Compact subspaces of moduli spaces of bundles over class VII surfaces

Towards the classification of class VII surfaces

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Definition 0.1

$VII := \{X \textit{ complex surface} \mid b_1(X) = 1,\ \mathrm{kod}(X) = -\infty\}.$

- Class VII surfaces with $b_2 = 0$ are classified. We are interested in minimal class VII surfaces with $b_2 > 0$. Let $X \in VII_{\min}^{b_2>0}$

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- Let $X \supset D > 0$ be an effective divisor with $\omega_D \simeq \mathcal{O}_D$. Then D contains a cycle of curves. Recall: $\omega_D := \mathcal{K}_D(D)$.

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- **Goal**: Prove that any $X \in VII_{\mathrm{min}}^{b_2>0}$ contains such a divisor. This would complete the classification up to deform. equivalence.

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- Let (X, g) be a complex surface endowed with a *Gauduchon metric*. The Gauduchon condition for surfaces : $\partial\bar\partial\omega_g = 0$.

$$\deg_g(\mathcal{L}) := \int_X c_1(\mathcal{L}, h) \wedge \omega_g,\ \deg_g(\mathcal{F}) := \deg_g(\det(\mathcal{F})).$$

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  $\deg_g : \mathrm{Pic}(X) \to \mathbb{R}$ is a morphism of Abelian Lie Groups.
- A holomorphic rank 2 bundle $\mathcal{E}$ on X is called
  - stable, if for every line bundle $\mathcal{L}$ and non-trivial morphism $\mathcal{L} \to \mathcal{E}$ one has $\deg(\mathcal{L}) < \frac{1}{2}\deg_g(\det(\mathcal{E}))$.
  - polystable, if is either stable or isomorphic to a direct sum $\mathcal{L} \oplus \mathcal{M}$ of line bundles with $\deg_g(\mathcal{L}) = \deg_g(\mathcal{M})$.

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- Let E be a $\mathcal{C}^\infty$ 2-bundle on X, $\mathcal{D}$ hol. struct. on $D := \det(E)$,

$$\mathcal{M}^{\mathrm{st}}_{\mathcal{D}}(E)\ ,\ \mathcal{M}^{\mathrm{pst}}_{\mathcal{D}}(E)$$

the moduli spaces of stable, polystable hol. structures $\mathcal{E}$ on E inducing $\mathcal{D}$ on $\det(E)$, modulo $\mathrm{Aut}_D(E) = \mathcal{G}^{\mathbb{C}} := \Gamma(X, \mathrm{SL}(E))$.

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Remarks:

- If $b_1(X)$ is odd then $\mathrm{Pic}^0(X)$ is non-compact and $\deg_g$ is not a topological invariant.

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- $\mathcal{M}^{\mathrm{st}}_{\mathcal{D}}(E)$ has a natural complex space structure obtained using classical deformation theory or complex gauge theory.
- The Kobayashi-Hitchin correspondence: Let a be the Chern connection of the pair $(\mathcal{D}, \det(h))$. We have isomorphisms
  $$\mathcal{M}^{\mathrm{pst}}_{\mathcal{D}}(E) \xrightarrow{\simeq KH} \mathcal{M}^{\mathrm{ASD}}_{a}(E), \ \mathcal{M}^{\mathrm{st}}_{\mathcal{D}}(E) \xrightarrow{\simeq KH^*} \mathcal{M}^{\mathrm{ASD}}_{a}(E)^*$$
  The points of the reduction space $\mathcal{R} = \mathcal{M}^{\mathrm{pst}}_{\mathcal{D}}(E) \setminus \mathcal{M}^{\mathrm{st}}_{\mathcal{D}}(E)$ have two interpretations: split polystable 2-bundles and reducible instantons.

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- From now on we assume
  - about X: $b_1(X) = 1$ and $p_g(X) = 0$ ($b_+(X) = 0$).
  - about E: $c_1(E) \not\in 2H^2(X, \mathbb{Z})$.

  Under these assumptions $\mathcal{R}$ is a finite disjoint union of circles.
- In general $\mathcal{M}^{\mathrm{pst}}_{\mathcal{D}}(E)$ is not a complex space around $\mathcal{R}$. The topological structure of the moduli space $\mathcal{M}^{\mathrm{pst}}_{\mathcal{D}}(E)$ around a circle R of regular reductions is simple: A cone bundle over R with fibre: cone (in the topological sense) over $\mathbb{P}^{d-1}_C$. Here

$$d = 4c_2(E) - c_1^2(E)$$

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- **Example:** For $d = 1$: $\mathcal{M}^{\mathrm{pst}}_{\mathcal{D}}(E)$ has the structure of a Riemann surface with boundary R around a circle R of reductions.

