Compact subspaces of moduli spaces of bundles over class VII surfaces
Towards the classification of class VII surfaces

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VBAC Berlin, September 4, 2014
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Class VII surfaces

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### Definition 0.1

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- Class VII surfaces with \( b_2 = 0 \) are classified. We are interested in minimal class VII surfaces with \( b_2 > 0 \). Let \( \text{X} \in \text{VII}_{\text{min}}^{b_2>0} \).
- \( \text{X} \) has \( b_2(\text{X}) \) rational curves \( \Rightarrow \text{X} \) is a Kato surface (it belongs to the list of known surfaces) [Dloussky-Oeljeklaus-Toma].
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- Let \( X \supset D > 0 \) be an effective divisor with \( \omega_D \simeq \mathcal{O}_D \). Then \( D \) contains a cycle of curves. Recall: \( \omega_D := \mathcal{K}_D(D) \).
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- Let \( X \supset D > 0 \) be an effective divisor with \( \omega_D \sim \mathcal{O}_D \). Then \( D \) contains a cycle of curves. Recall: \( \omega_D := \mathcal{K}_D(D) \).
- **Goal:** Prove that any \( X \in \text{VII}_{\min}^{b_2>0} \) contains such a divisor. This would complete the classification up to deform. equivalence.
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Let $(X, g)$ be a complex surface endowed with a Gauduchon metric. The Gauduchon condition for surfaces: $\partial \bar{\partial} \omega_g = 0$.

$$\deg_g(L) := \int_X c_1(L, h) \wedge \omega_g, \quad \deg_g(F) := \deg_g(\det(F)).$$

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- A holomorphic rank 2 bundle \(E\) on \(X\) is called
  - **stable**, if for every line bundle \(L\) and non-trivial morphism \(L \to E\) one has \(\deg(L) < \frac{1}{2} \deg_g(\det(E))\).
  - **polystable**, if is either stable or isomorphic to a direct sum \(L \oplus M\) of line bundles with \(\deg_g(L) = \deg_g(M)\).
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- Let \(E\) be a \(C^\infty\) 2-bundle on \(X\), \(\mathcal{D}\) hol. struct. on \(D := \det(E)\),

\[
\mathcal{M}_{\mathcal{D}}^{\text{st}}(E), \quad \mathcal{M}_{\mathcal{D}}^{\text{polyst}}(E)
\]

the moduli spaces of stable, polystable hol. structures \(\mathcal{E}\) on \(E\) inducing \(\mathcal{D}\) on \(\det(E)\), modulo \(\text{Aut}_D(E) = G^C := \Gamma(X, \text{SL}(E))\).
Remarks:

- If $b_1(X)$ is odd then $\text{Pic}^0(X)$ is non-compact and $\text{deg}_g$ is not a topological invariant.

**Example:** For a class VII surface one has

$$\text{Pic}^0(X) \cong \mathbb{C}^*, \quad \text{deg}_g(\mathcal{L}_\zeta) = C_g \log |\zeta|.$$
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- The **Kobayashi-Hitchin correspondence:** Let $a$ be the Chern connection of the pair $(\mathcal{D}, \det(h))$. We have isomorphisms

$$\mathcal{M}_{\text{p}^{\text{st}}_D}(E) \xrightarrow{\sim \text{KH}} \mathcal{M}_{a_{\text{ASD}}}(E), \quad \mathcal{M}_{\text{st}D}(E) \xrightarrow{\sim \text{KH}^*} \mathcal{M}_{a_{\text{ASD}}}(E)^*$$

The points of the reduction space $\mathcal{R} = \mathcal{M}_{\text{p}^{\text{st}}_D}(E) \setminus \mathcal{M}_{\text{st}D}(E)$ have two interpretations: split polystable 2-bundles and reducible instantons.
From now on we assume

- about $X$: $b_1(X) = 1$ and $p_g(X) = 0$ ($b_+(X) = 0$).
- about $E$: $c_1(E) \notin 2H^2(X, \mathbb{Z})$.

Under these assumptions $\mathcal{R}$ is a finite disjoint union of circles.

