The elliptic genus in conformal field theory

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[W14] *Snapshots of conformal field theory*; to appear in "Mathematical Aspects of Quantum Field Theories", Mathematical Physics Studies, Springer; arXiv:1404.3108 [hep-th]

[Taormina/W11] *The overarching finite symmetry group of Kummer surfaces in the Mathieu group M_{24}*, JHEP **1308**:152 (2013); arXiv:1107.3834 [hep-th]

[Taormina/W12] *A twist in the M_{24} moonshine story*; arXiv:1303.3221 [hep-th]

[Taormina/W13] *Symmetry-surfing the moduli space of Kummer K3s*; arXiv:1303.2931 [hep-th]

1. From indices to $U(1)$-equivariant loop space indices

[Hirzebruch78]

$$
\begin{aligned}
\chi_y(M) &:= \sum_{p,q}(-1)^q y^p h^{p,q}(M) \\
&= \sum_p y^p \sum_q (-1)^q \dim H^q(M, \Lambda^p T^*) \\
&= \sum_p y^p \chi(\Lambda^p T^*) \qquad = \int_M \mathrm{Td}(M) \sum_p y^p \,\mathrm{ch}(\Lambda^p T^*) \\
&= \int_M \mathrm{Td}(M)\,\mathrm{ch}(\Lambda_y T^*)
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Let $\mathcal{L}M = C^0(\mathbb{S}^1, M)$,

q_*: a topological generator of $U(1)$; $\mathcal{L}M^{U(1)} = M \hookrightarrow \mathcal{L}M$ (constant loops),

so for $p \in M$: $T_p(\mathcal{L}M) = \mathcal{L}(T_pM) = T_pM \oplus \mathcal{N}$, $\mathcal{N} = \bigoplus_{n\in\mathbb{Z}\setminus\{0\}} q^n T_pM$,

where $q^n T_pM \cong T_pM$: the eigenspace of q_* with eigenvalue q^n, $n \in \mathbb{Z}$,

$$\chi_y(q, \mathcal{L}M) := \int_M \prod_{j=1}^{D} \left\{ x_j \frac{1+ye^{-x_j}}{1-e^{-x_j}} \prod_{n=1}^{\infty} \left[\frac{1+q^n y e^{-x_j}}{1-q^n e^{-x_j}} \cdot \frac{1+q^n y^{-1} e^{x_j}}{1-q^n e^{x_j}} \right] (-y)^{\zeta(0)} \right\}$$

$$= \int_M \mathrm{Td}(M)\,\mathrm{ch}(\mathbb{E}_{q,y})$$

$$\mathbb{E}_{q,y} = (-y)^{-D/2} \Lambda_y T^* \otimes \bigotimes_{n=1}^{\infty} \left[\Lambda_{yq^n} T^* \otimes \Lambda_{y^{-1}q^n} T \otimes S_{q^n} T^* \otimes S_{q^n} T \right]$$

Definition of the elliptic genus

Theorem [Alvarez/Killingback/Mangano/Windey87, Hirzebruch88,Witten88,Krichever90,Borisov/Libgober00]

The elliptic genus

$$\mathcal{E}_M(\tau, z) \quad := \quad \int_M \mathrm{Td}(M)\,\mathrm{ch}(\mathbb{E}_{q,-y})$$

$$(\tau, z \in \mathbb{C},\ \mathrm{Im}(\tau) > 0,\ q = e^{2\pi i \tau},\ y = e^{2\pi i z})$$

of a Calabi-Yau D-fold M is a weak Jacobi form of weight 0 and index $\frac{D}{2}$,

that is, $\mathcal{E}_M(\tau, z) = e^{-2\pi i \frac{D}{2} \frac{cz^2}{c\tau+d}} \mathcal{E}_M(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}) = q^{Dn/2} y^{Dn} \mathcal{E}_M(\tau, z + m + n\tau)$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), \quad m, n \in \mathbb{Z}$,

and for all $\alpha, \beta \in \mathbb{Q}$, $\tau \mapsto \mathcal{E}_M(\tau, \alpha\tau + \beta)$ is bounded on the upper half plane.

$\mathcal{E}_M(\tau, z)$ only depends on the cobordism class of M.

