Freie Universität Berlin
Institut für Mathematik
Stand: 16. Oktober 2024

WiSe24/25
A. Constantinescu
J. Rückert

Algebra I – Homework 1

Deadline: **20:00 on Wednesday 23.10.2024.** (Uploads are possible until Friday at 23:55)

Submission: individually, on Whiteboard as `LASTname_A1_H1.pdf`

Full written proofs are required in support of your answers.

Problem 1.

Let R be a ring and $a, b \in R$.

1. Show that if a and b are nilpotent, then so is $a + b$.
2. Show that if a is nilpotent and b is a unit, then $a + b$ is a unit.

Problem 2.

Let $n \in \mathbb{Z}_{>0}$. Show that $\mathbb{C}[x]/(x^n)$ is a local ring, and describe all its prime ideals.

Total: 4 points

Extra Problems

These problems are neither to be graded nor need to be submitted. They will be discussed in the exercise session and are highly recommended for exam preparation.

Extra Porblem 3.

Let I, J be ideals of a ring R. Show that

$$I \cup J \text{ is an ideal } \Leftrightarrow I \subseteq J \quad \text{or} \quad J \subseteq I.$$

Extra Porblem 4.

Show that the rings $\mathbb{R}[x]/(x^2+1)$ and $\mathbb{C}$ are isomorphic.

Extra Porblem 5.

Let R be a ring, and $R[x]$ the ring of polynomials in one variable with coefficients in R. Let $f = c_0 + c_1 x + \cdots + c_n x^n \in R[x]$. Show that

1. f is nilpotent $\Leftrightarrow$ $c_0, \ldots, c_n$ are nilpotent.
2. f is a unit $\Leftrightarrow$ c_0 is a unit and $c_1, \ldots, c_n$ are nilpotent.
3. f is a zero divisor $\Leftrightarrow$ there exists $a \in R$ such that $af = 0$.
4. Determine the zero divisors, nilpotent elements, and units of $\mathbb{Z}/36\mathbb{Z}[x]$.

Extra Porblem 6.

Determine all $n \in \mathbb{N}$ such that $\mathbb{Z}/n\mathbb{Z}$ is a local ring.

Extra Porblem 7.

Consider the ring homomorphism $\varphi : \mathbb{C}[x, y] \longrightarrow \mathbb{C}[t]$ given by

$$x \mapsto t^2 \qquad \text{and} \qquad y \mapsto t^3.$$

Show that $\ker \varphi = (y^2 - x^3)$ and describe the image of φ.