A Systematic Account of Direction-Based Relations in Real Vector Spaces[21]

Albrecht Iseler, Freie Universität Berlin

The central Section A.2 of this document contains definitions, lemmas and corollaries, which make available some well known properties of linear orders and linear equivalence relations to situations, where a binary relation R is defined only on a subset of a real vector space, or where not all defining properties of linear orders can be taken for granted. The central Lemma A.6 is intended to be a toolbox such that suitable selections of a few assertions ('tools') enable corollaries for rather general application problems. In fact, the compilation of the toolbox has been motivated by the observation that proofs for some application oriented facts, which will be reported as Corollaries A.7, A.8 and A.9, can follow a rather parallel scheme and that a suitable conceptualisation (Definition A.5) can make this scheme available for further applications.

Since commonly accepted terminologies for cones, subspaces and linear orders are lacking in the pertinent literature, some basic concepts are summarized in the preliminary Section A.1. For situations of the kind described above, it is advantageous to combine a taxonomy of cones defined by Choquet (1969) with the taxonomy of linear orders in Holmes (1975) and Jameson (1970), since the basic concepts are weaker there than in the terminologies of most other authors. Of course, this approach requires a translation of Holmes' and Jameson's references to special classes of cones into Choquet's terminology.

A.1 Preliminaries

Definition A.1: A non-empty set F is a *real vector space* iff it is endowed with the operations addition ($\mathbf{x} + \mathbf{y}$, with $\mathbf{x}, \mathbf{y} \in F$) and scalar multiplication ($\lambda \cdot \mathbf{x}$, with $\lambda \in \mathbb{R}$ and $\mathbf{x} \in F$) such that the results of these operations are elements of F with the following properties for all elements $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ of F, and all real numbers λ and μ:

(1) $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$.

(2) $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$.

(3) There is an element $\mathbf{0} \in F$ such that $\mathbf{x} + \mathbf{0} = \mathbf{x}$ for every $\mathbf{x} \in F$.

(4) For every $\mathbf{x} \in F$ there is an element $-\mathbf{x} \in F$ such that $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$.

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[21] The present document is an extended version of the appendix of the chapter Iseler (1998), the extension consisting mainly in Part j of Definition A.2 and Section A.3. However, the present document is a self-contained, systematic presentation of the underlying mathematical theory, whereas the main text of the chapter Iseler (1998) is a more informal introduction. To facilitate shifting between the two documents, the numbering of sections, definitions, propositions, formulas and footnotes is maintained.

(5) $(\lambda + \mu) \cdot \mathbf{x} = \lambda \cdot \mathbf{x} + \mu \cdot \mathbf{x}$.
(6) $\lambda \cdot (\mathbf{x} + \mathbf{y}) = \lambda \cdot \mathbf{x} + \lambda \cdot \mathbf{y}$.
(7) $\lambda \cdot (\mu \cdot \mathbf{x}) = (\lambda \cdot \mu) \cdot \mathbf{x}$.
(8) $1 \cdot \mathbf{x} = \mathbf{x}$.

In this situation, the elements of F are called *vectors*, and the elements of $\mathbb{R}$ are called *scalars*.

For the following notational conventions, let λ be a scalar, $\mathbf{x}$ and $\mathbf{y}$ elements of F, and X and Y subsets of F:

$\mathbf{x} - \mathbf{y} := \mathbf{x} + (-\mathbf{y})$.
$\mathbf{x} / \lambda := (1/\lambda) \cdot \mathbf{x}$.
$\lambda \cdot X := \{\lambda \cdot \mathbf{x}: \mathbf{x} \in X\}$.
$-X := \{\mathbf{x}: -\mathbf{x} \in X\}$.
$X + Y := \{\mathbf{x} + \mathbf{y}: \mathbf{x} \in X \wedge \mathbf{y} \in Y\}$.
$X - Y := X + (-Y)$.

Furthermore, the operator $\cdot$ for scalar multiplication may be omitted, leading to $\lambda\, \mathbf{x} := \lambda \cdot \mathbf{x}$, and $\lambda\, X := \lambda \cdot X$.

The subsequent definition explicates properties which may be present in subsets, maps and relations. Note that some authors use materially different definitions.[22]

Definition A.2: The following definitions apply to every non-empty subset S of a real vector space F, every map $g{:}F\to\mathbb{R}$ and every relation R on a subset of F:

a) S is *symmetric* iff $S = -S$.
b) S is *convex* iff the vector $\alpha\, \mathbf{x} + (1-\alpha)\, \mathbf{y}$ is an element of S for all elements $\mathbf{x}$ and $\mathbf{y}$ of S and every $\alpha \in]0, 1[$.
c) S is a *cone* iff $\lambda\, \mathbf{x} \in S$ for every $\mathbf{x} \in S$ and every $\lambda > 0$. Furthermore, a cone S is *pointed* iff $\mathbf{0} \in S$, and it is a *proper cone* iff $(S \cap -S) \subseteq \{\mathbf{0}\}$. Finally, S is the cone *generated* by a given subset of F iff S is the smallest cone containing this subset.
d) S is *lineally closed* (in F) iff for all elements $\mathbf{x}_1$ and $\mathbf{x}_2$ of F with $\mathbf{x}_1 \notin S$ and $\mathbf{x}_2 \in S$ there is a scalar $\delta > 0$ such that $\mathbf{x}_1 + \alpha\, (\mathbf{x}_2 - \mathbf{x}_1) \notin S$ for every $\alpha \in]0, \delta[$.
e) S is a *linear subspace* of F iff it is closed under addition and scalar multiplication; i.e., iff the vectors $\mathbf{x}_1 + \mathbf{x}_2$ and $\lambda\, \mathbf{x}_1$ are elements of S for all elements $\mathbf{x}_1$ and $\mathbf{x}_2$ of S and every $\lambda \in \mathbb{R}$. Furthermore, S is an *affine subspace* of F iff the set $S-\{\mathbf{x}\}$ is a linear subspace of F for some $\mathbf{x} \in F$. Finally, S is the linear (resp. affine) subspace *generated* by a given subset of F iff S is the smallest linear (resp. affine) subset of F containing this subset.
f) S is *finite dimensional* iff there is a natural number n and a finite sequence $\{\mathbf{x}_i\}_{i=1..n}$ of elements of F such that every $\mathbf{x} \in S$ can be represented as $\mathbf{x} = \sum_{i=1..n} \lambda_i\, \mathbf{x}_i$ with suitable scalars λ_i.
g) A map $g{:}F\to\mathbb{R}$ is *linear* iff the equations $g(\mathbf{x}_1 + \mathbf{x}_2) = g(\mathbf{x}_1) + g(\mathbf{x}_2)$ and $g(\lambda\, \mathbf{x}_1) = \lambda\, g(\mathbf{x}_1)$ hold for

[22] In particular, the application of some concepts to the empty set varies in the literature. In the present context, we can afford to give criteria only for the application to a non-empty subset S of a real vector space. Note that Lemma A.6 explicitly introduces suitable assumptions of being non-empty for the sets under consideration.

all elements $\mathbf{x}_1$ and $\mathbf{x}_2$ of F and every real number λ.