In general $\mathcal{M}^{\text{pst}}_{\mathcal{D}}(E)$ is not a complex space around $\mathcal{R}$. The topological structure of the moduli space $\mathcal{M}^{\text{pst}}_{\mathcal{D}}(E)$ around a circle $R$ of regular reductions is simple: A cone bundle over $R$ with fibre: cone (in the topological sense) over $\mathbb{P}^{d-1}_C$. Here

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- **Example:** For $d = 1$: $\mathcal{M}^{p_{\text{st}}}_{D}(E)$ has the structure of a Riemann surface with boundary $R$ around a circle $R$ of reductions.
On non-algebraic surfaces: the appearance of *non-filtrable bundles* complicates the description of a moduli space $\mathcal{M}^{\text{pst}}_{D}(E)$. A rank 2 holomorphic bundle $\mathcal{E}$ on $X$ is called *filtrable* if there exists a sheaf mono-morphism

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A filtrable bundle $\mathcal{E}$ fits in a short exact sequence

\[ 0 \to \mathcal{M} \to \mathcal{E} \to \mathcal{N} \otimes \mathcal{I}_Z \to 0 , \]

for line bundles $\mathcal{M}$, $\mathcal{N}$ and a 0-dimensional l.c.i. $Z \subset X$. 
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A non-filtrable bundle is stable with respect to any Gauduchon metric. There exists no classification method for non-filtrable bundles.
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Let now $X$ be a class VII surface and $(E, h)$ a differentiable rank 2-bundle on $X$ with

$$c_2(E) = 0, \quad \det(E) = K_X \quad (\text{the underlying } \mathcal{C}^\infty \text{ bundle of } \mathcal{K}_X),$$

and let $a$ be the Chern connection of $(\mathcal{K}_X, \det(h))$. 
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Our fundamental object: the moduli space

$$\mathcal{M} := \mathcal{M}_{K}^{\text{pst}}(E) \overset{KH}{\leftarrow} \mathcal{M}_{a}^{\text{ASD}}(E).$$

and its open subspace $\mathcal{M}^{\text{st}} := \mathcal{M}^{\text{st}}_{K}(E)$ of stable bundles, which is a complex space of dimension $b := b_2(X)$. 
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Using bundles to prove existence of curves: prove that the same filtrable bundle can be written as an extension in two different ways. This yields a non-trivial (and non-isomorphic) morphism of line bundles, whose vanishing locus will be a curve.
Compactness: \( \mathcal{M} \) is compact (– and Buchdahl). This is proved using a the KH correspondence and a combination of gauge theoretical and complex geometric arguments.
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For \( b := b_2(X) \leq 3 \) the statement follows from Donaldson-Uhlenbeck compactness theorem for instantons: in this range the lower strata of the Uhlenbeck compactification are empty for topological reasons.
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• **Reductions:** $\mathcal{R}$ is a disjoint union of $2^{b-1}$ circles $R_{\{I,\overline{I}\}}$ indexed by unordered partitions $\{I, \overline{I}\}$ of $\{1, \ldots, b\}$.

• Every $R_{\{I,\overline{I}\}}$ has a compact neighborhood homeomorphic to $R_{\{I,\overline{I}\}} \times [\text{cone over } \mathbb{P}_C^{b-1}]$. 
Symmetry and twisted reductions: Natural involution on $\mathcal{M}$:

\[ \otimes \mathcal{L}_0 : \mathcal{M} \rightarrow \mathcal{M}, \quad \mathcal{L}_0^\otimes 2 \simeq \mathcal{O}_X, \text{ where } [\mathcal{O}_X] \neq [\mathcal{L}_0] \in \text{Pic}^0(X) . \]

This involution has finitely many fixed points, called twisted reductions. There are $2^{b-1}$ twisted reductions if $\pi_1(X, x_0) \cong \mathbb{Z}$. 

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A twisted reduction $\mathcal{E}$ can be written as $\pi_*(\mathcal{L})$, where

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is a double cover and $\mathcal{L}$ is a line bundle on $\tilde{X}$. Therefore $\pi^*(\mathcal{E})$ is split polystable (and corresponds to a reducible instanton).
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The filtrable bundles in our moduli space:

Put $b := b_2(X)$ and let $(e_1, \ldots, e_b)$ be a *Donaldson basis* of $H^2(X, \mathbb{Z})$, i.e., it a basis $(e_1, \ldots, e_b)$ such that

$$e_i \cdot e_j = -\delta_{ij}, \quad c_1(K_X) = \sum_{i=1}^b e_i.$$
Let $\mathcal{E}$ be rank 2 bundle on $X$ with $\det(\mathcal{E}) = K_X$, $c_2(\mathcal{E}) = 0$.

