 P) Prove that $\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Z} \oplus \mathbb{Z}/42\mathbb{Z}) \simeq \mathbb{Q}$.

d. (4 P) Show that a ring R is a field if and only if every R-module is free.

e. (5 P) Show that an integral domain R is a field if and only if every R-module is flat.

Problem 3 — 20 points

a. (3 P) Give the definition of Artinian ring.

b. (4 P) Prove that in an Artinian ring every prime ideal is maximal.

c. (4 P) Compute a primary decomposition of the ideal $(xy, xz, yz) \subseteq \mathbb{K}[x, y, z]$.

d. (3 P) Is the ring $S = \mathbb{K}[x, y, z]/(xy, xz, yz)$ Artinian?

e. (6 P) Compute the Hilbert function and the Hilbert polynomial of the ring S above.