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$$(\tau,z\in\mathbb{C},\ \mathrm{Im}(\tau)>0,\ q=e^{2\pi i\tau},\ y=e^{2\pi iz})$$

of a Calabi-Yau D-fold M is a weak Jacobi form of weight 0 and index $\frac{D}{2}$,

$$\begin{aligned}
\mathcal{E}_M(\tau,z=0) &= \chi(M),\\
\mathcal{E}_M(\tau,z=\tfrac{1}{2}) &= (-1)^{D/2}\sigma(M)+\mathcal{O}(q),\\
q^{D/4}\mathcal{E}_M(\tau,z=\tfrac{\tau+1}{2}) &= (-1)^{D/2}\chi(\mathcal{O}_M)+\mathcal{O}(q).
\end{aligned}$$

$\mathcal{E}_M(\tau,z)$ only depends on the cobordism class of M.

Chiral de Rham complex

Definition **[Malikov/Schechtman/Vaintrob99]**
For open $U \subset M$ with local holomorphic coordinates $z^1, \ldots, z^D$:
$\Omega^{ch}_M(U)$:= Fock space for the fields ϕ^j, p_j, ψ^j, ρ_j, $j \in \{1, \ldots, D\}$,
(D copies of a $bc - \beta\gamma$-system)
where $\phi^j \leftrightarrow z^j$, $p_j \leftrightarrow \frac{\partial}{\partial z^j}$, $\psi^j \leftrightarrow dz^j$, $\rho_j \leftrightarrow \frac{\partial}{\partial (dz^j)}$.
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Theorem **[Malikov/Schechtman/Vaintrob99;Borisov/Libgober00]**
There are globally well-defined fields on M,

$$L^{top} = -:p_j \partial \phi^j: - :\rho_j \partial \psi^j:, \; J = :\rho_j \psi^j:, Q = -:\psi^j p_j:, \; G = :\rho_j \partial \phi^j:,$$

which induce a topological $N = 2$ superconformal vertex algebra on $H^*(M, \Omega^{ch}_M)$.
The elliptic genus $\mathcal{E}_M(\tau, z)$ is the bigraded Euler characteristic of Ω^{ch}_M.

...and topologically half twisted sigma model

Result [Kapustin05]

There is a fine resolution $(\Omega_M^{ch,Dol}, d_{Dol})$ of Ω_M^{ch}, such that

$$\mathcal{E}_M(\tau, z) = \mathrm{sTr}_{H^*(\Omega_M^{ch,Dol})}\left(y^{J_0 - D/2} q^{L_0^{top}}\right).$$

$H^*(\Omega_M^{ch,Dol}) \cong \lim_{vol\to\infty}(\mathcal{H}_{NS}^{BRST})$, the large volume limit of the BRST-cohomology of Witten's half-twisted σ-model on M.

Conclusion

$$\mathcal{E}_M(\tau, z) = \mathrm{sTr}_{\mathcal{H}_R}\left(y^{J_0} q^{L_0 - D/8} \overline{q}^{\overline{L}_0 - D/8}\right) = \mathcal{E}_{CFT}(\tau, z),$$

$\mathcal{H}_R$: Ramond sector of any superconformal field theory associated to M,
$J_0, L_0, \overline{L}_0$: zero modes of the $U(1)$-current and Virasoro fields in the SCA.

2. The elliptic genus of K3

For every K3 surface M (i.e. M is a Calabi-Yau 2-fold, $h^{1,0} = 0$):

$$\mathcal{E}_{\mathrm{K3}}(\tau, z) = 8\left(\frac{\vartheta_2(\tau,z)}{\vartheta_2(\tau,0)}\right)^2 + 8\left(\frac{\vartheta_3(\tau,z)}{\vartheta_3(\tau,0)}\right)^2 + 8\left(\frac{\vartheta_4(\tau,z)}{\vartheta_4(\tau,0)}\right)^2.$$

For every $N = (2,2)$ SCFT at central charges $c = \overline{c} = 6$ with space-time SUSY and integral $U(1)$ charges:
its CFT elliptic genus either vanishes, or it agrees with $\mathcal{E}_{\mathrm{K3}}(\tau, z)$;
the theory has $N = (4,4)$ SUSY.

Definition (K3 THEORY)
An $N = (2,2)$ SCFT at $c = \overline{c} = 6$ with space-time SUSY, integral $U(1)$ charges and CFT elliptic genus $\mathcal{E}_{\mathrm{K3}}(\tau, z)$.