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One can prove that $\mathcal{E}/\mathcal{L}$ is then locally free and 
\[ c_1(\mathcal{L}) = e_I := \sum_{i \in I} e_i \text{ where } I \subset \{1, \ldots, b\}. \]

Therefore

**Proposition 2.1**

*Any filtrable bundle in our moduli space is an extension*

\[ 0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{K}_X \otimes \mathcal{L}^\vee \to 0, \quad (1) \]

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$\mathcal{M}^{\text{st}} \supset \mathcal{M}_I^{\text{st}} :=$ the subspace of stable bundles which are extensions of type (1) with fixed $c_1(\mathcal{L}) = e_I$. 
A crucial role is played by $\mathcal{M}_{I_m}^{st}$ associated with the maximal index set $I_m := \{1, \ldots, b\}$. Except for certain known (!) surfaces, one has

$$\mathcal{M}_{I_m}^{st} = \{ \mathcal{A}, \mathcal{A}' \}$$

where $\mathcal{A}$ is the *canonical extension* of $X$, defined as the essentially unique non-trivial extension of the form

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(note $h^1(\mathcal{K}_X) = 1$ by Serre duality), and $\mathcal{A}' := \mathcal{A} \otimes \mathcal{L}_0$. 
A crucial role is played by $\mathcal{M}_{l_m}^{\text{st}}$ associated with the maximal index set $l_m := \{1, \ldots, b\}$. Except for certain known (!) surfaces, one has

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(note $h^1(\mathcal{K}_X) = 1$ by Serre duality), and $\mathcal{A}' := \mathcal{A} \otimes \mathcal{L}_0$.

For $l \neq l_m$: If $X$ has no curves in certain homology classes (which we assume for simplicity!) $\mathcal{M}_l^{\text{st}}$ is a $\mathbb{P}^{b-|l|-1}_{\mathbb{C}}$-fibration over a punctured disk, these fibrations are pairwise disjoint, and the closure $\bar{\mathcal{M}}_l^{\text{st}}$ in $\mathcal{M}$ contains the circle $R\{l,\bar{l}\}$. 
Overview: What do we know about the moduli space \( \mathcal{M} \)?

- We know that \( \mathcal{M} \) is always compact. If certain simplifying conditions are satisfied (which we assume) then
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$$\bigcup_{I \subset I_m} \mathcal{M}_{I}^{\text{st}}, \text{ where } \mathcal{M}_{I_m}^{\text{st}} = \{A, A'\}, A' := A \otimes L_0$$
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- for $I \neq I_m$ the space $\mathcal{M}^{st}_I$ is a $\mathbb{P}_{\mathbb{C}}^{b-|I|-1}$-fibration over a punctured disk. The closure of $\mathcal{M}^{st}_I$ contains the circle $R\{I, \overline{I}\}$.
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I hope: the method generalizes for arbitrary $b_2$. This would complete the classification of class VII surfaces up to deformation equivalence.
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- I hope: the method generalizes for arbitrary $b_2$. This would complete the classification of class VII surfaces up to deformation equivalence.
- For $b_2 \in \{1, 2\}$ the result is proven in previous articles. The proof for $b_2 = 3$: partially available on the archive. Trying to go directly to arbitrary $b_2$.
- I will explain the proof for $b_2 = 1$, a new proof for $b_2 = 2$ and if I have the time, I will also explain briefly the case $b_2 = 3$. 

Strategy of the proof (in general): Use the following
Theorem 3.1

Any minimal class VII surface $X$ with $b_2(X) \in \{1, 2, 3\}$ has a cycle.

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- Strategy of the proof (in general): Use the following...
Proposition 3.2

If the canonical extension $A$ can be written as an extension in a different way, then $X$ has a cycle. In particular, if $A$ belongs to $M_{I}^{st}$ for $I \neq I_m$ or coincides with a twisted reduction, then $X$ has a cycle.
**Proposition 3.2**

If the canonical extension $\mathcal{A}$ can be written as an extension in a different way, then $X$ has a cycle. In particular, if $\mathcal{A}$ belongs to $\mathcal{M}_I^{st}$ for $I \neq I_m$ or coincides with a twisted reduction, then $X$ has a cycle.

