ALGEBRA 1 EXAM

1^{st} TRY , 19.2.2025

SKETCH OF SOLUTIONS (THIS IS NOT A GRADING SCHEME)

Problem 1. a. Let $I, J \subseteq R$.

$I \cdot J = (a \cdot b \mid a \in I, b \in J)$, the ideal generated by all products.

$\sqrt{I} = \{ r \in R \;;\; \exists n \in \mathbb{N} \text{ such that } r^n \in I \}$

b. For all ideals we have $IJ \subseteq I \cap J$. Thus also $\sqrt{IJ} \subseteq \sqrt{I \cap J}$. It remains to show $\sqrt{I \cap J} \subseteq \sqrt{IJ}$.

Let $f \in \sqrt{I \cap J}$. So there exists $n \in \mathbb{N}$ with $f^n \in I \cap J$.

Then $f^{2n} = f^n \cdot f^n \in I \cdot J$, so $f \in \sqrt{IJ}$.

c. In $A = \mathbb{C}[x]/(x^{2025})$ we have that x^k is nilpotent.

This implies that $x^{45} - 1$ is a unit, because it is the sum of a unit and a nilpotent. So $J = A$, which implies $I \cdot J = I \cdot A = I$.

We have $\sqrt{I} = \sqrt{(x^{13})} = (x)$. As (x) is maximal in A, $(A/(x) \cong \mathbb{C})$ this implies I is (x)-primary.

d. We have for any $r \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ that $r^2 = r$ and $r + r = 0$.

So, if $I = (r, s)$, then $I = (r + s + rs)$, because

- Clearly $r+s+rs \in I$.
- For the other inclusion:

$r\cdot(r+s+rs) = r^2 + \underbrace{rs+rs}_{=0} = r^2 = r$, thus $r \in (r+s+rs)$.

Similarly $s \in (r+s+rs)$, thus $I \subseteq (r+s+rs)$.

So, whenever we have two generators of an ideal, we may replace them by a single element.
This implies that any finitely generated ideal is principal.
The ring is finite, so we are done.

Alternative proof: - In general, for any two rings, the ideals of $R\times S$ are all of the form $I\times J$, with $I \subseteq R$ and $J \subseteq S$ ideals. (Quick check.)

In our case we have $\mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_2$, so there are eight ideals in total: All combinations of $I_1 \times I_2 \times I_3$ with $I_j \in \{(0), (1)\}$.

Each is generated by (a_1, a_2, a_3) with

$$a_j = \begin{cases} 0, & \text{if } I_j = (0) \\ 1, & \text{if } I_j = (1) \end{cases}.$$

e. As $b^2 = b\ \forall b \in B$, we have for every $\mathfrak{p} \in \operatorname{Spec} B$ that $b(b-1) = 0 \in \mathfrak{p}$. Thus $b \in \mathfrak{p}$ or $b-1 \in \mathfrak{p}$. This means that $B/\mathfrak{p} \simeq \mathbb{F}_2$, thus $\mathfrak{p}$ is maximal.

Problem 2. a. An R-module M is Artinian, if it satisfies the descending chain condition, that is if for every infinite sequence of R-submodules of M:

$$M_1 \supseteq M_2 \supseteq M_3 \supseteq \dots\dots$$

there exists $n_0 \in \mathbb{N}$, such that $M_{n_0} = M_{n_0+i}$, $\forall\, i \in \mathbb{N}$.

b. Let R be an Artinian domain, and let $a \in R \setminus \{0\}$.
We have the descending chain of ideals:

$$(a) \supseteq (a^2) \supseteq (a^3) \supseteq \dots$$

Because R is Artinian, there exist $n_0 \in \mathbb{N}$ such that

$(a^{n_0}) = (a^{n_0+1})$. In particular, $a^{n_0} \in (a^{n_0+1})$, thus there exists $r \in R$ with $a^{n_0} = r \cdot a^{n_0+1}$. This implies:

$$a^{n_0}(1 - r \cdot a) = 0.$$

As R is a domain and $a \neq 0$ (thus $a^{n_0} \neq 0$), we get $1 - r \cdot a = 0$, meaning a is invertible in R.
So R is a field.

