Freie Universität Berlin
Institut für Mathematik
Stand: 21. Oktober 2024

WiSe24/25
A. Constantinescu
J. Rückert

Algebra I – Homework 2

Deadline: **20:00 on Wednesday 30.10.2024.** (Uploads are possible until Friday at 23:55)

Submission: individually, on Whiteboard as `LASTname_A1_H2.pdf`

Full written proofs are required in support of your answers.

Problem 1. **2 points**

Let R be a ring. Show that in $R[x]$ the nilardical $\mathcal{N}_{R[x]}$ and the Jacobson radical $\mathcal{J}_{R[x]}$ coincide.

Problem 2. **2 points**

Decompose the ring $\mathbb{R}[x]/(x^3+1)$ and $\mathbb{C}[x]/(x^3+1)$ into products of fields.

Total: 4 points

Extra Problems

These problems are neither to be graded nor need to be submitted. They will be discussed in the exercise session and are highly recommended for exam preparation.

Extra Porblem 3.

Describe the ideals of the formal power series ring $\mathbb{K}[[x]]$.

Extra Porblem 4.

Let I, J, K, and for every i, I_i and J_i be ideals of a ring R. Prove the following:

(i) $I \subseteq I : J$

(ii) $(I : J)J \subseteq I$

(iii) $(I : J) : K = I : (JK) = (I : K) : J$

(iv) $(\cap_i I_i) : J = \cap_i (I_i : J)$

(v) $I : (\sum_j J) = \cap_j (I : J_j)$

(vi) $I \subseteq \sqrt{I}$

(vii) $\sqrt{I} = \sqrt{\sqrt{I}}$

(viii) $\sqrt{IJ} = \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$

(ix) $\sqrt{I} = (1) \Leftrightarrow I = (1)$

(x) $\sqrt{I + J} = \sqrt{\sqrt{I} + \sqrt{J}}$

(xi) If $\mathfrak{p}$ is a prime ideal, then $\sqrt{\mathfrak{p}^n} = \mathfrak{p}$ for all $n > 0$.

Extra Porblem 5.

1. What is the smallest positive multiple of 10 which has remainder 2 when divided by 3 and remainder 3 when divided by 7?
2. Let $\mathbb{K}$ be a field. Describe the ring $\mathbb{K}[x]/(x^2 - 1)$. Is it a product of fields?

Extra Porblem 6.

Let R be a ring and $\mathcal{N}_R$ its nilradical. Prove that the following statements are equivalent.

(i) R has exactly one prime ideal.

(ii) Every element of R is either a unit or nilpotent.

(iii) $R/\mathcal{N}_R$ is a field.

Extra Porblem 7.

1. Show that the set of all prime ideals of a ring has an inclusion-minimal element.
2. Show that minimal prime ideals consist entirely of zero divisors. (hard!)