Advanced statistical mechanics II

(SS 16, FU Berlin)

Problem sheet 1

Due date: May 3, 2016

Brief information

- The problem sheets will be distributed always on Tuesdays.
- Each problem sheet will allow for earning 16-20 points. The cool questions will be marked as bonus and will allow you to get even more points!
- The solutions can be handed in by groups of two students. You will have to present your solutions in the tutorials so both of you have to understand the solutions.
- You are permitted to work together with more students, but then indicate all their names on your completed homework.
- The solution is to be handed in right at the end of the lecture on the Tuesday one week after we have handed out the exercise. If solutions are handed in too late you will get zero points for your solution.
- Please indicate the tutorial in which you participate on your solution.
- It is necessary to have achieved at least 50 percent combined on all exercises to pass the tutorials.
- Tutorial page: `http://userpage.fu-berlin.de/~marekgluza/ASM2_16/` (short link: `bit.ly/1XTWwvn`)

Problems

1. **Fair dice (1+1+1+1+2+1=7)**
   We consider a fair die with six sides. We identify its sample space or state space with $\Gamma = \{1, 2, \ldots, 6\}$.

   a) What is the state space of two dice?

   b) Let X_1 and X_2 be the random variables associated to the outcomes of the first and second die respectively. What subset of the sample space corresponds to the event that the sum of the two dice yields 6, i.e., $X_1 + X_2 = 6$?

   c) What set/event corresponds to the event that at least one of the dice yields 2?

   In situations like this one, where there clearly is a natural probability measure μ, the uniform one in this case (we have fair dice!), one often writes $\mathbb{P}(A)$ for the probability of the event $A \subseteq \Gamma$. It is then also useful to introduce the *conditional probability* $\mathbb{P}(A|B)$ of A given B, for $\mathbb{P}(B) > 0$, which is defined by

   $$\mathbb{P}(A|B) := \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}. \tag{1}$$

   d) Calculate $\mathbb{P}(X_1 = 2 \text{ OR } X_2 = 2 | X_1 + X_2 = 6)$, i.e. the probability that at least one die yields a 2 conditioned on their sum being 6.

   A set of events $\{A_j\}$ is said to be *pairwise independent* iff

   $$\mathbb{P}(A_j \cap A_k) = \mathbb{P}(A_j)\, \mathbb{P}(A_k) \quad \forall\, j \neq k. \tag{2}$$

   and *mutually independent* iff

   $$\mathbb{P}\Big(\bigcap_{A \in \mathcal{A}} A\Big) = \prod_{A \in \mathcal{A}} \mathbb{P}(A) \quad \forall\, \mathcal{A} \subseteq \{A_j\}. \tag{3}$$

e) Suppose the events $\{A_j\}$ are pairwise independent. Compute $\mathbb{P}(A_j|A_k)$ for $j \neq k$. Furthermore prove that the three events

$$A_1 = \{X_1 \text{ is even}\} \tag{4}$$
$$A_2 = \{X_2 \text{ is even}\} \tag{5}$$
$$A_3 = \{X_1 + X_2 \text{ is even}\} \tag{6}$$

are pairwise independent but not mutually independent.

The notion of statistical independence can be extended from events to random variables. A set of random variables $\{X_j : \Gamma \to \mathbb{R}\}$ is said to be *pairwise/mutually independent* iff the associated set of events $\{X_j < \alpha_j\}$ is pairwise/mutually independent for all values of the parameters α_j.

f) Argue that the set of random variables $\{X_1, X_2, X_1 + X_2\}$ is neither pairwise, nor mutually independent.

2. **Efficiency of heat engines (1*+1+1+2+2+1*=2*+6)**
Carnot engines work quasi statically, i.e., infinity slowly and thus have no output power. For a real heat engine one is more interested in maximizing the output power than in maximizing the efficiency. A more realistic model of a heat engine, which takes into account the time necessary to transfer the heat from/to the reservoirs to/from the working medium, is the following:

The hot and cold reservoirs have temperatures T'_+ and T'_- respectively. The engine performs a Carnot cycle. During the isothermal expansion (compression) the working medium of the machine is held at $T_+ < T'_+$ ($T_- > T'_-$) and the heat Q_1 (Q_3) is transfered from (to) the reservoir in the time interval t_1 (t_3), i.e.,

$$Q_1 = k(T'_+ - T_+)t_1 \tag{7}$$
$$Q_3 = k(T'_- - T_-)t_3, \tag{8}$$

where k is the thermal conductivity. We want to find the T_+ and T_- for which the machine gives the optimal output power.

The time for the adiabatic processes is neglected. Even though the problem can be solved in the general case you can assume here for simplicity that $t_3 = t_1$.

a) *Bonus question:* Look up Fourier's law for heat transport. Do you see a sensible connection to (7) and (8)?

b) Does the power depend on the value of t_1?

c) Combine the first and second Law to write T_+ as a function of T_-, T'_+, and T'_-.
Hint: It turns out that $T_+ = T'_+/(2 - T'_-/T_-)$.

d) Use this expression to write the power P as a function of T_- and T'_+, T'_-.

e) Determine the optimal values of T_- and T_+ as a function of only T'_+ and T'_-.
Hint: It turns out that $T_- = (T'_- + \sqrt{T'_- T'_+})/2$.

f) *Bonus question:* Calculate the efficiency of the heat engine as a function of T'_+ and T'_-.

3. **Law of large numbers in toin cossing (1+1+1+2+1=6)**
Let us denote by $X_1, \ldots, X_n$ the result of tossing a coin n times, where $X_i \in \{H, T\}$ - heads and tails respectively. Let us assume that the coin is fair and that the behavior of each

tossing is independent of previous events, so that the probability of a given sequence fulfills $P(X_1, \ldots, X_n) = P(X_1) \times \ldots \times P(X_n)$ where $P(X_i) = \frac{1}{2}\ \forall i$. Use the Stirling bounds $\sqrt{2\pi n} n^n e^{-n} \leq n! \leq e\sqrt{n} n^n e^{-n}$ to show:

a) Take n even. Find a lower bound to the number of combinations $\{X_1, \ldots, X_n\}$ with $n/2$ heads.

b) Take $0 < p < \frac{1}{2}$ and n such that pn is an integer. Find an upper bound to the number of combinations with pn heads. Express it in terms of $H(p)$, where the so-called Shannon binary entropy is defined as $H(x) := -x \log_2 x - (1 - x) \log_2(1 - x)$.
Hint: It appears in the form of $2^{H(p)}$.

c) Use bound in b) to upper-bound the number of combinations that have less or equal than np heads, or more or equal than $n(1 - p)$ heads.

d) Weak law of large numbers: Using that $H(x) < 1\ \forall x \neq \frac{1}{2}$, show that $\forall\, p < \frac{1}{2}$, the probability of obtaining a fraction of heads that is between p and $1 - p$ tends to 1 as $n \to \infty$.

e) Find a lower-bound to the probability that the number of heads is between $40\% - 60\%$. Calculate the explicit values of the bound for n being $\{200, 400, 700\}$.