Freie Universität Berlin WiSe24/25
Institut für Mathematik A. Constantinescu
Stand: 29. Januar 2025 J. Rückert

Algebra I – Homework 14

Deadline: **20:00 on Wednesday 5.02.2025.** (Uploads are still possible until Friday 7.02 at 23:55)

Submission: individually, on Whiteboard as `LASTname_A1_H14.pdf`

Full written proofs are required in support of your answers.

Problem 1. **2 points**

Let $R \subseteq S$ be rings, with S integral over R. Let $f : R \longrightarrow \mathbb{K}$ be a homomorphism of R into an algebraically closed field $\mathbb{K}$. Prove that f can be extended to a homomorphism $\bar{f} : S \longrightarrow \mathbb{K}$.

Problem 2. **2 points**

Let G be a finite group of automorphisms of a ring R, and let R^G denote the subring of $G-$invariants, that is $R^G = \{x \in R \ : \ \sigma(x) = x \ \forall \sigma \in G\}$.

1. Prove that R is integral over R^G.

   (**Hint:** Observe that x is a root of the polynomial $\prod_{\sigma \in G}(t - \sigma(x))$.)
2. Let U be a multiplicatively closed subset of R such that $\sigma(U) \subseteq U$ for all $\sigma \in G$ and let $U^G = U \cap R^G$. Show that the action of G on R extends to an action on $U^{-1}R$ and that
$$(U^G)^{-1}R^G \simeq (U^{-1}R)^G.$$

Total: 4 points

Extra Problems

These problems are neither to be graded nor need to be submitted. They will be discussed in the exercise session and are highly recommended for exam preparation.

Extra Problem 3.

Let G be a finite group acting on the ring R and let $\mathfrak{p} \in \operatorname{Spec} R^G$. Let $P = \{\mathfrak{q} \in \operatorname{Spec} R \ : \ \mathfrak{q} \cap R^G = \mathfrak{p}\}$, the set of primes of R which contract to $\mathfrak{p}$ in R^G. Show that G acts transitively on P. In particular, P is finite.

Extra Problem 4.

For a semigroup H with neutral element $0 \in H$ we define the associated **semigroup algebra** $\mathbb{C}[H] := \bigoplus_{h \in H} \mathbb{C} \cdot \chi_h$ with multiplication among the basis vectors given by

$$\chi_h \cdot \chi_{h'} := \chi_{h+h'}.$$

Assume that $H \subseteq \mathbb{Z}^n$ is finitely generated with $\mathbb{Z}^n = H - H := \{h - h' \ : \ h, h' \in H\}$.

1. Show that $\mathbb{C}[H]$ is a normal ring if and only if $H = \mathbb{Z}^n \cap (\mathbb{Q}_{\geqslant 0} \cdot H)$ inside $\mathbb{Q}^n$. (This condition is referred to as "H is saturated").
2. Give an example where this condition does not hold true.

Hint: For the "$\Leftarrow$" part of Question 1 write H as an intersection of halfspaces, and thus reduce the claim to the special case $H = \mathbb{N} \times \mathbb{Z}^{n-1}$.

Extra Problem* 5.

Let $\mathbb{K}$ be an algebraically closed field. Let $S = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring in n variables over $\mathbb{K}$ and let $I \subseteq S$ be an ideal. Denote by

$$\begin{aligned} V &:= \mathcal{V}(I) = \{\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{A}^n(\mathbb{K}) \mid f(\mathbf{a}) = 0 \ \forall \ f \in I\} \quad \text{and} \\ \mathcal{I}(V) &:= \{g \in S \mid g(\mathbf{a}) = 0 \ \forall \ \mathbf{a} \in V\}. \end{aligned}$$

Prove that $\mathcal{I}(V) = \sqrt{I}$.

One inclusion was proven in Lemma 1.52. Consider the weak Nullstellensatz from Corollary 6.18 (in the lecture notes) and the indications given in Atiyah-Macdonald Exercise 14 after Chapter 7 (page 85).

Extra Problem 6.

Deduce the following result from the Nullstellensatz:
Let $\mathbb{K}$ be an algebraically closed field and $f, g \in \mathbb{K}[x_1, \ldots, x_n]$. Assume that f is irreducible and that $V(f) \subseteq V(g)$. Show that f divides g in $\mathbb{K}[x_1, \ldots, x_n]$.

Extra Problem 7.

Let X and Y be tow irreducible affine algebraic sets. Let $F : X \longrightarrow Y$ be a regular map between them and let $F^* : \mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$ be the corresponding homomorphism of algebras. Show that F is dominant (i.e. $\overline{F(X)} = Y$) if and only if F^* is injective.
(Recall that X is irreducible if and only if $\mathcal{O}(X) = \mathbb{K}[\mathbf{x}]/\mathcal{I}(X)$ is a domain.)