Decomposition into irreducible $N = 4$ characters

3 types of $N = 4$ irreps $\mathcal{H}_\bullet$ with $\chi_\bullet(\tau, z) = \mathrm{sTr}_{\mathcal{H}_\bullet}\left(y^{J_0} q^{L_0 - 1/4}\right)$:

- vacuum $\mathcal{H}_0$ with $\chi_0(\tau, 0) = -2$
- massless matter $\mathcal{H}_{m.m.}$ with $\chi_{m.m.}(\tau, 0) = 1$
- massive matter $\mathcal{H}_h$ $(h \in \mathbb{R}_{>0})$, $\chi_h(\tau, z) = q^h \widetilde{\chi}(\tau, z)$, $\chi_h(\tau, 0) = 0$

Ansatz:

$$\begin{aligned}\mathcal{H}_R \;=\; & \mathcal{H}_0 \otimes \overline{\mathcal{H}}_0 \oplus 20\, \mathcal{H}_{m.m.} \otimes \overline{\mathcal{H}}_{m.m.} \\ & \oplus \left(\bigoplus_{0<n\in\mathbb{N}} \left[f_n \mathcal{H}_n \otimes \overline{\mathcal{H}}_0 \oplus \overline{f}_n \mathcal{H}_0 \otimes \overline{\mathcal{H}}_n\right]\right) \\ & \oplus \left(\bigoplus_{0<m\in\mathbb{N}} \left[g_m \mathcal{H}_m \otimes \overline{\mathcal{H}}_{m.m.} \oplus \overline{g}_m \mathcal{H}_{m.m.} \otimes \overline{\mathcal{H}}_m\right]\right) \\ & \oplus \bigoplus_{0<h,\overline{h}\in\mathbb{R}} k_{h,\overline{h}} \mathcal{H}_h \otimes \overline{\mathcal{H}}_{\overline{h}}\end{aligned}$$

where all f_n, $\overline{f}_n$, g_m, $\overline{g}_m$, $k_{h,\overline{h}}$ are non-negative integers.

$$\Longrightarrow \quad \boxed{\begin{aligned}\mathcal{E}_{\mathrm{K3}}(\tau, z) \;&=\; -2\chi_0(\tau, z) + 20\chi_{m.m.}(\tau, z) + 2e(\tau)\widetilde{\chi}(\tau, z), \\ 2e(\tau) \;&=\; \sum_{n=1}^{\infty} (g_n - 2f_n) q^n\end{aligned}}$$

Conjecture **[Eguchi/Ooguri/Tachikawa10]** For all n, $g_n - 2f_n$ gives the dimension of a non-trivial representation of the Mathieu group M_{24}.

Mathieu Moonshine Phenomenon

Theorem [Gannon12] using results of **Cheng, Duncan, Gaberdiel, Hohenegger, Persson, Ronellenfitsch, Volpato**

There exists a representation $\mathcal{R}_n$ of M_{24} for every $n \in \mathbb{N}$, s.th.

$$\mathcal{R} := (-2)\mathcal{H}_0 \oplus 20\,\mathcal{H}_{m.m.} \oplus \bigoplus_{n=1}^{\infty} \mathcal{R}_n \otimes \mathcal{H}_n$$

has the twisted elliptic K3-genera as its graded characters.

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WHY?

HOW?

Is there an underlying structure of a vertex algebra?

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Theorem [Mukai88]

If G is a symmetry group of a K3 surface M,

that is, G fixes the two-forms that define the hyperkähler structure of M,

then G is isomorphic to a subgroup of the Mathieu group M_{24}, and $|G| \leq 960 \ll 244.823.040 = |M_{24}|$.

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[Gaberdiel/Hohenegger/Volpato11]

M_{24} cannot act as symmetry group of a K3 theory.

3. Symmetry surfing

Observation **[Taormina/W10-13]**
The map $\mathcal{H}_R \twoheadrightarrow \mathcal{H}^{BRST}_{R,\infty}$ depends on the choice of a geometric interpretation; SO: restrict to geometric symmetry groups.

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Observation [Taormina/W10-13]
The map $\mathcal{H}_R \twoheadrightarrow \mathcal{H}^{BRST}_{R,\infty}$ depends on the choice of a geometric interpretation; SO: restrict to geometric symmetry groups.

Conjecture [Taormina/W10-13]
In every geometric interpretation,

$$\mathcal{H}^{BRST}_{R,\infty} \cong (-2)\mathcal{H}_0 \oplus \mathcal{R}_{m.m.} \otimes \mathcal{H}_{m.m.} \oplus \bigoplus_{n=1}^{\infty} \mathcal{R}_n \otimes \mathcal{H}_n = \mathcal{R}$$

as a representation of the geometric symmetry group $G \subset M_{24}$; the rhs collects the symmetries from distinct points of the moduli space.

We call this procedure SYMMETRY SURFING.