h) The *algebraic dual* of F is the set of all linear maps $F \to \mathbb{R}$, endowed with pointwise addition and scalar multiplication, this endowment making the set a real vector space.

i) If R is a binary relation on S, $\lambda > 0$ a scalar, and $\mathbf{x}$ an element of F, then a binary relation $R^{\sim}$ on a subset $S^{\sim}$ of F is identical with R up to a *stretching* (by λ) and a *translation* (by $\mathbf{x}$) iff $S^{\sim} = \lambda\, S + \{\mathbf{x}\}$ and the relations $\mathbf{x}_1\, R\, \mathbf{x}_2$ and $\lambda\, \mathbf{x}_1 + \mathbf{x}\, R^{\sim}\, \lambda\, \mathbf{x}_2 + \mathbf{x}$ are equivalent for all elements $\mathbf{x}_1$ and $\mathbf{x}_2$ of S. The mentioning of a stretching by $\lambda = 1$ or a translation by $\mathbf{x} = \mathbf{0}$ may be omitted.

j) A vector $\mathbf{x}$ is *strongly separable* from S iff there exists a linear map $g{:}F \to \mathbb{R}$ and a scalar γ such that the inequality $g(\mathbf{x}) < \gamma < g(\mathbf{x}^*)$ holds for every $\mathbf{x}^* \in S$.

Definition A.3: A *linear order* on a real vector space F is a reflexive and transitive relation $\preceq$ fulfilling the equivalences

$$(\mathbf{x}_1 + \mathbf{x}) \preceq (\mathbf{x}_2 + \mathbf{x}) \;\Leftrightarrow\; \mathbf{x}_1 \preceq \mathbf{x}_2$$

and

$$(\lambda\, \mathbf{x}_1) \preceq (\lambda\, \mathbf{x}_2) \;\Leftrightarrow\; \mathbf{x}_1 \preceq \mathbf{x}_2$$

for all vectors $\mathbf{x}$, $\mathbf{x}_1$ and $\mathbf{x}_2$ and every scalar $\lambda > 0$. In this situation, the following additional definitions apply:

a) The set $\{\mathbf{x} \in F\text{: } \mathbf{0} \preceq \mathbf{x}\}$ is the *positive cone* of the relation $\preceq$.

b) A map $g{:}F \to \mathbb{R}$ is *monotonic* iff the implication $\mathbf{x}_1 \preceq \mathbf{x}_2 \;\Rightarrow\; g(\mathbf{x}_1) \le g(\mathbf{x}_2)$ holds for all vectors $\mathbf{x}_1$ and $\mathbf{x}_2$.

c) A linear order $\preceq$ is *Archimedean* iff the following implication holds for every $\mathbf{x} \in F$: If there exists a vector $\mathbf{y}$ such that $n\, \mathbf{x} \preceq \mathbf{y}$ for every natural number n, then $\mathbf{x} \preceq \mathbf{0}$.

Observe that the notion of a monotonic map is reserved for (weakly) increasing monotonicity by part b of the above definition.

A.2 Conceptualization and Results

The usual derivation of relations $\succeq$, $\sim$, $\prec$ and $\npreceq$ from an order relation $\preceq$ can be generalized to arbitrary binary relations in the following way:

Definition A.4: For every binary relation R on a set Y, the following terminology and notation are used for derived relations on Y (parts a through d, $\mathbf{y}_1$ and $\mathbf{y}_2$ always being elements of Y) or a subset $Y^{\sim}$ of Y:

a) The *inverse relation* R^{-1}: $\quad \mathbf{y}_1\, R^{-1}\, \mathbf{y}_2 \;:\Leftrightarrow\; \mathbf{y}_2\, R\, \mathbf{y}_1$.

b) The *symmetric part* R_s of R: $\quad \mathbf{y}_1\, R_s\, \mathbf{y}_2 \;:\Leftrightarrow\; (\mathbf{y}_1\, R\, \mathbf{y}_2 \wedge \mathbf{y}_2\, R\, \mathbf{y}_1)$.

c) The *asymmetric part* R_a of R: $\quad \mathbf{y}_1\, R_a\, \mathbf{y}_2 \;:\Leftrightarrow\; (\mathbf{y}_1\, R\, \mathbf{y}_2 \wedge \neg(\mathbf{y}_2\, R\, \mathbf{y}_1))$.

d) The *complementary relation* R_c: $\quad \mathbf{y}_1\, R_c\, \mathbf{y}_2 \;:\Leftrightarrow \neg(\mathbf{y}_1\, R\, \mathbf{y}_2)$.

e) Now let $Y^{\sim}$ be a subset of Y, and $R^{\sim}$ a binary relation on $Y^{\sim}$. Then $R^{\sim}$ is the *restriction* of R to $Y^{\sim}$ iff the relations $\mathbf{y}_1\, R^{\sim}\, \mathbf{y}_2$ and $\mathbf{y}_1\, R\, \mathbf{y}_2$ are equivalent for all elements $\mathbf{y}_1$ and $\mathbf{y}_2$ of $Y^{\sim}$. Furthermore, R is an *extension* of $R^{\sim}$ to Y iff $R^{\sim}$ is the restriction of R to $Y^{\sim}$.

Under the view of a binary relation R on a set Y as a subset of $Y \times Y$ consisting of all ordered pairs

$(\mathbf{y}_1, \mathbf{y}_2)$ with the property $\mathbf{y}_1\ R\ \mathbf{y}_2$,, the above definitions can be rewritten as

$R^{-1} := \{(\mathbf{y}_2, \mathbf{y}_1): (y_1, \mathbf{y}_2) \in R),$

$R_s := R \cap \Delta_{Y},$

$R_a := R \setminus \Delta_{Y},$

and

$R_c := (Y \times Y) \setminus R,$

with

$\Delta_Y := \{(\mathbf{y}, \mathbf{y}): \mathbf{y} \in Y\}.$

Furthermore, for $Y^{\sim} \subseteq Y$, a binary relation $R^{\sim}$ on $Y^{\sim}$ is the restriction of R to $Y^{\sim}$ (and R an extension of $R^{\sim}$ to Y) iff $R^{\sim} = R \cap (Y^{\sim} \times Y^{\sim})$.

The following definition reformulates some properties of linear orders such that their presence in binary relations on subsets of real vector spaces will enable further deductions.

