Freie Universität Berlin
Institut für Mathematik
Stand: 30. Oktober 2024

WiSe24/25
A. Constantinescu
J. Rückert

Algebra I – Homework 3

Deadline: **20:00 on Wednesday 6.11.2024.** (Uploads are possible until Friday at 23:55)

Submission: individually, on Whiteboard as `LASTname_A1_H2.pdf`

Full written proofs are required in support of your answers.

Problem 1. — 2 points

An element e in a ring R is idempotent if $e^2 = e$.

1. Find all the idempotent elements of $\mathbb{Z}/12\mathbb{Z}$.
2. Show that if R is a local ring, then 0 and 1 are its only idempotent elements.
3. Give an example of an infinite ring in which all elements are idempotent.

Problem 2. — 2 points

Let $\mathbb{K}$ be algebraically closed and let $I = \sqrt{I} \subseteq \mathbb{K}[x_1, \ldots, x_n]$ be a radical ideal. Show that[†]

$$V(I) \text{ is irreducible } \iff I \text{ is prime.}$$

Total: 4 points

[†] irreducible = not the union of two Zariski closed proper subsets. See Extra Problem for more details.

Extra Problems

These problems are neither to be graded nor need to be submitted. They will be discussed in the exercise session and are highly recommended for exam preparation.

Extra Problem 3.

Let $f : R \longrightarrow S$ be a ring homomorphism, and I_k and J_k be ideals of R respectively S. Prove the following.

(a) $(I_1 + I_2)^e = I_1^e + I_2^e$ and $(J_1 + J_2)^c \supseteq J_1^c + J_2^c$.

(b) $(I_1 \cap I_2)^e \subseteq I_1^e \cap I_2^e$ and $(J_1 \cap J_2)^c = J_1^c \cap J_2^c$.

(c) $(I_1 I_2)^e = I_1^e I_2^e$ and $(J_1 J_2)^c \supseteq J_1^c J_2^c$.

(d) $(I_1 : I_2)^e \subseteq I_1^e : I_2^e$ and $(J_1 : J_2)^c \subseteq J_1^c : J_2^c$.

(e) $(\sqrt{I}\,)^e \subseteq \sqrt{I^e}$ and $(\sqrt{J}\,)^c = \sqrt{J^c}$.

Extra Problem 4.

Show that an ideal $I \subseteq R$ is radical if and only if it is an intersection of prime ideals.

Extra Problem 5.

Determine whether the following sets are algebraic or not.

1. $\{(\sin t, \cos t) \ : \ t \in \mathbb{R}\}$.
2. $\{(t, \sin t) \ : \ t \in \mathbb{R}\}$.

Extra Problem 6.

A topological space X is called **irreducible** if it cannot be written as $X = X_1 \cup X_2$ with X_1, X_2 proper closed subsets of X. A subset A of X is called **dense** if its closure $\bar{A} = X$. Show that

$$X \text{ is irreducible} \iff \text{every nonempty open subset } U \subseteq X \text{ is dense.}$$

Extra Problem 7.

Let $\mathbb{K}$ be a field.

1. Describe the Zariski open subsets of $\mathbb{A}^1_{\mathbb{K}}$. (Recall that $\mathbb{K}[x]$ is a principal ideal domain).
2. As sets we have $\mathbb{A}^2_{\mathbb{K}} = \mathbb{A}^1_{\mathbb{K}} \times \mathbb{A}^1_{\mathbb{K}}$. Check that the Zariski topology on $\mathbb{A}^2_{\mathbb{K}}$ is not the product topology on $\mathbb{A}^1_{\mathbb{K}} \times \mathbb{A}^1_{\mathbb{K}}$, where each $\mathbb{A}^1_{\mathbb{K}}$ is endowed with the Zariski topology. (In the product topology on $X \times Y$ the open sets are unions of subsets $U \times V$ with U and V open subsets of X respectively Y.)