Freie Universität Berlin
Institut für Mathematik
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WiSe24/25
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Algebra I – Homework 15

Deadline: **20:00 on Wednesday 12.02.2025.** (Uploads are still possible until Friday 14.02 at 23:55)

Submission: individually, on Whiteboard as `LASTname_A1_H15.pdf`

Full written proofs are required in support of your answers.

Problem 1. **2 points**

Let $R = \bigoplus_{i \in \mathbb{N}} R_i$ be a graded rig.

1. Show that if I is a homogeneous ideal[‡] of R, then I is a graded R-module and that $R/I = \bigoplus_{i \in \mathbb{N}} R_i/I_i$ is graded ring.
2. Show that a homogeneous ideal $\mathfrak{p} \subseteq R$ is prime, if for any *homogeneous* elements $a, b \in R$ we have $ab \in \mathfrak{p} \Rightarrow a \in \mathfrak{p}$ or $b \in \mathfrak{p}$.

Problem 2. **3 points**

Let $n \in \mathbb{N}_{>0}$ and $S = \mathbb{C}[x_1, \ldots, x_n] = \bigoplus_{d \in \mathbb{N}} S_d$ be the polynomial ring with the standard $\mathbb{N}-$grading (the one induced by $\deg x_i = 1$).

1. Compute for each $d \in \mathbb{N}$ the value of the **Hilbert function** of S at d:
$$\mathrm{HF}_S(d) := \dim_{\mathbb{C}} S_d.$$
2. Show that there is a polynomial with rational coefficients $\mathrm{HP}_S(t) \in \mathbb{Q}[t]$ such that $\mathrm{HP}_S(d) = \mathrm{HF}_S(d)$ for all $d \in \mathbb{N}$. This is called the **Hilbert Polynomial**[†] of S.
3. Show that the **Hilbert Series** of S: $\mathrm{HS}_S(t) = \sum_{d \in \mathbb{N}} \mathrm{HF}_S(d) \cdot t^d \in \mathbb{Z}[[t]]$ is a rational function of the form [‡]
$$\mathrm{HS}_S(t) = \frac{1}{(1-t)^n}.$$

Try $n = 2$ first. You can obtain half of the points for a correct solution in the case $n = 3$.

Total: 4 points + 1 bonus point

[‡] i.e. generated by homogeneous elements of R

[†] In general, the Hilbert polynomial agrees with the Hilbert function from a certain value on. For the polynomial ring this value is zero.

[‡] The Hilbert series is a rational function in general, but the numerator may be a polynomial in t of degree higher than zero, and the exponent of the numerator is the Krull dimension of the graded ring.

Extra Problems

These problems are neither to be graded nor need to be submitted. They will be discussed in the exercise session and are highly recommended for exam preparation.

Extra Problem 3.

Prove a graded version of Nakayama's Lemma:

Let R be a ($\mathbb{N}$-)graded ring, $I \subseteq R$ a homogeneous ideal, and M a ($\mathbb{Z}$-)graded $R-$module. Assume that $I = \bigoplus_{i \geqslant i_0} I_i$ with $i_0 > 0$ and that $M = \bigoplus_{n \geqslant n_0} M_n$ for some $n_0 \in \mathbb{Z}$. Show that if $IM = M$, then $M = 0$.

Extra Problem 4.

Consider the polynomial rings $R = \mathbb{K}[x_1, \dots, x_n] \subseteq S = \mathbb{K}[x_0, x_1, \dots, x_n]$. For every polynomial $f \in R$ of degree d, the **homogenization** of f (w.r.t. the variable x_0) is defined as

$$f^{hom} := x_0^d \cdot f({}^{x_1}/{}_{x_0}, \dots, {}^{x_n}/{}_{x_0}) \in S.$$

For every ideal $I \subseteq R$, the homogenization of I with respect to x_0 is the ideal $I^{hom} \subseteq S$ generated by the homogenizations of all the polynomials in I.

Prove or disprove: If $I = (g_1, \dots, g_r)$, then $I^{hom} = (g_1^{hom}, \dots, g_r^{hom})$.

Extra Problem 5.

Consider the domain $R = \mathbb{C}[x, y]/(x^3 - y^2)$. Abuse notation and write $x, y \in R$ for the residue classes of the variables as well. Consider then the maximal ideal $\mathfrak{m} = (x, y) \subseteq R$ and show that $y^2 = 0 \in \mathfrak{gr}_{\mathfrak{m}} R$.

Replace now R with $S = \mathbb{C}[x, y]/(x^3 + x^2 - y^2)$, and write $\mathfrak{n} = (x, y) \subseteq S$ for the maximal ideal at the origin. Show that there is no degree one element in $\mathfrak{gr}_{\mathfrak{n}} S$ whose square is zero. Conclude thus that the rings R and S are not isomorphic.

Extra Problem 6.

Let $R = \mathbb{C}[x, y]$ and consider the ideal $I = (x, y) \subseteq R$.

1. Describe R^*, the blowup algebra of R at I, as a quotient ring of some polynomial ring†.

2. Do the same for the associated graded ring $\mathfrak{gr}_I R$.

3. Describe the inclusion $R \hookrightarrow R^*$ in terms of the description of R^* as a quotient of a polynomial ring.

† *Hint:* Recall that we defined the blowup algebra as $R^* = \bigoplus_{i \in \mathbb{N}} I^i = R \oplus I \oplus I^2 \oplus \dots$. In order not to confuse the degree zero element $x \in (R^*)_0$ and the degree one element $x \in (R^*)_1$, it is convenient to introduce a dummy variable t and define $R^* := R[I \cdot t]$. This way $x \in (R^*)_0$ and $xt \in (R^*)_1$ fixes the confusion.