Definition A.5: For a relation R on a subset Y of a real vector space, the following terminology is used:

a) R is *direction-based* iff it is binary and the equivalence

$$\mathbf{y}_{11}\ R\ \mathbf{y}_{12} \Leftrightarrow \mathbf{y}_{21}\ R\ \mathbf{y}_{22} \tag{A.1}$$

is valid whenever $\mathbf{y}_{11}$, $\mathbf{y}_{12}$, $\mathbf{y}_{21}$ and $\mathbf{y}_{22}$ are elements of Y such that

$$\mathbf{y}_{22}-\mathbf{y}_{21} = \lambda\ (\mathbf{y}_{12}-\mathbf{y}_{11}) \tag{A.2}$$

for some scalar $\lambda > 0$. In this situation, the *direction cone* of R is the smallest cone in F where the relation $\mathbf{y}_1\ R\ \mathbf{y}_2$ holds if and only if the difference $\mathbf{y}_2-\mathbf{y}_1$ is an element of the cone.

b) R is *pre-cancellative* iff it is binary and the equivalence

$$\mathbf{y}_1 + \alpha\ (\mathbf{y}-\mathbf{y}_1)\ R\ \mathbf{y}_2 + \alpha\ (\mathbf{y}-\mathbf{y}_2) \Leftrightarrow \mathbf{y}_1\ R\ \mathbf{y}_2 \tag{A.3}$$

holds for $\alpha \in\ ]0, 1[$ whenever $\mathbf{y}$, $\mathbf{y}_1$ and $\mathbf{y}_2$ are elements of Y such that the vector $\mathbf{y}_i + \alpha\ (\mathbf{y}-\mathbf{y}_i)$ is an element of Y for $i \in \{1, 2\}$.

c) R is *pre-Archimedean* iff it is binary and for all elements $\mathbf{y}$, $\mathbf{y}_1$ and $\mathbf{y}_2$ of Y with $\mathbf{y}\ R_c\ \mathbf{y}_1$ and $\mathbf{y}\ R\ \mathbf{y}_2$ there is a real number $\delta > 0$ such that the implication $\mathbf{z}_\alpha \in Y \Rightarrow \mathbf{y}\ R_c\ \mathbf{z}_\alpha$ holds for every $\alpha \in\ ]0, \delta[$, the vector $\mathbf{z}_\alpha$ being given by

$$\mathbf{z}_\alpha := \mathbf{y}_1 + \alpha\ (\mathbf{y}_2-\mathbf{y}_1). \tag{A.4}$$

The notion of a direction-based relation is motivated by the view that the directions from $\mathbf{y}_{11}$ to $\mathbf{y}_{12}$ and from $\mathbf{y}_{21}$ to $\mathbf{y}_{22}$ are identical iff Equation (A.2) holds with $\lambda > 0$. The existence and uniqueness of a cone with the properties required in the definition of a direction cone are stated in Assertion (5) of the subsequent lemma. This lemma also shows that pre-cancellative and pre-Archimedean relations preform (at least in some situations) cancellation and Archimedean properties of an extended relation R' on F. The lemma systematises further correspondences of this kind. Note that some claims referring to linear orders hold only for the $\leq$-orders underlying Definition A.3 and Lemma A.12, but not for $\geq$-orders.[23]

[23] In particular, if R stands for $\geq$, then the psoitive cone of the order is $-C_R$ instead of C_R.

Lemma A.6: Let F be a real vector space, Y a non-empty[24] subset of F, and R a binary relation on Y, the derived relations R^{-1}, R_s, R_a and R_c being given by Definition A.4. Furthermore, let X^* be the affine subspace of F generated by Y, and X the linear subspace of F generated by the set $Y-Y$.
Let the sets D_R, C_R and C_D be defined by

$$D_R := \{\mathbf{y}_2-\mathbf{y}_1: \mathbf{y}_1 \in Y \wedge \mathbf{y}_2 \in Y \wedge \mathbf{y}_1 \, R \, \mathbf{y}_2\}, \tag{A.5}$$

$$C_R := \bigcup_{\lambda>0} \lambda \, D_R, \tag{A.6}$$

and

$$C_D := \bigcup_{\lambda>0} \lambda \, (Y-Y), \tag{A.7}$$

and assume that the sets D_R and C_R are non-empty.
Let the binary relation R' on F be defined by

$$\mathbf{x}_1 \, R' \, \mathbf{x}_2 \;:\Leftrightarrow\; \mathbf{x}_2-\mathbf{x}_1 \in C_R \tag{A.8}$$

for every $\mathbf{x}_1, \mathbf{x}_2 \in F$, and let R^* and R_X be the restrictions of R' to X^* resp. to X.
Let B' and B^* be the sets of those direction-based relations on F resp. on X^* which are extensions of R.
Let G be the set of all linear maps $g: F \to \mathbb{R}$ where the implication

$$\mathbf{y}_1 \, R \, \mathbf{y}_2 \Rightarrow g(\mathbf{y}_1) \le g(\mathbf{y}_2) \tag{A.9}$$

holds for all elements $\mathbf{y}_1$ and $\mathbf{y}_2$ of Y, and $G_{|X}$ the set of their restrictions to X. Finally, if G^* is a non-empty subset of G, then the set C_{G^*} is defined by

$$C_{G^*} := \bigcap_{g \in G^*} \{\mathbf{x} \in C_D: g(\mathbf{x}) \ge 0\}, \tag{A.10}$$

and the elements of G^* will be said to represent R jointly[25] iff the equivalence

$$\mathbf{y}_2-\mathbf{y}_1 \in C_{G^*} \;\Leftrightarrow\; \mathbf{y}_1 \, R \, \mathbf{y}_2 \tag{A.11}$$

holds for all elements $\mathbf{y}_1$ and $\mathbf{y}_2$ of Y.
For purposes of notational convenience, the vector $\mathbf{z}_\alpha$ defined in Equation (A.4) will be used in the subsequent assertions, the scalar α and the vectors $\mathbf{y}_1$ and $\mathbf{y}_2$ being always given by the local context.

In this situation, the following properties follow:

(1) If one of the relations R, R^{-1} or R_c is pre-cancellative (resp. direction-based) or the relations R_s and R_a are both pre-cancellative (resp. direction-based), then the relations R, R^{-1}, R_s, R_a and R_c are all pre-cancellative (resp. direction-based).
(2) If R is transitive, then R^{-1}, R_a and R_s are transitive.
(3) $D_R \subseteq C_R \subseteq C_G \subseteq C_{G^*} \subseteq C_D \subseteq X = X^*-X^*$, where G^* may be every non-empty subset of G.
(4) C_R is the cone generated by D_R, and C_D is the cone generated by the set $Y-Y$. Furthermore, if G^* is a non-empty subset of G, then C_{G^*} is a pointed cone, which is convex if C_D is convex.
(5) The following assertions are equivalent:

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[24] Some deviations from the assertions of Lemma A.6 for situations with $Y = \varnothing$ are mentioned after the proof.

[25] For this terminology, observe that (for elements $\mathbf{y}_1$ and $\mathbf{y}_2$ of Y) the difference vector $\mathbf{y}_2-\mathbf{y}_1$ is an element of C_{G^*} iff $g(\mathbf{y}_1) \le g(\mathbf{y}_2)$ for all $g \in G^*$.

(5.a) The relation R is direction-based

(5.b) The equivalence $\lambda\, \mathbf{y}_1 + \mathbf{x}\ R\ \lambda\, \mathbf{y}_2 + \mathbf{x} \Leftrightarrow \mathbf{y}_1\ R\ \mathbf{y}_2$ holds for all elements $\mathbf{y}_1$ and $\mathbf{y}_2$ of Y, every $\mathbf{x} \in F$ and every scalar $\lambda > 0$ whenever the linear combinations $\lambda\, \mathbf{y}_i + \mathbf{x}$ are elements of Y.