Symmetries of $\mathbb{Z}_2$-orbifold CFTs on K3

G: geometric symmetry group of a $\mathbb{Z}_2$-orbifold CFT with geometric interpretation on $X = T/\mathbb{Z}_2$, $T = \mathbb{C}^2/\Lambda$

using [Fujiki88] $\Longrightarrow$

$$G = (\mathbb{Z}_2)^4 \rtimes G_T \subset (\mathbb{Z}_2)^4 \rtimes \mathrm{GL}_4(\mathbb{F}_2) \overset{\text{[Jordan1870]}}{\cong} (\mathbb{Z}_2)^4 \rtimes A_8,$$

$$\mathbb{F}_2^4 \cong \tfrac{1}{2}\Lambda/\Lambda,\ G_T \subset \mathrm{SO}(3)$$

- $G_T \subset (G_T)_k$, one of three maximal finite groups $(G_T)_1 = A_4$, $(G_T)_2 = S_3$, $(G_T)_0 = \mathbb{Z}_2^2$
- there exists a smooth deformation, preserving the symmetry G, from Λ into Λ_k with

$$\begin{aligned}
\Lambda_1 &= \mathrm{span}_{\mathbb{Z}} \left\{(1,0,0,0), (0,1,0,0), (0,0,1,0), \tfrac{1}{2}(1,1,1,1)\right\}, \\
\Lambda_2 &= \mathrm{span}_{\mathbb{Z}} \left\{(1,0,0,0), \tfrac{1}{2}(-1,\sqrt{3},0,0), (0,0,1,0), \tfrac{1}{2}(0,0,-1,\sqrt{3})\right\}, \\
\Lambda_0 &= \mathrm{span}_{\mathbb{Z}} \left\{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\right\}.
\end{aligned}$$

Symmetry surfing the moduli space of Kummer K3s

Result **[Taormina/W11&12]**
For the $\mathbb{Z}_2$-orbifold CFTs on K3 with geometric interpretation on some $X = \widetilde{T/\mathbb{Z}_2}$ with $T = \mathbb{C}^2/\Lambda$, the joint action of all symmetry groups yields the maximal subgroup $\mathrm{Aff}(\mathbb{F}_2^4) = (\mathbb{Z}_2)^4 \rtimes A_8 \subset M_{24}$.

Note: $\mathbb{Z}_2^4 \rtimes A_8$ is not a subgroup of M_{23}.

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Note: $\mathbb{Z}_2^4 \rtimes A_8$ is not a subgroup of M_{23}.

Recall:

$$\mathcal{H}^{BRST}_{R,\infty} \cong (-2)\mathcal{H}_0 \oplus \mathcal{R}_{m.m.} \otimes \mathcal{H}_{m.m.} \oplus \bigoplus_{n=1}^{\infty} \mathcal{R}_n \otimes \mathcal{H}_n = \mathcal{R}$$

Result [Taormina/W13]
$\mathcal{R}_1$ can be constructed as a 90-dim. space of states common to all K3-theories that are $\mathbb{Z}_2$-orbifolds of toroidal theories.
As common representation space of all geometric symmetry groups of Kummer K3s, $\mathcal{R}_1$ carries an action of $\mathbb{Z}_2^4 \rtimes A_8$ induced from $\mathcal{R}_1 \cong \mathbf{45} \oplus \overline{\mathbf{45}}$ with irreps $\mathbf{45}$, $\overline{\mathbf{45}}$ of M_{24}.

4. Conclusions: A simpler open (?) conjecture

Recall:

$$\begin{aligned}\mathcal{E}_{\mathrm{K3}}(\tau,z) &= \int_{\mathrm{K3}} \mathrm{Td}(\mathrm{K3})\mathrm{ch}(\mathbb{E}_{q,-y}) \\ &= -2\chi_0(\tau,z) + 20\chi_{m.m.}(\tau,z) + \sum_{n=1}^{\infty}(g_n - 2f_n)\chi_n(\tau,z)\end{aligned}$$

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Conjecture [W13]

There are polynomials p_n for every $n \in \mathbb{N}$, such that

$$\mathbb{E}_{q,-y} = -\mathcal{O}_{\mathrm{K3}}\chi_0(\tau,z) - T\chi_{m.m.}(\tau,z) + \sum_{n=1}^{\infty} p_n(T)\chi_n(\tau,z),$$

where $\dim(\mathcal{R}_n) = g_n - 2f_n = \int_{\mathrm{K3}} \mathrm{Td}(\mathrm{K3})p_n(T) = \chi(p_n(T))$ for all $n \in \mathbb{N}$.

Moreover, $p_n(T) \twoheadrightarrow \mathcal{R}_n$ carries a natural action of every geometric symmetry group $G \subset M_{24}$ of K3.

The End

Thank you
for your attention!