4

- On non-algebraic surfaces: the appearance of *non-filtrable bundles* complicates the description of a moduli space $\mathcal{M}^{\mathrm{pst}}_{\mathcal{D}}(E)$. A rank 2 holomorphic bundle $\mathcal{E}$ on X is called *filtrable* if there exists a sheaf mono-morphism

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- A filtrable bundle $\mathcal{E}$ fits in a short exact sequence

$$0 \to \mathcal{M} \to \mathcal{E} \to \mathcal{N} \otimes \mathcal{I}_Z \to 0 \ ,$$

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- A non-filtrable bundle is stable with respect to *any* Gauduchon metric. There exists no classification method for non-filtrable bundles.

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- Our fundamental object: the moduli space

  $$\mathcal{M} := \mathcal{M}^{\rm pst}_{\mathcal{K}}(E) \xleftarrow[]{\overset{KH}{\simeq}} \mathcal{M}^{\rm ASD}_a(E)\ .$$

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- Using bundles to prove existence of curves: prove that the same filtrable bundle can be written as en extension in two different ways. This yields a non-trivial (and non-isomorphic) morphism of line bundles, whose vanishing locus will be a curve.

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- Compactness: $\mathcal{M}$ is compact (– and Buchdahl).
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- Every $R_{\{I,\bar{I}\}}$ has a compact neighborhood homeomorphic to $R_{\{I,\bar{I}\}} \times$ [cone over $\mathbb{P}^{b-1}_{\mathbb{C}}$].

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- Symmetry and twisted reductions: Natural involution on $\mathcal{M}$:

$$\otimes\mathcal{L}_0 : \mathcal{M} \to \mathcal{M},\ \mathcal{L}_0^{\otimes 2} \simeq \mathcal{O}_X,\ \text{where } [\mathcal{O}_X] \neq [\mathcal{L}_0] \in \mathrm{Pic}^0(X).$$

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- A twisted reduction $\mathcal{E}$ can be written as $\pi_*(\mathcal{L})$, where

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  is a double cover and $\mathcal{L}$ is a line bundle on $\tilde{X}$. Therefore $\pi^*(\mathcal{E})$ is split polystable (and corresponds to a reducible instanton).
- *The filtrable bundles in our moduli space:*
  Put $b := b_2(X)$ and let $(e_1, \dots, e_b)$ be a *Donaldson basis* of $H^2(X, \mathbb{Z})$, i.e., it a basis $(e_1, \dots, e_b)$ such that

$$e_i \cdot e_j = -\delta_{ij},\ c_1(\mathcal{K}_X) = \sum_{i=1}^{b} e_i\ .$$

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- Let $\mathcal{E}$ be rank 2 bundle on X with $\det(\mathcal{E}) = \mathcal{K}_X$, $c_2(\mathcal{E}) = 0$ $\mathcal{L}$ a line bundle and $0 \to \mathcal{L} \to \mathcal{E}$ a sheaf monomorphism with torsion free quotient.

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- One can prove that $\mathcal{E}/\mathcal{L}$ is then locally free and

$$c_1(\mathcal{L}) = e_I := \sum_{i \in I} e_i \text{ where } I \subset \{1, \dots, b\}.$$

Therefore

Proposition 2.1

Any filtrable bundle in our moduli space is an extension

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{K}_X \otimes \mathcal{L}^\vee \to 0 \ , \tag{1}$$

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where $c_1(\mathcal{L}) = e_I$ for an index set $I \subset \{1, \ldots, b\}$.

- $\mathcal{M}^{\mathrm{st}} \supset \mathcal{M}^{\mathrm{st}}_I :=$ the subspace of stable bundles which are extensions of type (1) with fixed $c_1(\mathcal{L}) = e_I$.