(5.c) There is a cone $C^{\sim}$ in F such that the properties $\mathbf{y}_1\ R\ \mathbf{y}_2$ and $\mathbf{y}_2 - \mathbf{y}_1 \in C^{\sim}$ are equivalent for all elements $\mathbf{y}_1$ and $\mathbf{y}_2$ of Y.

(5.d) The set B' is non-empty.

(5.e) The set B^* is non-empty.

(5.f) The relation R' is an extension of R.

(5.g) The relation R^* is an extension of R.

If these assertions hold, then C_R is the (unique) direction cone of R, and a given cone $C^{\sim}$ in F has the property described in (5.c) iff

$$C^{\sim} \cap\ C_D = C_R. \qquad \text{(A.12)}$$

(6) If $Y^{\sim}$ is a subset of F, and $C^{\sim}$ is a cone in F, then a direction-based relation on $Y^{\sim}$ with direction cone $C^{\sim}$ exists iff $C^{\sim} \subseteq \bigcup_{\lambda>0} \lambda\ (Y^{\sim} - Y^{\sim})$. Furthermore, a relation with these properties is unique, if it exists.

(7) The relations R' , R^* and R_X are the unique direction-based relations on F resp. X^* resp. X with direction cone C_R. Furthermore, R_X is identical with R^* up to a translation by $-\mathbf{x}$ iff $\mathbf{x} \in X^*$.

(8) B^* is the set of all restrictions to X^* of elements of B'.

(9) The direction cone of every element of B^* is a subset of X.

(10) A linear map $g{:}F\rightarrow\mathbb{R}$ is an element of G iff $g(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in C_R$.

(11) G is a pointed, convex cone in the algebraic dual of F.

(12) If G^* is a subset of G whose elements jointly represent the relation R, then the relation R is reflexive, transitive, direction-based (with direction cone $C_{G^*} = C_R$), and pre-Archimedean, and it is also connected if G^* has only one element.

The subsequent Assertions (13) through (15) are valid if Y is convex[26]:

(13) $C_D = X$.

(14) For every finite sequence $\{\mathbf{x}_i\}_{i=1..n}$ of elements of X, there are elements $\mathbf{y}$ and $\{\mathbf{y}_i\}_{i=1..n}$ of Y and a scalar $\lambda > 0$ such that $\mathbf{x}_i = \lambda\ (\mathbf{y}_i - \mathbf{y})$ for i = 1..n.

(15) The following properties are equivalent:

(15.a) The relation R is pre-Archimedean.

(15.b) For all elements $\mathbf{y}$, $\mathbf{y}_1$ and $\mathbf{y}_2$ of Y, the set

$$A := \{\alpha \in [0, 1]{:}\ \mathbf{y}\ R\ \mathbf{z}_\alpha\}, \qquad \text{(A.13)}$$

is a closed subset of $\mathbb{R}$.

(15.c) For all elements $\mathbf{y}$, $\mathbf{y}_1$ and $\mathbf{y}_2$ of Y with $\mathbf{y}\ R_c\ \mathbf{y}_1$ and $\mathbf{y}\ R\ \mathbf{y}_2$, the above set A is a closed subset of $\mathbb{R}$.

If R is direction-based, then Assertions (16) through (28) hold.

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[26] For some following assertions, the assumption of a convex set Y can be replaced by weaker conditions, which are explicated in the proofs of (13) and (29).

(16) R is pre-cancellative.
(17) The equivalence $\mathbf{y}_1 \, R \, \mathbf{y}_2 \;\Leftrightarrow\; \lambda\,(\mathbf{y}_2 - \mathbf{y}_1) \in C_R$ holds for $\mathbf{y}_1, \mathbf{y}_2 \in Y$ and $\lambda > 0$.
(18) R^{-1} is the only direction-based relation on Y with direction cone $-C_R$.
(19) C_R is a pointed cone iff R is reflexive.
(20) C_R is a proper cone iff the implication $\mathbf{y}_1 \, R_s \, \mathbf{y}_2 \;\Rightarrow\; \mathbf{y}_1 = \mathbf{y}_2$ holds for all elements $\mathbf{y}_1$ and $\mathbf{y}_2$ of Y.
(21) C_R is symmetric iff R is symmetric.
(22) $C_R \cup (-C_R) = C_D$ iff R is connected.
(23) If a relation $R^{\sim}$ on a subset $Y^{\sim}$ of F is identical with R up to a stretching and a translation, then $R^{\sim}$ is a direction-based relation with direction cone C_R.
(24) For every subset $Y^{\sim}$ of Y, the restriction of R to $Y^{\sim}$ is direction-based, the direction cone of this restriction being $C_R \cap \bigcup_{\lambda>0} \lambda\,(Y^{\sim} - Y^{\sim})$.
(25) If $Y^{\sim}$ is a subset of F with $Y \subseteq Y^{\sim}$, and $R^{\sim}$ is a direction-based relation on $Y^{\sim}$ with direction cone $C^{\sim}$, then $R^{\sim}$ is an extension of R iff $C^{\sim} \cap C_D = C_R$.
(26) A direction-based relation on F (resp. on X^*) with direction cone $C^{\sim}$ is an element of B' (resp. of B^*) iff $C^{\sim} \cap C_D = C_R$. In particular, R' and R^* are the only elements of B' resp. of B^* with direction cone C_R.
(27) If $\mathbf{y}_1$ and $\mathbf{y}_2$ are elements of Y, and α and α' real numbers with $\alpha < \alpha'$ such that the vectors $\mathbf{z}_\alpha$ and $\mathbf{z}_{\alpha'}$ are elements of Y, then $\mathbf{z}_\alpha \, R \, \mathbf{z}_{\alpha'}$ iff $\mathbf{y}_1 \, R \, \mathbf{y}_2$.
(28) If C_R is convex resp. lineally closed, then R is transitive resp. pre-Archimedean.

Assertions (29) through (33) are valid if Y is convex and R is pre-cancellative.
(29) R is direction-based.
(30) The relation R^* is the only element of the set B^*. Furthermore, R' is the only element of B' iff $X=F$.
(31) The following assertions are equivalent:
    (31.a) The relation R is transitive.
    (31.b) The set D_R is convex.
    (31.c) C_R is a convex cone.
    (31.d) $(\mathbf{y} \, R \, \mathbf{y}_1 \wedge \mathbf{y} \, R \, \mathbf{y}_2) \Rightarrow \mathbf{y} \, R \, \mathbf{z}_\alpha$ for every $\alpha \in [0, 1]$ and all elements $\mathbf{y}, \mathbf{y}_1, \mathbf{y}_2$ of Y.
    (31.e) $(\mathbf{y}_1 \, R \, \mathbf{y} \wedge \mathbf{y}_2 \, R \, \mathbf{y}) \Rightarrow \mathbf{z}_\alpha \, R \, \mathbf{y}$ for every $\alpha \in [0, 1]$ and all elements $\mathbf{y}, \mathbf{y}_1, \mathbf{y}_2$ of Y.
(32) The following properties are equivalent:
    (32.a) The relation R is pre-Archimedean.
    (32.b) The cone C_R is lineally closed.
    (32.c) The relation R^{-1} is pre-Archimedean.
(33) If R is pre-Archimedean, then R is reflexive.

