2 September 2014

Projective Normality of Toric weak Fano 3-folds

VBAC 2014

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Let $X$ be a nonsingular projective algebraic variety. $X$ is called Fano if its anti-canonical divisor $-K_X$ is ample. For a Fano variety $X$, the number

$$i_X := \{i \in \mathbb{N}; -K_X = iD \text{ for an ample divisor } D\}$$

is called the Fano index.

**Theorem (Kobayashi-Ochiai)**

1. A Fano $n$-fold with index $n + 1$ is $\mathbb{P}^n$.
2. A Fano $n$-fold with index $n$ is the hyperquadric in $\mathbb{P}^{n+1}$. 
A projective algebraic variety $X$ is called **Gorenstein** if its dualizing sheaf $\omega_X$ is invertible.

We can define a Gorenstein variety to be Fano in the same way.

**Theorem** Let $X$ be a projective toric $n$-fold.

1. If $X$ has an ample line bundle $L$ with $H^0(L^\otimes n \otimes \omega_X) = 0$, then $X \cong \mathbb{P}^n$.
2. If $X$ has an ample line bundle $L$ with $\dim H^0(L^\otimes n \otimes \omega_X) = 1$, then $X$ is Gorenstein toric Fano, and it coincides with one of $\mathbb{P}(1, 1, 2, \ldots, 2)$, which is a cone over the plane conic, and a cone over $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ in $\mathbb{P}^{n+1}$. 
Theorem (Kobayashi-Ochiai)
If a nonsingular projective variety of dimension $n$ has an ample line bundle $L$ satisfying the self-intersection number $(L^n) = 1$ and $\dim H^0(L) \geq n + 1$, then it is $\mathbb{P}^n$.

We also have analogous results in toric varieties. We note that for an ample line bundle $L$ on a toric $n$-fold, we have $\dim H^0(L) \geq n + 1$. We may assume $n \geq 2$.

Theorem Let $X$ be a projective toric $n$-fold with an ample line bundle $L$ with $\dim H^0(L) = n + 1$.
(1) If $H^0(L^{\otimes(n-1)} \otimes \omega_X) = 0$, then $X \cong \mathbb{P}^n$.
(2) If $\dim H^0(L^{\otimes(n-1)} \otimes \omega_X) = 1$, then $X$ is Gorenstein toric Fano.
2 Normal Generation

Let $X \subset \mathbb{P}^N$ be a projective algebraic variety. $X$ is called projectively normal if its homogeneous coordinate ring is normal.

Let $L$ be a very ample line bundle on $X$. When $\mathbb{P}^N = \mathbb{P}(\Gamma(X, L))$, $X$ is projectively normal if and only if $X$ is normal and if

$$\Gamma(\mathbb{P}^N, \mathcal{O}(k)) \rightarrow \Gamma(X, L^\otimes k)$$

is surjective for all integers $k \geq 1$. 
An ample line bundle $L$ is called normally generated if the multiplication map

$$\text{Sym}^k \Gamma(X, L) \to \Gamma(X, L^\otimes k)$$

(1)
is surjective for all integers $k \geq 1$.

If $L$ is normally generated, then it is very ample. The converse is not true.

We have examples of very ample but not normally generated line bundles on toric varieties of all dimensions $n \geq 3$. 
3 History of Normality in Toric Variety

- Koelman (1993) When $\dim X = 2$, an ample $L$ is normally generated.
- Ewald-Wessels (1991) When $\dim X = n$, $L^\otimes k$ is very ample for $k \geq n - 1$.
- Nakagawa (1994), Bruns-Gubeladze-Trung (1997) When $\dim X = n$, $L^\otimes k$ is normally generated for $k \geq n - 1$. Moreover, Nakagawa shows

$$\Gamma(L^\otimes k) \otimes \Gamma(L) \to \Gamma(L^\otimes k+1)$$

is surjective for $k \geq n - 1$. 
4 Very Ample, not Normally Generated

- Demazure(1970) An ample line bundle on a nonsingular toric variety is always very ample.
- Bruns-Gubeladze(2002) There exists a $(X, L)$ of dimension 5 such that $L$ is very ample but not normally generated.
- O(2011), Higashitani(2012) For all dimension $n \geq 3$, there are such examples.
5 Normality of Nonsingular Toric Variety

We have the three classes of nonsingular projective toric 3-folds so that an ample line bundle on a member is always normally generated.