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- A crucial role is played by $\mathcal{M}^{\mathrm{st}}_{I_m}$ associated with the maximal index set $I_m := \{1, \dots, b\}$. Except for certain known (!) surfaces, one has

$$\mathcal{M}^{\mathrm{st}}_{I_m} = \{\mathcal{A}, \mathcal{A}'\}$$

where $\mathcal{A}$ is the *canonical extension* of X, defined as the essentially unique non-trivial extension of the form

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(note $h^1(\mathcal{K}_X) = 1$ by Serre duality), and $\mathcal{A}' := \mathcal{A} \otimes \mathcal{L}_0$.
- For $I \neq I_m$: If X has no curves in certain homology classes (which we assume for simplicity!) $\mathcal{M}^{\rm st}_I$ is a $\mathbb{P}^{b-|I|-1}_{\mathbb{C}}$-fibration over a punctured disk, these fibrations are pairwise disjoint, and the closure $\bar{\mathcal{M}}^{\rm st}_I$ in $\mathcal{M}$ contains the circle $R_{\{I,\bar{I}\}}$.

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- $\mathcal{M}^{\mathrm{st}}$ is a smooth b-dimensional manifold and the local structure around a circle of reductions is known (topologically).
- The locus of filtrable stable bundles decomposes as

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- for $I\neq I_m$ the space $\mathcal{M}^{\mathrm{st}}_I$ is a $\mathbb{P}^{b-|I|-1}_{\mathbb{C}}$-fibration over a punctured disk. *The closure of $\mathcal{M}^{\mathrm{st}}_I$ contains the circle $R_{\{I,\bar{I}\}}$.*

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Any minimal class VII surface X with $b_2(X) \in \{1, 2, 3\}$ has a cycle.

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- For $b_2 \in \{1, 2\}$ the result is proven in previous articles. The proof for $b_2 = 3$: partially available on the archive. Trying to go directly to arbitrary b_2.
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- Strategy of the proof (in general): Use the following

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Proposition 3.2

If the canonical extension $\mathcal{A}$ can be written as an extension in a different way, then X has a cycle. In particular, if $\mathcal{A}$ belongs to $\mathcal{M}_I^{\mathrm{st}}$ for $I \neq I_m$ or coincides with a twisted reduction, then X has a cycle.

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Proposition 3.2

If the canonical extension $\mathcal{A}$ can be written as an extension in a different way, then X has a cycle. In particular, if $\mathcal{A}$ belongs to $\mathcal{M}_I^{\mathrm{st}}$ for $I \neq I_m$ or coincides with a twisted reduction, then X has a cycle.

Proof.

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{K}_X \arrow[r, "i"] & \mathcal{A} \arrow[r, "p"] & \mathcal{O}_X \arrow[r] & 0 \\ & & \mathcal{L} \arrow[u, "j"] \arrow[ur, "p \circ j"'] & & \end{tikzcd}$$

$p \circ j$ is non-zero (because $\mathcal{L}$ is a different kernel) and non-isomorphism, because the canonical extension is non-split. Therefore $\mathrm{im}(p \circ j) = \mathcal{O}_X(-D)$ where $D > 0$ is the vanishing divisor of $p \circ j$. Restrict the diagram to D taking into account that j is a bundle embedding. We get $\omega_D := \mathcal{K}_X(D)_D \simeq \mathcal{O}_D$, so D contains a cycle.

■

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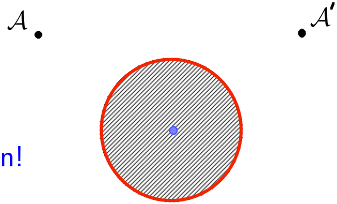

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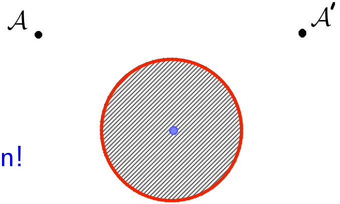

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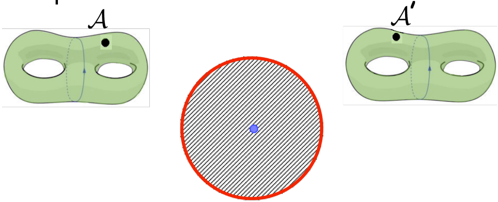

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Proposition 3.3

Suppose that X is a complex surface with $a(X) = 0$, E a differentiable rank 2 bundle over X, Y a closed Riemann surface and

$$f : Y \to \mathcal{M}^{\mathrm{simple}}(E)\ ,\ y \mapsto [\mathcal{E}_y]$$

a holomorphic map. Then the bundles $\mathcal{E}_y$ are either all filtrable or all non-filtrable.

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Proof.