If Y is convex and R is pre-cancellative and transitive, then Assertions (34) through (37) are also valid:
(34) For all elements $\mathbf{y}$, $\mathbf{y}_1$ and $\mathbf{y}_2$ of Y, there are real numbers δ and δ' such that $0 \le \delta \le \delta' \le 1$, and $]\delta, \delta'[\subseteq A \subseteq [\delta, \delta']$ for the set A given by Equation (A.13).
(35) If R is reflexive, then the relation R' (resp. R_X) is the only linear order on F (resp. on X) with positive cone C_R, and G (resp. $G_{|X}$) is the set of all monotonic maps $g{:}F \rightarrow \mathbb{R}$ (resp. $g{:}X \rightarrow \mathbb{R}$).

(36) The following assertions a through d are equivalent:

(36.a) The relation R is pre-Archimedean.

(36.b) R_X is an Archimedean linear order on X.

(36.c) R' is an Archimedean linear order on F.

(36.d) For all elements $\mathbf{y}$, $\mathbf{y}_1$ and $\mathbf{y}_2$ of Y, the set A given by Equation (A.13) is either empty or a closed subinterval of [0, 1].

If C_R is finite dimensional[27], then the following assertion is also equivalent with the foregoing ones:

(36.e) For all elements $\mathbf{y}_1$ and $\mathbf{y}_2$ of Y, the relation $\mathbf{y}_1 \; R \; \mathbf{y}_2$ is present iff the inequality $g(\mathbf{y}_1) \le g(\mathbf{y}_2)$ holds for every $g \in G$. (I.e., the elements of G jointly represent the relation R.)

(37) If R is symmetric, then the relations R, R', R^* and R_X are equivalence relations, and C_R is a linear subspace of X.

Finally, the remaining Assertions (38) through (40) follow under the assumption that Y is convex and R is a pre-cancellative weak order:

(38) R_X is a linear weak order on X, and $X = C_R \cup (-C_R) = C_R - C_R$

(39) If $\mathbf{y}$, $\mathbf{y}_1$ and $\mathbf{y}_2$ are elements of Y such that $\mathbf{y}_1 \; R_a \; \mathbf{y}_2$, then there is a unique number $\delta \in [0, 1]$ such that $\mathbf{z}_\alpha \; R_a \; \mathbf{y}$ for $0 \le \alpha < \delta$, and $\mathbf{y} \; R_a \; \mathbf{z}_\alpha$ for $\delta < \alpha \le 1$.

(40) The following properties are equivalent:

(40.a) The relation R is pre-Archimedean

(40.b) R_X is an Archimedean linear weak order on X.

(40.c) There is a linear map $g: F \to \mathbb{R}$ such that the equivalence

$$\mathbf{y}_1 \; R \; \mathbf{y}_2 \;\Leftrightarrow\; g(\mathbf{y}_1) \le g(\mathbf{y}_2) \qquad \text{(A.14)}$$

holds for all elements $\mathbf{y}_1$ and $\mathbf{y}_2$ of Y.

(40.d) For all elements $\mathbf{y}$, $\mathbf{y}_1$ and $\mathbf{y}_2$ of Y with $\mathbf{y}_1 \; R_a \; \mathbf{y}$ and $\mathbf{y} \; R_a \; \mathbf{y}_2$, the relation $\mathbf{y} \; R_s \; \mathbf{z}_\alpha$ holds for some $\alpha \in \,]0, 1[$.

If these properties are given, then the number α fulfilling (40.d) is unique for given elements $\mathbf{y}$, $\mathbf{y}_1$ and $\mathbf{y}_2$ of Y with the assumed properties.

The following corollaries are examples of application oriented facts following immediately from Lemma A.6.

Corollary A.7: Let F be a real vector space, Y a convex subset of F, and $\sim$ an equivalence relation on Y. Then the following assertions are equivalent:

(1) There is a linear subspace X of F such that the equivalence

$$\mathbf{y'} \sim \mathbf{y''} \Leftrightarrow (\mathbf{y''} - \mathbf{y'}) \in X$$

holds for all elements $\mathbf{y'}$ and $\mathbf{y''}$ of Y.

(2) The equivalence

$$\mathbf{y'} + \alpha\,(\mathbf{y} - \mathbf{y'}) \;\sim\; \mathbf{y''} + \alpha\,(\mathbf{y} - \mathbf{y''}) \;\Leftrightarrow\; \mathbf{y'} \sim \mathbf{y''}$$

holds for all elements $\mathbf{y}$, $\mathbf{y'}$ and $\mathbf{y''}$ of Y and every $\alpha \in \,]0, 1[$.

[27] Weaker sufficient conditions for the equivalence claimed in (36.e) are outlined in the proof.

Corollary A.8: Let F be a real vector space, Y a convex subset of F, and $\preceq$ a weak order on Y. Then the equivalence (1) $\Leftrightarrow$ ((2) $\wedge$ (3)) holds for the following properties:

(1) There is a linear map $g{:}F \to \mathbb{R}$ such that the equivalence

$$\mathbf{y}' \preceq \mathbf{y}'' \Leftrightarrow g(\mathbf{y}') \leq g(\mathbf{y}'')$$

holds for all elements $\mathbf{y}'$ and $\mathbf{y}''$ of Y.

(2) The equivalence

$$\mathbf{y}' + \alpha\,(\mathbf{y}-\mathbf{y}') \;\preceq\; \mathbf{y}'' + \alpha\,(\mathbf{y}-\mathbf{y}'') \;\Leftrightarrow\; \mathbf{y}' \preceq \mathbf{y}''$$

holds for all elements $\mathbf{y}$, $\mathbf{y}'$ and $\mathbf{y}''$ of Y and every $\alpha \in\,]0, 1[$.

(3) For all allements $\mathbf{y}$, $\mathbf{y}'$ and $\mathbf{y}''$ of Y with $\mathbf{y}' \prec \mathbf{y} \prec \mathbf{y}''$, there is some $\alpha \in\,]0, 1[$ with the property $\mathbf{y} \sim \mathbf{y}' + \alpha\,(\mathbf{y}''-\mathbf{y}')$.

If these properties are given, then a map $\phi{:}Y \to \mathbb{R}$ fulfills the equivalence

$$\phi(\mathbf{y}_1) \leq \phi(\mathbf{y}_2) \;\Leftrightarrow \mathbf{y}_1 \preceq \mathbf{y}_2$$

for all elements $\mathbf{y}_1$ and $\mathbf{y}_2$ of Y if and only if there is a linear map $g{:}F \to \mathbb{R}$ with the property described in Assertion (1), and a strictly increasing map $\psi{:}\mathbb{R} \to \mathbb{R}$ such that ϕ is the restriction of $\psi \circ g$ to Y.