**Proof.**

\[
\begin{array}{cccccc}
0 & \rightarrow & \mathcal{K}_X & \overset{i}{\rightarrow} & \mathcal{A} & \overset{p}{\rightarrow} & \mathcal{O}_X & \rightarrow & 0 \\
\downarrow j & & \downarrow p \circ j & & & & \mathcal{L} \\
\end{array}
\]

$p \circ j$ is non-zero (because $\mathcal{L}$ is a different kernel) and non-isomorphism, because the canonical extension is non-split. Therefore $\text{im}(p \circ j) = \mathcal{O}_X(-D)$ where $D > 0$ is the vanishing divisor of $p \circ j$. Restrict the diagram to $D$ taking into account that $j$ is a bundle embedding. We get $\omega_D := \mathcal{K}_X(D)_D \simeq \mathcal{O}_D$, so $D$ contains a cycle.
In order to complete the proof it "suffices" to prove

**The remarkable incidence:** The bundle $\mathcal{A}$ belongs to

$$\{\text{twisted reductions}\} \cup \left( \bigcup_{l \neq l_m} \mathcal{M}_l^{st} \right)$$

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How will this RI be proved?
For $b_2 = 1$: $I \in \{\emptyset, I_m\}$.

red locus: the circle of reductions
grey locus (punctured disk): $\mathcal{M}_{st}^\emptyset$
the blue point: a twisted reduction!
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Therefore we have the **dichotomy:** either (1) the remarkable incidence holds (*and the conjecture is proved*), or (2) the connected component of $\mathcal{A}$ in $\mathcal{M}$ is a closed Riemann surface $Y \subset \mathcal{M}^{\text{st}}$ which has at most two filtrable points.
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**Proposition 3.3**

Suppose that $X$ is a complex surface with $a(X) = 0$, $E$ a differentiable rank 2 bundle over $X$, $Y$ a closed Riemann surface and $f : Y \rightarrow \mathcal{M}_{simple}(E)$, $y \mapsto [\mathcal{E}_y]$ a holomorphic map. Then the bundles $\mathcal{E}_y$ are either all filtrable or all non-filtrable.
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**Proof.**

(Idea) Change the roles, i.e. construct a family of holomorphic bundles on $Y$ parameterized by $X$. Use the fact that $Y$ is algebraic and $a(X) = 0$. 

Abdul Teleman

Compact subspaces of moduli spaces
For $b_2 \geq 2$, $\dim(\mathcal{M}) = b_2$ and $\mathcal{M}$ contains $2^{b_2-1}$ circles of reductions.

**Remark 3.4**

*For $b_2 \leq 3$ all circles of reductions belong to the same component $\mathcal{M}_0$ of $\mathcal{M}$. This component comes with a natural stratification.*
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**Remark 3.4**

*For $b_2 \leq 3$ all circles of reductions belong to the same component $M_0$ of $M$. This component comes with a natural stratification.*

We give the proof in the case $b_2 = 2$, in particular we show how the *canonical stratification* of $M_0$ is obtained. The method generalizes to $b_2 = 3$ and (it seems) to arbitrary $b_2$. 
Proof.

For $b_2 = 2$ we have 2 circles of reductions $R_{\{\emptyset, I_m\}}, R_{\{1, 2\}}$. We prove that there does not exist any connected component $Y$ of $\mathcal{M}$ containing exactly one circle of reductions $R$. 
Proof.

For $b_2 = 2$ we have 2 circles of reductions $R_{\{\emptyset, l_m\}}, R_{\{1\},\{2\}}$. We prove that there does not exist any connected component $Y$ of $\mathcal{M}$ containing exactly one circle of reductions $R$. 

Let $N$ be a standard compact neighborhood of $R$. The boundary $\partial N$ is also the boundary of $Y \setminus \partial N \subset B^* a(E)$ (the moduli space of irreducible connections with fixed determinant $a$), so $\partial N$ would be homologically trivial in $B^* a(E)$. On the other hand the restriction to $\partial N$ of a Donaldson class $\eta \in H^3(B^* a(E), \mathbb{Q})$ is nontrivial. Contradiction.
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- Let $\mathcal{N}$ be standard compact neighborhood of $R$. The boundary $\partial \mathcal{N}$ is also the boundary of $Y \setminus \mathcal{N} \subset \mathcal{B}_a^*(E)$ (the moduli space of irreducible connections with fixed determinant $a$), so $\partial \mathcal{N}$ would be homologically trivial in $\mathcal{B}_a^*(E)$.
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$\mathcal{M}_0$ contains both circles of reductions, so also the strata $\mathcal{M}^{st}_0$, $\mathcal{M}^{st}_{\{1\}}$, $\mathcal{M}^{st}_{\{2\}}$. We can build the connected component $\mathcal{M}_0$ from the known pieces as in a puzzle game.
\( \mathcal{M}_0 \) contains both circles of reductions, so also the strata \( \mathcal{M}_{0}^{st} \), \( \mathcal{M}_{\{1\}}^{st} \), \( \mathcal{M}_{\{2\}}^{st} \).
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Unfortunately there is no way to prove directly that the obvious solution is the correct solution, because we don’t know if we have all the pieces. Classification of surfaces: a minimal ruled surface is a locally trivial $\mathbb{P}^1$-bundle, but our component $\mathcal{M}_0$ might be non-minimal.
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The correct solution: $M_0$ is obtained from $D \times \mathbb{P}^1_C$ by applying an iterated blow up above the origin of $D$ and afterwards collapsing to points the projective lines above the boundary of $D$. 