(Idea) Change the roles, i.e. construct a family of holomorphic bundles on Y parameterized by X. Use the fact that Y is algebraic and $a(X)=0$. ∎

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- For $b_2 \geq 2$, $\dim(\mathcal{M}) = b_2$ and $\mathcal{M}$ contains 2^{b_2-1} circles of reductions.

Remark 3.4

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- We give the proof in the case $b_2 = 2$, in particular we show how the *canonical stratification* of $\mathcal{M}_0$ is obtained. The method generalizes to $b_2 = 3$ and (it seems) to arbitrary b_2.

17

Proof.

- For $b_2 = 2$ we have 2 circles of reductions $R_{\{\emptyset, I_m\}}$, $R_{\{\{1\},\{2\}\}}$. We prove that there does not exist any connected component Y of $\mathcal{M}$ containing exactly one circle of reductions R.

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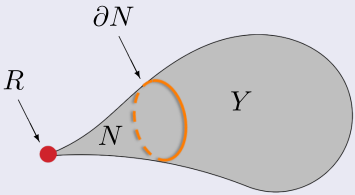

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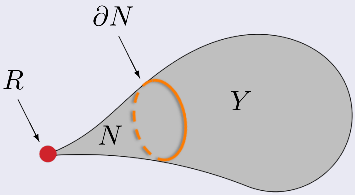

- Let N be standard compact neighborhood of R. The boundary ∂N is also the boundary of $Y \setminus \mathring{N} \subset \mathcal{B}_a^*(E)$ (the moduli space of irreducible connections with fixed determinant a), so ∂N would be homologically trivial in $\mathcal{B}_a^*(E)$.

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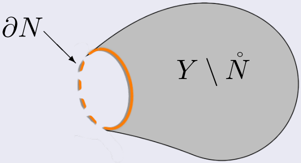

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- On the other hand the restriction to ∂N of a Donaldson class $\eta \in H^3(\mathcal{B}_a^*(E), \mathbb{Q})$ is nontrivial. Contradiction.

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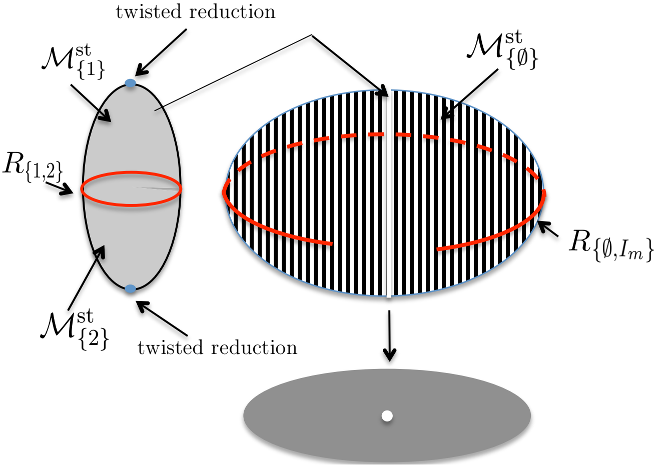

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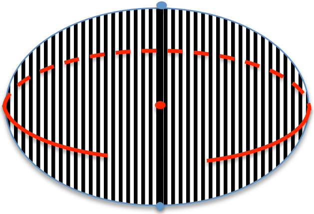

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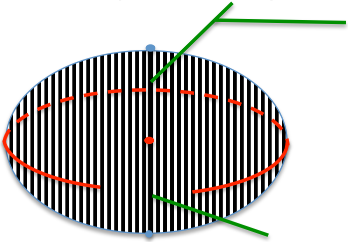

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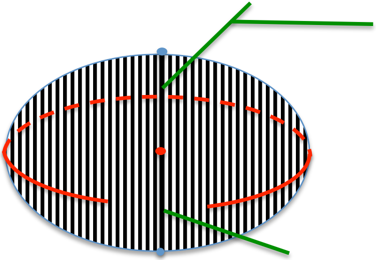

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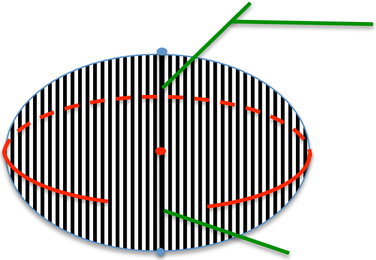

- The fiber over $0 \in D$ is a tree of rational curves: the known curve $\mathcal{M}^{\mathrm{st}}_{\{1\}} \cup \mathcal{M}^{\mathrm{st}}_{\{2\}} \cup R_{\{\{1\},\{2\}\}} \cup \{$two twisted reductions$\}$ and unknown ("green") curves, whose generic points must be non-filtrable.