- If $X$ has a surjective morphism onto $\mathbb{P}^1$, then all ample line bundles on $X$ are normally generated.
- (2014) Let $(X, A)$ be a nonsingular polarized toric 3-fold. (1) If $H^0(A + K_X) = 0$, then all ample line bundles on $X$ are normally generated. (2) Even if $H^0(A + K_X) \neq 0$ and if $A + K_X$ is not big, then all ample line bundles on $X$ are normally generated.
- (2014) Let $X$ be a toric weak Fano 3-fold. Then all ample line bundles on $X$ are normally generated.
6 Gorenstein toric Fano 3-fold

A nonsingular projective variety is called weak Fano if its anti-canonical divisor is nef and big.

For a weak Fano toric $n$-fold $X$, we have a Gorenstein toric Fano $n$-fold $Y$ and a surjective morphism $\pi : X \to Y$ such that $K_X = \pi^*K_Y$.

Since $-K_Y$ is ample, it corresponds to a lattice polytope $P$ of dimension 3.

By a classification of lattice polytopes, we see that there exist 4,319 Gorenstein toric Fano 3-folds.

Moreover, since $X$ is a crepant resolution of $Y$, from one $Y$ we have several $X$'s.
Example. Let $M = \mathbb{Z}^3$. Let

$$P := \text{Conv}\{0, (1, 0, 0), (0, 1, 0), (1, 1, 2)\}.$$ 

Then $P$ defines the toric 3-fold $Y = X(\Delta)$, where the fan $\Delta$ consists of four 3-simplicial cones and their faces. A typical cone is

$$\sigma = \mathbb{R}_{\geq 0}\{(2, 0, -1), (0, 2, -1), (0, 0, 1)\}.$$
Since $\sigma$ is singular, we have to make a nonsingular refinement. In the triangle $R(\sigma) := \text{Conv}\{(2, 0, -1), (0, 2, -1), (0, 0, 1)\}$, we have three lattice points $(1, 1, -1), (1, 0, 0)$ and $(0, 1, 0)$. We have 4 triangulations of this triangle $R(\sigma)$ with these lattice points as vertices.
the cone $\sigma$ and the cross section $R(\sigma)$
7 Main Results

(Key) Proposition Let $X$ be a weak Fano toric 3-fold. If a nef and big line bundle $L$ on $X$ satisfies that $h^0(L + K_X) \neq 0$ and that $h^0(L + 2K_X) = 0$, then $L$ is normally generated.

Theorem 1 Let $X$ be a weak Fano toric 3-fold. If a nef and big line bundle $L$ on $X$ satisfies that $h^0(L + K_X) \neq 0$, then $L$ is normally generated.
Lemma 1 Let $X$ be a nonsingular projective toric 3-fold and $L$ be a nef and big line bundle on $X$ with $h^0(L + K_X) \neq 0$. If $L + K_X$ is not nef, then its fixed part is a reduced divisor $\sum_i E_i$ with $(E_i, L|_{E_i}) \cong (\mathbb{P}^2, \mathcal{O}(1))$ and $\mathcal{O}_X(E_i)|_{E_i} \cong \mathcal{O}_{\mathbb{P}^2}(-1)$, and $L + K_X - \sum_i E_i$ is nef.

We assume that $h^0(L + rK_X) \neq 0$ and $h^0(L + (r + 1)K_X) = 0$ for $r \geq 2$. Let $F_1$ be the fixed part of $L + K_X$. Then $L_1 := L + K_X - F_1$ is nef and big and $h^0(L_1 + (r - 1)K_X) \neq 0$ and $h^0(L_1 + rK_X) = 0$. 