Corollary A.9: Let W be a finite, non-empty set, Y the set of all probability measures on the power set of W, and $\preceq$ a weak order on Y with Properties (2) and (3) of Corollary A.8. Finally, define for every map $v{:}W \to \mathbb{R}$ the map $g_v{:}Y \to \mathbb{R}$ by

$$g_v(\mathbf{y}) := \sum_{w \in W} \mathbf{y}(\{w\}) \cdot v(w) \qquad \text{(A.15)}$$

for every $\mathbf{y} \in Y$. Then there is a map $v{:}W \to \mathbb{R}$ such that the equivalence

$$\mathbf{y}' \preceq \mathbf{y}'' \;\Leftrightarrow\; g_v(\mathbf{y}') \leq g_v(\mathbf{y}'') \qquad \text{(A.16)}$$

holds for all elements $\mathbf{y}'$ and $\mathbf{y}''$ of Y, this map being unique up to linear transformations $\alpha\, v + \beta$ with $\alpha > 0$.

Applications of Corollary A.9 can be based on the following interpretation of Equation (A.15): If Z is a W-valued random variable with distribution $\mathbf{y}$, then $g_v(\mathbf{y})$ is the expectation of the real valued random variable $v(Z)$.

The uniqueness up to positive linear transformations of the map v fulfilling Equivalence (A.16) follows from an application of the following lemma.

Lemma A.10: Let $\mathcal{A}$ be a σ-algebra in a set W. Furthermore, let $u{:}W \to \mathbb{R}$ and $v{:}W \to \mathbb{R}$ be measurable maps, and Y a convex set of probability measures on $\mathcal{A}$ with the following property: For every $w \in W$, there is an element $\mathbf{y}_w$ of Y such that $\mathbf{y}_w(u^{-1}(u(w)) \cap v^{-1}(v(w))) = 1$. Then the following assertions are equivalent:

(1) There are real numbers α and β with $\alpha > 0$ such that $u = \alpha\, v + \beta$.

(2) The equivalence

$$\int u \,d\mathbf{y}' \leq \int u \,d\mathbf{y}'' \;\Leftrightarrow\; \int v \,d\mathbf{y}' \leq \int v \,d\mathbf{y}''$$

holds for all elements $\mathbf{y}'$ and $\mathbf{y}''$ of Y.

A.3 Proofs

A.3.1 Some Basic Facts.

The lemmas in this section summarise some basic facts referring to real vector spaces and linear orders, which are proved in many pertinent textbooks and will be used for proofs in later sections.

The following properties of subsets and linear maps are immediate consequences of Definition A.2. For (10) and (11), see e.g. Holmes (1975, p. 3, 5 and 64).

Lemma A.11: Let A and S be non-empty subsets of a real vector space F.

(1) If S is an intersection of convex sets, then S is convex.
(2) The linear space generated by S is finite dimensional iff S is finite dimensional.
(3) S is the cone generated by A iff $S = \bigcup_{\lambda>0} \lambda A$. If these properties are given and A is convex, then S is convex.
(4) If S is a cone, then S is convex iff it is closed under addition (i.e., iff $\mathbf{x}_1+\mathbf{x}_2 \in S$ for all elements $\mathbf{x}_1$ and $\mathbf{x}_2$ of S).
(5) If S is a lineally closed cone, then S is pointed.
(6) If S is a linear subspace of F, then a subset of S is lineally closed in F iff it is lineally closed in S.
(7) S is a linear subspace of F iff S is a symmetric and convex cone.
(8) If S is the affine subspace of F generated by A, then the linear subspace generated by $A-A$ is $S-S$. Furthermore, that linear subspace is $S-\{\mathbf{x}\}$ iff $\mathbf{x} \in S$.
(9) If S is the cone generated by $A-A$ with convex A, then S is also the linear subspace generated by $A-A$.
(10) If S is a linear subspace of F, then every linear map $S\to\mathbb{R}$ can be extended to a linear map $F\to\mathbb{R}$.
(11) If S is a (linear or affine) subspace of F or a lineally closed, convex subset of a finite dimensional F, then every $\mathbf{x} \in F\backslash S$ is strongly separable from S.

Proofs of the following correspondences between linear orders and their positive cones are given e.g. by Jameson (1970). Since this author uses a different terminology for cones (see pp. 2 and 3), the results have been translated into the taxonomy of our Definition A.2.c

Lemma A.12: Let $\preceq$ be a linear order on a real vector space F, and let the set $C := \{\mathbf{x} \in F\colon \mathbf{0} \preceq \mathbf{x}\}$ be the positive cone of the order. Then the following properties apply:

(1) For all elements $\mathbf{x}_1$ and $\mathbf{x}_2$ of F, the relation $\mathbf{x}_1 \preceq \mathbf{x}_2$ holds iff $\mathbf{x}_2-\mathbf{x}_1 \in C$. Furthermore, C is the only subset of F with this property.
(2) C is a pointed, convex cone.
(3) For every pointed convex cone in F, there is a unique linear order on F such that the given cone is the positive cone of the order.
(4) The order $\preceq$ is antisymmetric iff C is a proper cone.
(5) The order $\preceq$ is connected (i.e., a weak order) iff $F = C \cup (-C) = C-C$.
(6) The order $\preceq$ is Archimedean iff C is lineally closed.
(7) The order $\preceq$ is connected and Archimedean iff there exists a linear map $g\colon F\to\mathbb{R}$ such that

$C = \{\mathbf{x} \in F\colon g(\mathbf{x}) \geq 0\}.$

A.3.2 Proof of Lemma A.6.

Before we prove the assertions of the lemma, we introduce the convention that a range of i = 1,2 and j = 1,2 has to be assumed for terms with subscript i or j, unless a different range is specified in the respective context. Furthermore, λ will always be a scalar such that $\lambda > 0$. We will also frequently refer to elements $\mathbf{x} = \lambda\,(\mathbf{y}_2 - \mathbf{y}_1)$ or $\mathbf{x}_\mathrm{i} = \lambda\,(\mathbf{y}_\mathrm{i} - \mathbf{y})$ of X or a subset of X. Let it be said once and for all that this stands for the following sentences: Let $\mathbf{x}$ resp. $\mathbf{x}_\mathrm{i}$ be element(s) of X (or a locally specified subset of X). Furthermore, let a scalar $\lambda > 0$ and elements ($\mathbf{y}$ and) $\mathbf{y}_\mathrm{i}$ of Y be given such that $\mathbf{x} = \lambda\,(\mathbf{y}_2 - \mathbf{y}_1)$ resp. $\mathbf{x}_\mathrm{i} = \lambda\,(\mathbf{y}_\mathrm{i} - \mathbf{y})$, the existence of suitable λ and $\mathbf{y}_\mathrm{i}$ being granted by (13) or (14) or Equation (A.6) or (A.7).

Finally, take it as a convention that during the proof of an implication the antecedent part of the implication is assumed to be valid without saying.

For situations with sufficiently specified elements $\mathbf{y}$, $\mathbf{y}_1$ and $\mathbf{y}_2$ of Y, observe that the definition of $\mathbf{z}_\alpha$ in Equation (A.4) implies

$$\mathbf{z}_\alpha - \mathbf{z}_{\alpha'} = (\alpha - \alpha')\,(\mathbf{y}_2 - \mathbf{y}_1) \qquad \text{(A.17)}$$

and

$$\mathbf{z}_\alpha - \mathbf{y}_\mathrm{i} = (\alpha - \mathrm{i} + 1) \cdot (\mathbf{y}_2 - \mathbf{y}_1) \qquad \text{(A.18)}$$

for all real numbers α and α'.

