Advanced statistical mechanics II

(SS 16, FU Berlin)

Problem sheet 10

Due date: July 5th, 2016

Problems

26. Fermions on a ring (1+3+1+1+1+3+2=12)

In this exercise we look at a very simple model for Fermions on a ring. We assume non-interacting and spin-less Fermions that can hop on a discretized line made up of N "sites". Each site j only has one single-particle state, so our system is described by N fermionic annihilation and creation operators $\{c_j, c_j^\dagger\}$, $j = 1, \dots, N$. We assume periodic boundary conditions, so that $c_{N+1} = c_1$, and that the Hamiltonian of the system conserves the number of Fermions and only couples nearest neighbours. Then its general form is given by

$$H = \frac{1}{2}\sum_{j=1}^{N}\left(\epsilon\left(c_j^\dagger c_j - c_j c_j^\dagger\right) + \lambda_1\left(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j\right) + \mathrm{i}\lambda_2\left(c_j^\dagger c_{j+1} - c_{j+1}^\dagger c_j\right)\right) \tag{1}$$

a) Show that, if U is any unitary $N \times N$ matrix, the operators f_k defined by

$$c_j = \sum_{k=1}^{N} U_{jk} f_k \tag{2}$$

and their adjoints $f_k^\dagger$ fulfill the canonical anti-commutation relations $\{f_k, f_{k'}^\dagger\} = \delta_{kk'}$, $\{f_k, f_{k'}\} = 0$.

b) Show that, by chosing $U_{jk} = \frac{1}{\sqrt{N}}\exp(-2\pi\mathrm{i}\frac{jk}{N})$, i.e., the discrete Fourier transform, the Hamiltonian can be written in the form

$$H = \sum_{k=1}^{N} E_k f_k^\dagger f_k + C\mathbb{1}, \tag{3}$$

where C is independent of k, and the single-particle energies E_k are of the form

$$E_k = d_0 + d_1\cos\left(\frac{2\pi k}{N}\right) + d_2\sin\left(\frac{2\pi k}{N}\right) \tag{4}$$

with constants d_0, d_1, d_2. What are the constants?

c) How does the term C in eq. 3 depend on N? What happens in the thermodynamic limit $N \to \infty$? Discuss.

We will now drop the constant term C in the Hamiltonian and for simplicity set $\lambda_1 = -1, \lambda_2 = 0$.

d) Sketch the single-particle energies as function of k in the three regimes given by $\epsilon > 1, |\epsilon| < 1, \epsilon < -1$.

e) How do you obtain the ground-state of the system?

f) Compute the average ground-state energy per site ϵ_0 in the thermodynamic limit for the three regimes given by $\epsilon > 1, |\epsilon| < 1, \epsilon < -1$.

g) Is the ground-state energy per site analytic at $|\epsilon| = 1$?

27. Jordan-Wigner transformation (1+1.5+1.5+1+2=7)

The fermionic Fock-space of N single-particle states and the Hilbert-space of N spin-1/2 particles both have dimension 2^N and are hence isomorphic as vector-spaces. In this exercise we look at a method, called *Jordan-Wigner transformation* , to represent the spin-operators as fermionic operators. To do that we use the raising and lowering operators for spin-1/2, defined as $\sigma^{\pm} = \frac{1}{2}(\sigma^x \pm \mathrm{i}\sigma^y)$.

a) Show $\{\sigma^{\pm}, \sigma^z\} = 0$ and $\{\sigma^-, \sigma^+\} = \mathbb{1}$, where $\{\cdot, \cdot\}$ is the anti-commutator.

We see that a single spin-1/2 can be described by operators fulfilling the fermionic anti-commutation relations. It is thus suggestive to denote the spin-up state as $|1\rangle$ and the spin-down state as $|0\rangle$. We now enumerate our N spin-1/2 particles in an arbitrary way, so that the Pauli-matrices get an index j running from 1 to N. Then we define the operators

$$\tilde{N}_j := \sigma_j^+ \sigma_j^-, \quad \tilde{c}_j := \exp(\mathrm{i}\pi \sum_{k=1}^{j-1} \tilde{N}_k)\sigma_j^-. \tag{5}$$

b) Show $\{\exp(\mathrm{i}\pi\tilde{N}_j), \sigma_j^{\pm}\} = 0$ and $\tilde{N}_j = \tilde{c}_j^\dagger \tilde{c}_j$.

c) Show $\{\tilde{c}_j, \tilde{c}_k^\dagger\} = \delta_{j,k}\mathbb{1}$ and $\{\tilde{c}_j, \tilde{c}_k\} = 0$.

We thus obtained a description of our spin-1/2 system by "virtual" fermions. While the dynamics will be 'truly fermionic', it's good to know that states coming from spins may appear rather unconventional from the fermionic perspective. Real Fermions obey the *parity superselection-rule*, which says that their joint-state cannot be a superposition of states where one of them has an even and the other one an odd number of Fermions. In general they still can be in a super-position of different numbers of particles, though. This happens, for example, in the description of superconductors. Unlike real Fermions however, the virtual Fermions we have constructed do not need to obey this rule since for a spin-1/2 particle there is no reason not to be in a state $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

Let us now take a look at a chain of N spin-1/2 particles with periodic boundary conditions, i.e., $\sigma_{N+1}^\alpha = \sigma_1^\alpha$ for $\alpha \in \{x, y, z\}$. The *fermionic number operator* is defined by $N_F = \sum_{j=1}^N \tilde{N}_j$.

d) Show that the total spin-component in z-direction $S^z = \sum_{j=1}^N \frac{\sigma_j^z}{2}$ is given by $N_F - \frac{N}{2}$.

e) Prove that for $j < N$ we have $\sigma_j^+ \sigma_{j+1}^- = \tilde{c}_j^\dagger \tilde{c}_{j+1}$. What happens for $j = N$?

We thus see that the Jordan-Wigner transformation is useful for nearest-neighbour Hamiltonians but we have to be careful when dealing with periodic boundary conditions.