Freie Universität Berlin
Institut für Mathematik
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WiSe24/25
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Algebra I – Homework 6

Deadline: **20:00 on Wednesday 27.11.2024.** (Uploads are still possible until Friday 29.11 at 23:55)

Submission: individually, on Whiteboard as `LASTname_A1_H6.pdf`

Full written proofs are required in support of your answers.

Problem 1. — 1 point

Let $I \subseteq R$ be a nilpotent ideal[†] and M an $R-$module which is not necessarily finitely generated. Show that $IM = M$ implies $M = 0$.

Problem 2. — 3 points

Let R be a ring. An R-module M is **simple** if the only proper R-submodule of M is 0.

1. Prove that an R-module M is simple if and only if $M \cong R/\mathfrak{m}$ for some maximal ideal $\mathfrak{m} \subset R$. (Isomorphic as R-modules!)
2. Let I be an ideal of R such that for every nonzero finitely generated R-module M we have $IM \neq M$. Prove that I is contained in the Jacobson radical of R.
   (Hint: "Less is more"[‡])
3. Give an example of a triple $(R, \mathfrak{m}, M)$, where R is a ring, $\mathfrak{m}$ a maximal ideal of R and $M \neq 0$ a finitely generated R-module with $\mathfrak{m}M = M$.

Total: 4 points

[†] that is there exists $n \in \mathbb{N}$ such that $I^n = 0$.
[‡] In other words, focus on simple things.

Extra Problems

These problems are neither to be graded nor need to be submitted. They will be discussed in the exercise session and are highly recommended for exam preparation.

Extra Problem 3.

Let $(R, \mathfrak{m})$ be a local ring and $I \subsetneq R$. Suppose $x \in \mathfrak{m}$ is a nonzerodivisor on R/I. Show that any minimal set of generators for I reduces modulo (x) to a minimal set of generators for the image of I in $R/(x)$. Show by example that this can fail if x is a zerodivisor on R/I.

Extra Problem 4.

Let R be a ring and M_i, N_i be R-modules for $i = 1, \dots 5$. Consider the following commutative diagram with exact rows:

$$\begin{tikzcd}
M_1 \arrow[r, "f_1"] \arrow[d, "\varphi_1"] & M_2 \arrow[r, "f_2"] \arrow[d, "\varphi_2"] & M_3 \arrow[r, "f_3"] \arrow[d, "\varphi_3"] & M_4 \arrow[r, "f_4"] \arrow[d, "\varphi_4"] & M_5 \arrow[d, "\varphi_5"] \\
N_1 \arrow[r, "g_1"] & N_2 \arrow[r, "g_2"] & N_3 \arrow[r, "g_3"] & N_4 \arrow[r, "g_4"] & N_5
\end{tikzcd}$$

Prove that if φ_1 is an epimorphism, φ_2 and φ_4 are isomorphisms, and φ_5 is a monomorphism, then φ_3 is an isomorphism. (Exact 1st row means that $\operatorname{Im} f_i = \operatorname{Ker} f_{i+1}$ for $i = 1, 2, 3$.)

Extra Problem 5.

Let $f : M \longrightarrow N$ be an R-module homomorphism. Show that $\operatorname{Coker}(f)$ and $\operatorname{Ker}(f)$ are uniquely determined by the following universal properties, respectively[§].

The Universal Property of the Cokernel. For every R-linear map $g : N \longrightarrow P$, with the property that $g \circ f = 0$, there exists a unique R-linear map $\overline{g} : \operatorname{Coker}(f) \longrightarrow P$, such that $g = \overline{g} \circ \pi$, i.e. such that the following diagram commutes.

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[dr, "0"'] & N \arrow[r, "\pi"] \arrow[d, "g"] & \operatorname{Coker}(f) \arrow[dl, red, "\exists!"', "\overline{g}"] \\
& P &
\end{tikzcd}$$

The Universal Property of the Kernel. For every R-linear map $h : Q \longrightarrow M$ with the property that $f \circ h = 0$, there exists a unique R-linear map $\overline{h} : Q \longrightarrow \operatorname{Ker}(f)$, such that $h = i \circ \overline{h}$, i.e. such that the following diagram commutes.

$$\begin{tikzcd}
\operatorname{Ker}(f) \arrow[r, "i"] & M \arrow[r, "f"] & N \\
& Q \arrow[ul, red, "\exists!", "\overline{h}"'] \arrow[u, "h"] \arrow[ur, "0"'] &
\end{tikzcd}$$

Extra Problem 6.

Let $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ be a short exact sequence of R-modules. Show that if M_1 and M_3 are finitely generated, then so is M_2.

[§] The maps $\pi : N \longrightarrow \operatorname{Coker}(f) = N/\operatorname{Im} f$ and $i : \operatorname{Ker}(f) \longrightarrow M$ are the canonical projection and injection, respectively. In particular, they satisfy $\pi \circ f = 0$ and $f \circ i = 0$. So the universal property states that every module with similar maps ($g : N \longrightarrow P$ with $g \circ f = 0$ and $h : Q \longrightarrow M$ with $f \circ h = 0$) factors through $\operatorname{Coker} f$ and $\operatorname{Ker} f$, respectively.