To facilitate the application of results to other binary relations $R^{\sim}$ on a subset $Y^{\sim}$ of F, let the sets $C_R(\mathrm{Y}\sim, \mathrm{R}\sim)$ and $C_D(\mathrm{Y}\sim)$ be defined for such situations by an analogous application of Equations (A.6) and (A.7). Furthermore, let $X(\mathrm{Y}\sim)$ be the subspace of F generated by $Y^{\sim} - Y^{\sim}$. The following applications to R', R^* and R_X follow immediately:

$$X(F) = C_D(F) = F, \qquad \text{(A.19)}$$

$$X(X^*) = C_D(X^*) = X(X) = C_D(X) = X, \qquad \text{(A.20)}$$

and

$$C_R(F, R') = C_R(X^*, R) = C_R(X, R_X) = C_R. \qquad \text{(A.21)}$$

Conclusions based on such correspondeces will be called immediate generalizations. In particular, after the proof of (5), results referring to the relation R and the set C_R can be generalized to every direction-based relation $R^{\sim}$ on a set $Y^{\sim} \subseteq F$ and its direction cone, which ist $C_R(Y^{\sim}, R^{\sim})$.

Now the assertions of the lemma will be proved.

(1) and (2): These assertions are immediate consequences of Definitions A.4 and A.5.

(3): See Equations (A.5), (A.6), (A.7) and (A.10). For the inclusion $C_R \subseteq C_G$, let an element $\mathbf{x} = \lambda\,(\mathbf{y}_2 - \mathbf{y}_1)$ of C_R be given such that $\mathbf{y}_1\ R\ \mathbf{y}_2$. Then $g(\mathbf{y}_1) \leq g(\mathbf{y}_2)$ follows from Implication (A.12) for every $g \in G$, and this implies $g(\mathbf{x}) \geq 0$, since elements of G are linear.

(4): Combine Lemma A.11.(3) with Equations (A.6) and (A.7) for C_R resp. C_D. The claim referring to C_{G*} follows immediately from Equation (A.10). (Note that this equation can be rewritten as $C_{G*} = C_D \cap (\bigcap_{g \in G*} \{\mathbf{x} \in F\colon g(\mathbf{x}) \geq 0\})$ to represent C_{G*} as an intersection of convex sets in situations with convex C_D. Hence C_{G*} is convex in such situations by Lemma A.11.(1).)

(5): We will first prove the implications a⇒b⇒c⇒d⇒e⇒a.

a⇒b: For the vectors and the scalar in b, Equation (A.2) holds with the definitions $\mathbf{y}_{1j}$:= b and $\mathbf{y}_{2j} := \lambda\, \mathbf{y}_j + \mathbf{x}$. Hence Equivalence (A.1) follows, if R is direction-based.

b⇒c: To obtain useful side results, we will show that (under the assumtion of b) the equivalence in (5.c) holds for all elements $\mathbf{y}_1$ and $\mathbf{y}_2$ of Y, if $C^{\sim}$ is a cone fulfilling equation (A.12). So let $C^{\sim}$ be a such cone (the existence being granted by the fact that C_R is a cone with this property; see (3) and (4)). Furthermore, let $\mathbf{y}_1$ and $\mathbf{y}_2$ be arbitrary elements of Y. Then $\mathbf{y}_2 - \mathbf{y}_1 \in C_D$ follows from Equation (A.7). Furthermore, if $\mathbf{y}_1\ R\ \mathbf{y}_2$, then $\mathbf{y}_2 - \mathbf{y}_1 \in C_R$ follows from Equation (A.6). Combining both results, we get $\mathbf{y}_2 - \mathbf{y}_1 \in C^{\sim}$. Conversely, if $\mathbf{y}_2 - \mathbf{y}_1 \in C^{\sim}$, then $\mathbf{y}_2 - \mathbf{y}_1 \in C_R$ follows from $\mathbf{y}_2 - \mathbf{y}_1 \in C_D$. So Equation (A.6) implies the existence of elements $\mathbf{y}_3$ and $\mathbf{y}_4$ and a scalar $\lambda > 0$ such that $\mathbf{y}_3\ R\ \mathbf{y}_4$, and $\mathbf{y}_2 - \mathbf{y}_1 = (\mathbf{y}_4 - \mathbf{y}_3) / \lambda$. Then the definition $\mathbf{x} := \mathbf{y}_3 - \lambda\, \mathbf{y}_1$ allows to derive $\mathbf{y}_1\ R\ \mathbf{y}_2$ from $\mathbf{y}_3\ R\ \mathbf{y}_4$ if b holds.

c⇒d: Let $C^{\sim}$ be a cone in F fulfilling property c, and let the relation $R^{\sim}$ on F be defined by $\mathbf{x}_1\ R^{\sim}\ \mathbf{x}_2 \ :\Leftrightarrow\ \mathbf{x}_2 - \mathbf{x}_1 \in C^{\sim}$. Then it is easily verified that $R^{\sim}$ is an extension of R, and a direction-based relation on F. Hence $R^{\sim}$ is an element of B', and B' is non-empty.

d⇒e: Let $R^{\sim}$ be an element of B'. Then it follows immediately that the restriction of $R^{\sim}$ to X^* is an element of B^*.

e⇒a: Let $R^{\sim}$ be an element of B^*, and let elements $\mathbf{y}_{ij}$ of Y and a scalar $\lambda > 0$ be given such that Equation (A.2) holds. Then $\mathbf{y}_{11}\ R^{\sim}\ \mathbf{y}_{12} \ \Leftrightarrow\ \mathbf{y}_{21}\ R^{\sim}\ \mathbf{y}_{22}$, since $R^{\sim}$ (being an element of B^*) must be direction-based. So Equivalence (A.1) follows, since all elements of B^* are extensions of R.

For the proof of the additional claims referring to C_R, assume the validity of (5.a) through (5.e). As a side result of the proof of the implication b⇒c, Equation (A.12) is sufficient for a cone $C^{\sim}$ to fulfill the equivalence in (5.c) for all elements $\mathbf{y}_1$ and $\mathbf{y}_2$ of Y. In particular, C_R has this property. To show that Equation (A.12) is also necessary for this property, let $C^{\sim}$ be such a cone in F. It suffices to verify that the properties $\mathbf{x} \in C^{\sim}$ and $\mathbf{x} \in C_R$ are equivalent for every element $\mathbf{x}$ of C_D; then equation (A.12) follows, since C_R is a subset of C_D. So let $\mathbf{x} = \lambda\, (\mathbf{y}_2 - \mathbf{y}_1)$ be an arbitrary element of C_D. Since $C^{\sim}$ and C_R are cones, we get the outer equivalences in the following chain:

$$\mathbf{x} \in C^{\sim} \ \Leftrightarrow\ \mathbf{y}_2 - \mathbf{y}_1 \in C^{\sim} \ \Leftrightarrow\ \mathbf{y}_1\ R\ \mathbf{y}_2 \ \Leftrightarrow\ \mathbf{y}_2 - \mathbf{y}_1 \in C_R \ \Leftrightarrow\ \mathbf{x} \in C_R.$$

In summary, Equation (A.12) must hold for a cone $C^{\sim}$ fulfilling the equivalence in (5.c) for all elements $\mathbf{y}_1$ and $\mathbf{y}_2$ of Y. But this implies $C_R \subseteq C^{\sim}$ for every such cone. Hence C_R (being one of them) is the smallest one, i.e., the unique direction cone of R according to Definition A.5.a.