21

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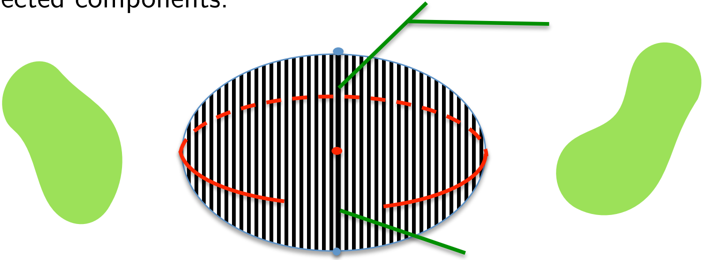

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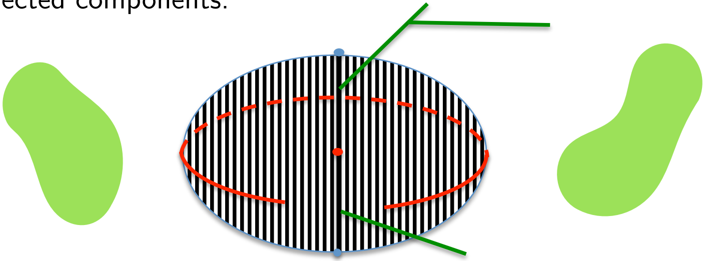

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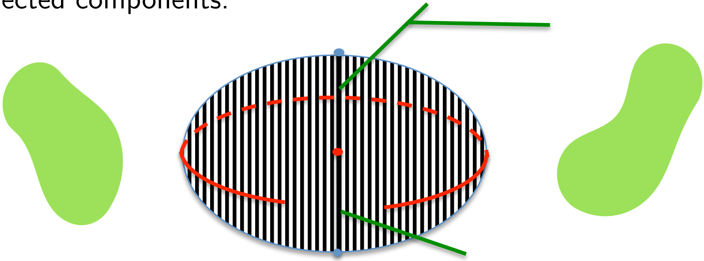

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- The natural question is: What is the position of $a := [\mathcal{A}]$?

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- We have again a **dichotomy**: Either the remarkable incidence holds (hence X has a cycle), or $a := [\mathcal{A}]$ belongs to a compact subspace of $\mathcal{M}^{\mathrm{st}}$ (of dimension 1 or 2) consisting generically of non-filtrable bundles.

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- The latter possibility is ruled out by the main result of a recent article ("*Compact subspaces of moduli* . . ." `arXiv:1309.0350`):

Theorem 3.5

There does not exist any positive dimensional compact subspace $Y \subset \mathcal{M}^{\mathrm{st}}$ containing the point a and an open neighborhood $a \in Y_a \subset Y$ such that $Y_a \setminus \{a\}$ consists only of non-filtrable bundles.

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- In other words, within any positive dimensional compact subspace $a \in Y \subset \mathcal{M}^{\mathrm{st}}$, the point a can be approached by filtrable bundles, *it cannot be surrounded only by non-filtrables.*

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- Apply our non-existence theorem, which rules out the second possibility.

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  Main difficulty: rule out the situation when $\mathcal{M}_0$ contains an unknown stratum Y (consisting generically of non-filtrable bundles) which *contains* a circle of reductions. For such a stratum the non-existence theorem does not apply.

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- The proof uses "reductio ad absurdum" and a cobordism argument.

bibliography

- G. Dloussky, K. Oeljeklaus, M. Toma: *Class* VII_0 *surfaces with* b_2 *curves*, Tohoku Math. J. (2) 55 no. 2, 283-309 (2003)
- I. Nakamura : *Towards classification of non-Kählerian surfaces*, Sugaku Expositions vol. 2, No 2 , 209-229 (1989)
- A. Teleman: *Donaldson theory on non-Kählerian surfaces and class VII surfaces with* $b_2 = 1$, Invent. math. 162, 493-521 (2005)
- A. Teleman: *Instantons and holomorphic curves on class VII surfaces*, Annals of Mathematics, 172 (2010), 1749-1804.
- A. Teleman: *A variation formula for the determinant line bundle. Compact subspaces of moduli spaces of stable bundles over class VII surfaces*, `arXiv:1309.0350`.