Freie Universität Berlin WiSe24/25
Institut für Mathematik A. Constantinescu
Stand: 22. Januar 2025 J. Rückert

Algebra I – Homework 13

Deadline: **20:00 on Wednesday 29.01.2025.** (Uploads are still possible until Friday 31.01 at 23:55)

Submission: individually, on Whiteboard as `LASTname_A1_H13.pdf`

Full written proofs are required in support of your answers.

Problem 1. **2 points**

Show that every unique factorization domain is integrally closed (in its field of fractions).

Problem 2. **2 points**

Let $f : R \longrightarrow S$ be an integral homomorphism of rings. Prove that $f^* : \mathrm{Spec}(S) \longrightarrow \mathrm{Spec}(R)$ is a *closed* map (i.e. maps closed sets to closed sets).

Total: 4 points

Extra Problems

These problems are neither to be graded nor need to be submitted. They will be discussed in the exercise session and are highly recommended for exam preparation.

Extra Problem 3.

Let $A \subset B$ be a ring extension such that $B \setminus A$ is closed under multiplication. Show that A is integrally closed in B.

Extra Problem 4.

Prove that $\mathbb{Z}[\sqrt{5}]$ not a UFD is and show that $\mathbb{Z}[\frac{1+\sqrt{5}}{2}]$ is its integral closure. (Use the fact that the latter ring is Euclidean, and thus a UFD).
(You may also check out [Stark, An Introduction to Number Theory (1978) p.292])

Extra Problem 5.

Determine the field of fractions and the integral closure (a.k.a. normalization) for the following domains:

1. $\mathbb{Z}[2i] = \{a + b2i \mid a, b \in \mathbb{Z},\ i = \sqrt{-1}\}$.
2. $\mathbb{C}[x^2, xy, y^2]$.

Extra Problem 6.

Let $\mathbb{Z}[i] := \{a + bi \in \mathbb{C} \mid a, b \in \mathbb{Z}\}$ be the ring of Gaussian integers. The operations are the usual ones on complex numbers.

1. Determine all invertible elements of $\mathbb{Z}[i]$.
2. Which of the Gaussian integers $2, 5, 7, 17$ are irreducible?
(An element a is irreducible it is not a unit and if $a = bc$ implies that either b or c is a unit.)

Extra Problem 7.

Consider the following ideals of $\mathbb{Z}[i]$:

$$I_1 = \langle 1 + i \rangle\,, \quad I_2 = \langle 2 \rangle\,, \quad I_3 = \langle 3 \rangle\,.$$

1. Determine a system of representatives for the equivalence modulo I_k for every k.
2. For each $k = 1, 2, 3$ determine the inverse of 5 in $\mathbb{Z}[i]/I_k$.
3. Which of the three ideals I_1, I_2, I_3 is maximal?
4. For those I_k which are maximal, determine to which "known" filed the quotient $\mathbb{Z}[i]/I_k$ is isomorphic to. If this field is finite, then present it as $\mathbb{F}_p[x]/\langle f \rangle$, where $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ and f is an irreducible polynomial and determine explicitly an isomorphism

$$\mathbb{Z}[i]/I_k \simeq F_p[x]/\langle f \rangle\,.$$