c. We have in $\mathbb{Z}/_{12\mathbb{Z}}$: $(0) \subseteq (6) \subseteq (3) \subseteq (1)$ which we claim is a composition series of length 3:

$$(6)/_{(0)} = {6 \cdot \mathbb{Z}}/_{12\mathbb{Z}} \simeq \mathbb{Z}/_{2\mathbb{Z}} \simeq \mathbb{F}_2 \text{, which is simple.}$$

$$(3)/_{(6)} = \frac{3\mathbb{Z}/_{12\mathbb{Z}}}{6\mathbb{Z}/_{12\mathbb{Z}}} \simeq \frac{3\mathbb{Z}}{6\mathbb{Z}} \simeq \mathbb{Z}/_{2\mathbb{Z}} \simeq \mathbb{F}_2 \text{, which is simple.}$$

$$(1)/_{(3)} = \frac{\mathbb{Z}/_{12\mathbb{Z}}}{3\mathbb{Z}/_{12\mathbb{Z}}} \simeq \mathbb{Z}/_{3\mathbb{Z}} \simeq \mathbb{F}_3 \text{, which is simple.}$$

So $\ell(\mathbb{Z}/_{12\mathbb{Z}}) = 3$.

d. Quick solution: Use $R/I \otimes_R M \simeq M/IM$.

$$\mathbb{Z}/12\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/35\mathbb{Z} \simeq \frac{\mathbb{Z}/35\mathbb{Z}}{(12)\cdot \mathbb{Z}/35\mathbb{Z}}.$$

Because 12 is invertible mod 35, the quotient is zero.

Elementary solution: show $[a]_{12} \otimes [b]_{35} = 0$, $\forall a,b \in \mathbb{Z}$.

12 is coprime to 35, so $[12]_{35}$ is invertible in $\mathbb{Z}/35\mathbb{Z}$.

(Explicitly: $[12]_{35} \cdot [3]_{35} = [36]_{35} = [1]_{35}$)

So $[a]_{12} \otimes [b]_{35} = [a]_{12} \otimes 12 \cdot [3b]_{35} = 12 [a]_{12} \otimes [3b]_{35} =$

$= [12a]_{12} \otimes [3b]_{35} = [0]_{12} \otimes [3b]_{35} = 0$.

e. Using the associativity of the tensor product and the general fact that $L \otimes_S S \simeq L$ we get

$$0 = R/\underline{m} \otimes_R 0 = R/\underline{m} \otimes_R (M \otimes_R N) \simeq (R/\underline{m} \otimes_R M) \otimes_R N$$

$$\simeq \left[(R/\underline{m} \otimes_R M) \otimes_{R/\underline{m}} R/\underline{m}\right] \otimes_R N \simeq (R/\underline{m} \otimes_R M) \otimes_{R/\underline{m}} (R/\underline{m} \otimes_R N),$$

Now apply $R/\underline{m} \otimes_R M \simeq M/\underline{m}M$, and $R/\underline{m} \simeq \mathbb{k}$ - a field:

$$M/\underline{m}M \otimes_{\mathbb{k}} N/\underline{m}N = 0.$$

The tensor product of two vector spaces is zero only if one of the factors is zero, so we get

$$M/\underline{m}M = 0 \quad \text{or} \quad N/\underline{m}N = 0.$$

From Nakayama's lemma for local rings it follows that $M = 0$ or $N = 0$.

Problem 3. **a.** On $\operatorname{Spec} R = \{\mathfrak{p} \subseteq R : \mathfrak{p} = \text{prime ideal}\}$ we define for every ideal I, the subset of $\operatorname{Spec} R$:

$$V(I) = \{\mathfrak{p} \in \operatorname{Spec} R, \text{ with } I \subseteq \mathfrak{p}\}.$$

The Zariski topology on $\operatorname{Spec} R$ is defined by taking $V(I)$ for every ideal $I \subseteq R$ as the closed sets.

b. We have to show that for every $J \subseteq S$, there exists $I \subseteq R$ such that $f^*(V(J)) = V(I)$.