8 Theorem 1 from Key Proposition

We have a useful lemma.
If $X$ is weak Fano, since $-K_X$ is nef, then $L_0 := L - F_1 = L_1 + (-K_X)$ is also nef and big.

We have another useful lemma.

**Lemma 2** Let $S$ be a nonsingular toric surface. Let $A$ and $B$ be nef line bundles on $S$. Then the multiplication map

\[ \Gamma(A) \otimes \Gamma(A + B) \rightarrow \Gamma(2A + B) \]

is surjective.

Let $D := T \setminus T$ be the boundary divisor of $X$. Then $D \sim -K_X$. 
From an exact sequence

$$0 \rightarrow L_1 \rightarrow L_0 \rightarrow L_0|_D \rightarrow 0,$$

we have a commutative diagram

$$\begin{array}{cccccc}
\Gamma(L_1) \otimes \Gamma(L_1) & \rightarrow & \Gamma(L_1) \otimes \Gamma(L_0) & \rightarrow & \Gamma(L_1) \otimes \Gamma(L_0|_D) \\
\downarrow & & \downarrow & & \downarrow \\
\Gamma(2L_1) & \rightarrow & \Gamma(L_1 + L_0) & \rightarrow & \Gamma((L_1 + L_0)|_D)
\end{array}$$

The vertical arrow in the right is surjective from Lemma 2. If $L_1$ is normally generated, then the arrow in the middle is surjective.

We also have an exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow -K_X \rightarrow (-K_X)|_D \rightarrow 0.$$
From this we can see the subjectivity of

\[ \Gamma(-K_X) \otimes \Gamma(L_1 + L_0) \rightarrow \Gamma(2L_0). \]

From a commutative diagram

\[
\begin{array}{ccc}
\Gamma(-K_X) \otimes \Gamma(L_1) \otimes \Gamma(L_0) & \rightarrow & \Gamma(L_0) \otimes \Gamma(L_0) \\
\downarrow & & \downarrow \\
\Gamma(-K_X) \otimes (L_1 + L_0) & \rightarrow & \Gamma(2L_0),
\end{array}
\]

we see that the projective normality of \( L_1 \) induces that of \( L_0 \).

Next consider the exact sequence

\[ 0 \rightarrow L_0 \rightarrow L \rightarrow L|_{F_1} \rightarrow 0. \]
From Lemma 1, we have $F_1 = \sum_i E_i, E_i \cong \mathbb{P}^2$ and $L|_{E_i} \cong \mathcal{O}(1)$.

After taking global sections, by tensoring with $\Gamma(L_0)$ and $\Gamma(L)$, the projective normality of $L_0$ implies the subjectivity of the multiplication maps of $\Gamma(L_0) \otimes \Gamma(L) \to \Gamma(L_0 + L)$ and $\Gamma(L)^{\otimes 2} \to \Gamma(2L)$.

By induction on $r$, we have a proof of Theorem.
9 Applications

Theorem 2 Let $X$ be a toric weak Fano 3-fold. All ample line bundles on $X$ are normally generated.

Theorem 3 Let $Z$ be a nonsingular toric Fano 4-fold and $Y$ be a general member of the anti-canonical system $|-K_Z|$. Then all ample line bundles on $Y$ are very ample, moreover, normally generated.
As a direct corollary of Theorem 1, we have

**Theorem 4** Let $X$ be a Gorenstein toric Fano 3-fold. If an ample line bundle $L$ on $X$ satisfies that $h^0(L + K_X) \neq 0$, then $L$ is normally generated.

**Remark 3** We have an example of a polarized Gorenstein toric Fano 3-folds $(X, L)$ with not very ample $L$. The corresponding polytope is

$$\text{Conv}\{0, (1, 0, 0), (0, 1, 0), (1, 1, 2)\}.$$
10 References

11. O.-Zhao, A characterization of toric Fano varieties with higher index, arXiv.AG/ 1404.6870.