The fiber over $0 \in D$ is a tree of rational curves: the known curve $M_{st}\{1\} \cup M_{st}\{2\} \cup \mathcal{R}\{\{1\},\{2\}\} \cup \{two\ twisted\ reductions\}$ and unknown ("green") curves, whose generic points must be non-filtrable.
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The moduli space $\mathcal{M}$ consists of $\mathcal{M}_0$ and possibly other connected components:
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- All positive dimensional of stable bundles appear in grey, all reductions in red, and all twisted reductions in blue. *Any green component (curve or surface) consists generically of non-filtrable bundles.*
The moduli space $\mathcal{M}$ consists of $\mathcal{M}_0$ and possibly other connected components:

- All positive dimensional of stable bundles appear in grey, all reductions in red, and all twisted reductions in blue. Any green component (curve or surface) consists generically of non-filtrable bundles.
- The natural question is: What is the position of $a := [\mathcal{A}]$?
We have again a **dichotomy**: Either the remarkable incidence holds (hence $X$ has a cycle), or $a := [A]$ belongs to a compact subspace of $\mathcal{M}^{\text{st}}$ (of dimension 1 or 2) consisting generically of non-filtrable bundles.
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The latter possibility is ruled out by the main result of a recent article ("Compact subspaces of moduli ..." arXiv:1309.0350):

**Theorem 3.5**

*There does not exist any positive dimensional compact subspace $Y \subset \mathcal{M}^{\text{st}}$ containing the point $a$ and an open neighborhood $a \in Y_a \subset Y$ such that $Y_a \setminus \{a\}$ consists only of non-filtrable bundles.*
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- In other words, within any positive dimensional compact subspace $a \in Y \subset \mathcal{M}^{st}$, the point $a$ can be approached by filtrable bundles, *it cannot be surrounded only by non-filtrables.*
general strategy

The strategy of the proof (for any $b_2$) is:

- We showed (easy!) that the RI implies the existence of a cycle.
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- We showed (easy!) that the RI implies the existence of a cycle.
- The connected component $\mathcal{M}_0$ of $\mathcal{M}$ which contains the circles of reductions comes with a natural stratification. Studying this stratification one comes to the dichotomy: Either (1) the remarkable incidence holds, or (2) $a := [\mathcal{A}]$ belongs to a compact complex subspace $Y \subset \mathcal{M}^{\text{st}}$ in which it is surrounded only by non-filtrable points.
The strategy of the proof (for any $b_2$) is:

- We showed (easy!) that the RI implies the existence of a cycle.
- The connected component $M_0$ of $M$ which contains the circles of reductions comes with a natural stratification. Studying this stratification one comes to the dichotomy: Either (1) the remarkable incidence holds, or (2) $a := [\mathcal{A}]$ belongs to a compact complex subspace $Y \subset M^{st}$ in which it is surrounded only by non-filtrable points.
- Apply our non-existence theorem, which rules out the second possibility.
The difficulty for $b_2 \geq 3$:

Main difficulty: rule out the situation when $\mathcal{M}_0$ contains an unknown stratum $Y$ (consisting generically of non-filtrable bundles) which contains a circle of reductions. For such a stratum the non-existence theorem does not apply.
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**Theorem 3.6**

Suppose $b_2 = 3$. Any bidimensional stratum $\mathcal{Z}$ of the canonical stratification of $\mathcal{M}_0$ whose closure contains the circle $R_{\{\{i\},\{j,k\}\}}^{\{a\},\{b,c\}}$ coincides with a known stratum $\mathcal{M}_{\text{st}}^{\{a\},\{b,c\}}$. 

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Compact subspaces of moduli spaces
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The proof uses “reductio ad absurdum" and a cobordism argument.
bibliography


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