For the equivalence of assertions (5.f) and (5.g) with the foregoing ones, it can be taken as a side result that R' and R^* are extensions of R iff C_R is a cone with the properties of (5.c).

(6): Let $Y^{\sim}$ be a subset of F, and $C^{\sim}$ a cone in F. A reference to Definition A.5.a shows: If a direction-based relation on $Y^{\sim}$ with direction cone $C^{\sim}$ exists, then it must be the (unique) relation $R^{\sim}$ where the properties $\mathbf{y}_1\ R^{\sim}\ \mathbf{y}_2$ and $\mathbf{y}_2 - \mathbf{y}_1 \in C^{\sim}$ are equivalent for all elements $\mathbf{y}_1$ and $\mathbf{y}_2$ of $Y^{\sim}$. So let $R^{\sim}$ be this relation. Then it is an immediate generalization of (5) that $R^{\sim}$ is direction-based, and that $C^{\sim}$ is its direction cone iff $C^{\sim}$ is a subset of $C_D(Y^{\sim})$.

(7): It follows from Equivalence (A.8) and immediate generalizations of (5) that the relations R', R^* and R_X are direction-based, since the definition $C^{\sim} := C_R$ gives a cone with the property required for

(5.c). So another reference to (5) and to Equation (A.21) shows that C_R is the direction cone of these relations. - Now $X = X^* + \{-\mathbf{x}\}$ is equivalent with $\mathbf{x} \in X^*$ by Lemma A.11.(8), and the chain of equivalences $\mathbf{x}_1\ R_X\ \mathbf{x}_2 \Leftrightarrow \mathbf{x}_1\ R'\ \mathbf{x}_2 \Leftrightarrow \mathbf{x}_1 + \mathrm{x}\ R'\ \mathbf{x}_2 + \mathrm{x} \Leftrightarrow \mathbf{x}_1 + \mathrm{x}\ R^*\ \mathbf{x}_2 + \mathrm{x}$ follows for $\mathbf{x} \in X^*$ and $\mathbf{x}_1,\ \mathbf{x}_2 \in X$ from the definitions of R_X and R^* (outer equivalences) and from an immediate generalization of (5.b). Hence R_X is identical with R^* up to a translation by $-\mathbf{x}$ iff $\mathbf{x} \in X^*$ (see Definition A.2.i).

(8): It follows immediately from the definitions of B' and B^* that every restriction to X^* of an element of B' is an element of B^*. Conversely, for a given element $R^\sim$ of B^* with direction cone $C^\sim$, let $R_\sim$ be the direction-based relation on F with direction cone $C^\sim$ (the existence and uniqueness of this relation being granted by (6)). Then the relations $\mathbf{x}_1\ R^\sim\ \mathbf{x}_2$ and $\mathbf{x}_1\ R_\sim\ \mathbf{x}_2$ are equivalent for all elements $\mathbf{x}_1$ and $\mathbf{x}_2$ of X^*, since both relations are equivalent to $\mathbf{x}_2 - \mathbf{x}_1 \in C^\sim$. So $R_\sim$ is a direction-based extension of $R^\sim$ to F, and it is also an extension of R, since $R^\sim \in B^*$. But then $R_\sim$ is an element of B', and $R^\sim$ is the restriction of $R_\sim$ to X^*.

(9): This assertion is an immediate generalization of the set inclusion $C_R \subseteq X$ in (3). See Equation (A.20) for $X(X^*) = X$.

(10): Let a linear map $g{:}F \to \mathbb{R}$ be an element of G, and $\mathbf{x} = \lambda\ (\mathbf{y}_2 - \mathbf{y}_1)$ an arbitrary element of C_R. Then Equation (A.6) implies $\mathbf{y}_1\ R\ \mathbf{y}_2$, and $g(\mathbf{x}) = \lambda\ (g(\mathbf{y}_2) - g(\mathbf{y}_1)) \geq 0$ follows for $\lambda > 0$ from the linearity of g and from $g \in G$. - Conversely, if a linear map $g{:}F \to \mathbb{R}$ fulfills $g(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in C_R$, then the chain of implications $\mathbf{y}_1\ R\ \mathbf{y}_2 \Rightarrow \mathbf{y}_2 - \mathbf{y}_1 \in C_R \Rightarrow g(\mathbf{y}_2 - \mathbf{y}_1) \geq 0 \Rightarrow g(\mathbf{y}_1) \leq g(\mathbf{y}_2)$ is granted by Equation (A.6) and the linearity of g. Hence $g \in G$.

(11): The set G is non-empty, since the non-empty set C_R is a subset of G by (3). Furthermore, it follows from (10) that G is closed under addition and under multiplication by non-negative scalars. Hence it is a pointed, convex cone by Definition A.2.c and Lemma A.11.(4).

(12): Let G^* be a subset of G whose elements jointly represent the relation R. To verify that R is reflexive, transitive, and direction-based with direction cone $C_{G^*} = C_R$, combine Equivalence (A.11) with (3), (4) and (5) and Lemma A.11.(4). For a proof of the pre-Archimedian property of R, let $\mathbf{y}$, $\mathbf{y}_1$ and $\mathbf{y}_2$ be elements of Y such that $\mathbf{y}\ R_c\ \mathbf{y}_1$ and $\mathbf{y}\ R\ \mathbf{y}_2$, and let g be an element of G^* such that $g(\mathbf{y}_1 - \mathbf{y}) < 0$, the existence following from Equivalence (A.11) and Equation (A.10). Then $g(\mathbf{y}_2 - \mathbf{y}) \geq 0$ follows from Implication (A.9), since g is linear. This linearity also implies that the scalar δ defined by $\delta := (g(\mathbf{y}) - g(\mathbf{y}_1))\ /\ (g(\mathbf{y}_2) - g(\mathbf{y}_1))$ is greater than 0, and that the inequality $g(\mathbf{z}_\alpha - \mathbf{y}) < 0$ holds for every $\alpha \in\ ]0, \delta[$. Hence $\mathbf{z}_\alpha - \mathbf{y} \notin C_{G^*}$ follows from Equation (A.10), and Equivalence (A.11) yields $\mathbf{y}\ R_c\ \mathbf{z}_\alpha$ for $\mathbf{z}_\alpha \in Y$. So δ is a number with the properties required by Definition A.5.c. - Finally, if g is the only element of G^*, then the linearity of g implies for all elements $\mathbf{y}_1$ and $\mathbf{y}_2$ of Y that either $\mathbf{y}_2 - \mathbf{y}_1$ or $\mathbf{y}_1 - \mathbf{y}_2$ is an element of C_{G^*}. But then the connectedness of R follows from Equivalence (A.11).

For the proof of (13) through (15), assume that Y is convex.

(13): See (4) and Lemma A.11.(9). Note that convexity of Y is sufficient, but not necessary for the