Freie Universität Berlin
Institut für Mathematik
Stand: 18. November 2024

WiSe24/25
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Algebra I – Homework 5

Deadline: **20:00 on Wednesday 20.11.2024.** (Uploads are still possible until Friday 22.11 at 23:55)

Submission: individually, on Whiteboard as `LASTname_A1_H5.pdf`

Full written proofs are required in support of your answers.

Problem 1.

Let R be a ring and $I \subseteq \sqrt{(0)}$. Let M be an R-module, N be a finitely generated R-module, and $\varphi : M \longrightarrow N$ be a homomorphism. Show that if the induced homomorphism $\overline{\varphi} : M/IM \longrightarrow N/IN$ is surjective, then φ is surjective.

Problem 2.

Let M be a finitely generated R-module.

1. Find an example of a finitely generated module, which has a submodule that is not finitely generated.
2. Show that if $M = M_1 \oplus M_2$, then M_1 and M_2 are finitely generated.
3. Let $\varphi : M \longrightarrow R^n$ a surjective homomorphism. Show that $\mathrm{Ker}(\varphi)$ is finitely generated.[†]

Total: 4 points

[†]Hint: Show that $\mathrm{Ker}(\varphi)$ is a direct summand of M.

Extra Problems

These problems are neither to be graded nor need to be submitted. They will be discussed in the exercise session and are highly recommended for exam preparation.

Extra Problem 3.

Let M, N be R-modules. Prove that:

1. $\operatorname{Ann}(M+N) = \operatorname{Ann}(M) \cap \operatorname{Ann}(N)$.
2. $M : N = \operatorname{Ann}\left(\frac{N+M}{M}\right)$.

Extra Problem 4.

Let $r, s \in \mathbb{N}$ and R be a ring.

1. Show that if there exists a surjective R-linear map $\varphi : R^r \longrightarrow R^s$, then $r \geq s$.
2. Show that if $R^r \cong R^s$ (as R-modules), then $r = s$.
3. Is it true that if $\varphi : R^r \longrightarrow R^s$ is injective, then $r \leq s$? (Either prove it or find a counter example).

Extra Problem 5.

Let M be an R-module and $n \in \mathbb{Z}_{>0}$. For every $i = 1, \ldots n$ let $p_i \in \operatorname{End}_R(M)$ with

- $p_i^2 = p_i$ for every $i = 1, \ldots, n$,
- $p_i \circ p_j = 0$ for all $i \neq j$ and
- $\operatorname{id}_M = p_1 + \cdots + p_n$.

1. Prove that $M = \bigoplus_{i=1}^n \operatorname{Im} p_i$.
2. Prove the converse: If $M = \bigoplus_{i=1}^n M_i$, there exist $p_1, \ldots, p_n$ with the above properties, such that $\operatorname{Im} p_i = M_i$ for $i = 1, \ldots, n$.

Extra Problem* 6.

A **directed set** is a partially ordered set $\mathcal{I}$ such that for each pair $i, j \in \mathcal{I}$, there exists $k \in \mathcal{I}$ such that $i \leq k$ and $j \leq k$. Let R be a ring and $(M_i)_{i \in \mathcal{I}}$ be a family of R-modules indexed by a directed set $\mathcal{I}$. For each pair $i, j \in \mathcal{I}$ with $i \leq j$, let $\mu_{ij} : M_i \longrightarrow M_j$ be an R-linear map, such that the following axioms are satisfied:

(DS1) $\mu_{ii} = \operatorname{id}_{M_i}$ for all $i \in \mathcal{I}$.

(DS2) $\mu_{ik} = \mu_{jk} \circ \mu_{ij}$ whenever $i \leq j \leq k$.

The modules M_i together with the homomorphisms μ_{ij} are said to form a **direct system** $\mathcal{M} = (M_i, \mu_{ij})$ over the directed set $\mathcal{I}$.

Let $C = \bigoplus_{i \in \mathcal{I}} M_i$. We identify M_i with its canonical image under the canonical injection $j_i : M_i \longrightarrow C$, and define the R-submodule of C:

$$D := \langle x_i - \mu_{ij}(x_i) \mid \forall\, x_i \in M_i, \quad \forall\, j \geq i \rangle \,.$$

Denote by $M := C/D$ the quotient module, by $\mu : C \longrightarrow M$ the canonical projection, and for each $i \in \mathcal{I}$ by $\mu_i := \mu|_{M_i}$ the restriction to M_i. The homomorphisms μ_i are part of the data, and the **direct limit** of the direct system $\mathcal{M} = (M_i, \mu_{ij})$ is the pair

$$\varinjlim M_i := (M, (\mu_i)_{i \in \mathcal{I}}) \,.$$

1. Can you interpret $\mathbb{Q}$ as a direct limit of finitely generated $\mathbb{Z}$-modules?[†]
2. Show that $\forall x \in M$ there exist $i \in \mathcal{I}$ and $x_i \in M_i$ sucht that $x = \mu_i(x_i)$.
3. Show that if $\mu_i(x_i) = 0$ for some $i \in \mathcal{I}$, then there exists $j \geq i$ such that $\mu_{ij}(x_i) = 0$.
4. Show that the direct limit is characterized up to isomorphism[‡] by the following universal property: For any R-module N and for any family $(\alpha_i : M_i \longrightarrow N)_{i \in \mathcal{I}}$ of R-linear maps with the property that $\alpha_i = \alpha_j \circ \mu_{ij}$ whenever $i \leq j$, there exists a unique homomorphism $\alpha : M \longrightarrow N$ such that $\alpha_i = \alpha \circ \mu_i$ for all $i \in \mathcal{I}$.
5. Solve exercises 21 and 22 at the end of Chapter 2 in Atiyah-Macdonald.

[†] Hint: Try $\frac{1}{n}\mathbb{Z}$, indexed by the directed set $\mathbb{N} \setminus \{0\}$ with divisibility as partial order.

[‡] i.e. the direct limit satisfies this, and if any other pair $(M', (\mu_i')_{i \in \mathcal{I}})$ satisfies the property, then it is canonically isomorphic to $\varinjlim M_i$.