<u>Claim</u>: This holds for $I := f^{-1}(J)$.

(Identifying R with $f(R)$, because f = inj., we have $f^{-1}(J) = J \cap R$)

$\boxed{\subseteq}$ $f^*(V(J)) = \{f^{-1}(\mathfrak{q}) : \mathfrak{q} \in \operatorname{Spec} S \text{ and } \mathfrak{q} \supseteq J\}$.

$\mathfrak{q} \supseteq J$ implies $f^{-1}(\mathfrak{q}) \supseteq f^{-1}(J) = I$, so $f^*(V(J)) \subseteq V(I)$.

$\boxed{\supseteq}$ Let $\mathfrak{p} \in V(I)$. That is $\mathfrak{p} \in \operatorname{Spec} R$, with $\mathfrak{p} \supseteq f^{-1}(J) = J \cap R$.

From the lecture (Prop. 8.9.a): $R \subseteq S$ integral $\Rightarrow$ $R/_{J\cap R} \subseteq S/_J$ is also integral. Write $\overline{\mathfrak{p}} \in \operatorname{Spec} R/_{J\cap R}$, for the image of $\mathfrak{p}$ in $R/_{J\cap R}$.

By the "Lying over Theorem" (Thm. 8.13 for $R/_{J\cap R} \subseteq S/_J$) we have $\exists\, \overline{\mathfrak{q}} \in \operatorname{Spec} S/_J$, such that $\overline{\mathfrak{q}} \cap R/_{J\cap R} = \overline{\mathfrak{p}}$.

The one-to-one correspondence between ideals of $S/_J$ and ideals of S which contain J, induces a bijection between $\operatorname{Spec} S/_J$ and $V(J)$. The corresponding ideal of $\overline{\mathfrak{q}}$ is then $\mathfrak{q} \in V(J)$ with $\mathfrak{q} \cap R = f^*(\mathfrak{q}) = \mathfrak{p}$.

Thus $\mathfrak{p} \in f^*(V(J))$.

c. We have $\mathbb{C}[x^3, x^2y, xy^2, y^3] \subseteq \mathbb{C}[x,y]$ (subring).

Furthermore, $\mathbb{C}[x,y]$ is integral over A, because

x is a solution of $t^3 - x^3 \in A[t]$ and

y is a solution of $t^3 - y^3 \in A[t]$ and

$\mathbb{C}[x,y] = A[x,y]$.

So $\dim_{\text{Krull}} A = \dim_{\text{Krull}} \mathbb{C}[x,y] = 2$.

d. If $R \subseteq S$ is an integral ring extension, then for every $\mathfrak{p} \in \operatorname{Spec} R$, there exists $\mathfrak{q} \in \operatorname{Spec} S$

such that $\mathfrak{q} \cap R = \mathfrak{p}$.

Furthermore: $\mathfrak{p}$ is maximal if and only if $\mathfrak{q}$ is maximal.

Applying this to the integral extension $A \subseteq \mathbb{C}[x,y]$ we get:

So $\operatorname{MaxSpec} A = \{\mathfrak{m} \cap A : \mathfrak{m} \in \operatorname{MaxSpec} \mathbb{C}[x,y]\}$

As $\mathbb{C}$ is algebraically closed, we have by Hilbert's Nullstellensatz, that:

$$\operatorname{Max Spec} \mathbb{C}[x,y] = \{(x-a, y-b) : (a,b) \in \mathbb{C}^2\}.$$

Finally, $(x-a, y-b) \cap \mathbb{C}[x^3, x^2y, xy^2, y^3] =$

$$= (x^3 - a^3,\ x^2y - a^2b,\ xy^2 - ab^2,\ y^3 - b^3).$$