Freie Universität Berlin WiSe24/25
Institut für Mathematik A. Constantinescu
Stand: 7. November 2024 J. Rückert

Algebra I – Homework 4

Deadline: 20:00 on Wednesday 13.11.2024. (Uploads are still possible until Friday 15.11 at 23:55)

Submission: individually, on Whiteboard as `LASTname_A1_H2.pdf`

Full written proofs are required in support of your answers.

Problem 1. 2 points

A regular map $f : X \longrightarrow Y$ of affine algebraic sets is an isomorphism if it has an inverse $f^{-1} : Y \longrightarrow X$ which is also a regular map.

Consider the affine algebraic subsets of $\mathbb{A}^2_{\mathbb{C}}$: $Z = V(y - x^2)$ and $W = V(y^2 - x^3)$, and the regular map $\varphi : Z \longrightarrow W$ given by $\phi(a, b) = (b, a^3)$.

1. Describe the pullback $\varphi^* : \mathcal{O}(W) \longrightarrow \mathcal{O}(Z)$.
2. Show that φ is a bijective regular map, but that it is not an isomorphism of affine algebraic sets.

Problem 2. 2 points

Let R be a ring, and consider the Zariski topology on $\operatorname{Spec}(R)$. Let $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec}(R)$. Show that

1. $\mathfrak{q} \in \overline{\{\mathfrak{p}\}} \iff \mathfrak{p} \subseteq \mathfrak{q}$.
2. The closure of a point is $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$. Which are the closed points of $\operatorname{Spec} R$?

Total: 4 points

Extra Problems

These problems are neither to be graded nor need to be submitted. They will be discussed in the exercise session and are highly recommended for exam preparation.

Extra Problem3.

Let $\mathbb{K}$ be an algebraically closed field. Consider the regular map $f : \mathbb{A}^2_{\mathbb{K}} \longrightarrow \mathbb{A}^2_{\mathbb{K}}$ defined by $f(x, y) = (x, xy)$.

1. Find the image $f(A^2_{\mathbb{K}})$; is it open in $A^2_{\mathbb{K}}$? Is it dense? Is it closed?
2. The same question for the map $f : \mathbb{A}^3_{\mathbb{K}} \longrightarrow \mathbb{A}^3_{\mathbb{K}}$ defined by $f(x, y, z) = (x, xy, xyz)$.

Extra Problem 4.

Draw pictures of $\mathrm{Spec}(\mathbb{R})$, $\mathrm{Spec}(\mathbb{K}[[x]]/(x^n))$, $\mathrm{Spec}(\mathbb{Z})$, $\mathrm{Spec}(\mathbb{R}[x])$, $\mathrm{Spec}(\mathbb{C}[x])$, $\mathrm{Spec}(\mathbb{Z}[x])$.

Extra Problem 5.

Let R be a ring, and consider the varieties $V(I) = \{\mathfrak{p} \;:\; I \subseteq \mathfrak{p}\} \subseteq \mathrm{Spec}(R)$. Show that

$$V(I) = V(J) \iff \sqrt{I} = \sqrt{J}\,,$$

for any two ideals $I, J \subseteq R$.

Extra Problem 6.

For any ring homomorphism $\varphi : R \longrightarrow S$, we define $\mathrm{Spec}(\varphi) := \varphi^* : \mathrm{Spec}(S) \longrightarrow \mathrm{Spec}(R)$, given by $\varphi^*(\mathfrak{q}) := \varphi^{-1}(\mathfrak{q})$ (the fibre over $\mathfrak{q}$).

1. Show[‡] that Spec defines a contravariant functor from the category of commutative rings to the category of topological spaces (where $\mathrm{Spec}(R)$ is endowed with the Zariski topology).
2. Show that if $\varphi : R \longrightarrow S$ is injective, then $\varphi^*(\mathrm{Spec}(S))$ is dense in $\mathrm{Spec}(R)$.
3. Show that if $\varphi : R \longrightarrow S$ is surjective, then φ^* is a homeomorphism of $\mathrm{Spec}(S)$ onto the closed subset $V(\mathrm{Ker}(\varphi))$ of $\mathrm{Spec}(R)$.

Extra Problem* 7.

A topological space X is T_0 if for any $x, y \in X$ there exists an open set U such that

$$(x \in U \text{ and } y \notin U) \quad \text{or} \quad (x \notin U \text{ and } y \in U).$$

A topological space X is T_1 if for any $x, y \in X$ there exist open sets U and V such that

$$(x \in U \text{ and } y \notin U) \quad \text{and} \quad (x \notin V \text{ and } y \in V).$$

Let R be a ring. Prove the following.

[‡]Hint: for continuity use distinguished open sets

1. $\operatorname{Spec}(R)$ is a T_0, but that in general $\operatorname{Spec}(R)$ is not T_1.
2. $\operatorname{Spec}(R)$ is irreducible $\iff \sqrt{0}$ is a prime ideal.
3. The irreducible components of $\operatorname{Spec}(R)$ are the closed sets $V(\mathfrak{p})$, where $\mathfrak{p}$ is a minimal prime ideal of R.