2 September 2014

Projective Normality of Toric weak Fano 3-folds

VBAC 2014

Tohoku University, Shoetsu Ogata

1 Fano Variety

Let X be a nonsingular projective algebraic variety.
X is called Fano if its anti-canonical divisor $-K_X$ is ample.
For a Fano variety X, the number

$$i_X := \{i \in \mathbb{N}; -K_X = iD \text{ for an ample divisor } D\}$$

is called the Fano index.

Theorem(Kobayashi-Ochiai)
(1) A Fano n-fold with index $n+1$ is $\mathbb{P}^n$.
(2) A Fano n-fold with index n is the hyperquadric in $\mathbb{P}^{n+1}$.

A projective algebraic variety X is called Gorenstein if its dualizing sheaf ω_X is invertible.

We can define a Gorenstein variety to be Fano in the same way.

Theorem Let X be a projective toric n-fold.
(1) If X has an ample line bundle L with $H^0(L^{\otimes n} \otimes \omega_X) = 0$, then $X \cong \mathbb{P}^n$.
(2) If X has an ample line bundle L with $\dim H^0(L^{\otimes n} \otimes \omega_X) = 1$, then X is Gorenstein toric Fano, and it coincides with one of $\mathbb{P}(1, 1, 2, \ldots, 2)$, which is a cone over the plane conic, and a cone over $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ in $\mathbb{P}^{n+1}$.

Theorem(Kobayashi-Ochiai)
If a nonsingular projective variety of dimension n has an ample line bundle L satisfying the self-intersection number $(L^n) = 1$ and $\dim H^0(L) \geq n+1$, then it is $\mathbb{P}^n$.

We also have analogous results in toric varieties. We note that for an ample line bundle L on a toric n-fold, we have $\dim H^0(L) \geq n+1$. We may assume $n \geq 2$.

Theorem Let X be a projective toric n-fold with an ample line bundle L with $\dim H^0(L) = n+1$.
(1) If $H^0(L^{\otimes(n-1)} \otimes \omega_X) = 0$, then $X \cong \mathbb{P}^n$.
(2) If $\dim H^0(L^{\otimes(n-1)} \otimes \omega_X) = 1$, then X is Gorenstein toric Fano.

2 Normal Generation

Let $X \subset \mathbb{P}^N$ be a projective algebraic variety.
X is called projectively normal if its homogeneous coordinate ring is normal.

Let L be a very ample line bundle on X.
When $\mathbb{P}^N = \mathbb{P}(\Gamma(X, L))$, X is projectively normal if and only if X is normal and if

$$\Gamma(\mathbb{P}^N, \mathcal{O}(k)) \to \Gamma(X, L^{\otimes k})$$

is surjective for all integers $k \geq 1$.

An ample line bundle L is called normally generated if the multiplication map

$$\mathrm{Sym}^k\Gamma(X,L) \to \Gamma(X,L^{\otimes k}) \tag{1}$$

is surjective for all integers $k \geq 1$.

If L is normally generated, then it is very ample.
The converse is not true.

We have examples of very ample but not normally generated line bundles on toric varieties of all dimensions $n \geq 3$.

3 History of Normality in Toric Variety

- Koelman(1993)
  When $\dim X = 2$, an ample L is normally generated.
- Ewald-Wessels(1991) When $\dim X = n$, $L^{\otimes k}$ is very ample for $k \geq n - 1$.
- Nakagawa(1994), Bruns-Gubeladze-Trung(1997) When $\dim X = n$, $L^{\otimes k}$ is normally generated for $k \geq n - 1$.
  Moreover, Nakagawa shows

$$\Gamma(L^{\otimes k}) \otimes \Gamma(L) \to \Gamma(L^{\otimes k+1})$$

is surjective for $k \geq n - 1$.

4 Very Ample, not Normally Generated

- Demazure(1970) An ample line bundle on a nonsingular toric variety is always very ample.
- Bruns-Gubeladze(2002) There exists a (X, L) of dimension 5 such that L is very ample but not normally generated .
- Bruns-Gubeladze(2009) An example of such (X, L) in dimension 3.
- O(2011), Higashitani(2012) For all dimension $n \geq 3$, there are such examples.

5 Normality of Nonsingular Toric Variety

We have the three classes of nonsingular projective toric 3-folds so that an ample line bundle on a member is always normally generated.

- If X has a surjective morphism onto $\mathbb{P}^1$, then all ample line bundles on X are normally generated.
- (2014) Let (X, A) be a nonsingular polarized toric 3-fold.
  (1) If $H^0(A + K_X) = 0$, then all ample line bundles on X are normally generated.
  (2) Even if $H^0(A + K_X) \neq 0$ and if $A + K_X$ is not big, then all ample line bundles on X are normally generated.
- (2014) Let X be a toric weak Fano 3-fold. Then all ample line bundles on X are normally generated.

6 Gorenstein toric Fano 3-fold

A nonsingular projective variety is called weak Fano if its anti-canonical divisor is nef and big.

For a weak Fano toric n-fold X, we have a Gorenstein toric Fano n-fold Y and a surjective morphism $\pi : X \to Y$ such that $K_X = \pi^* K_Y$.

Since $-K_Y$ is ample, it corresponds to a lattice polytope P of dimension 3.
By a classification of lattice polytopes, we see that
there exist 4,319 Gorenstein toric Fano 3-folds.
Moreover, since X is a crepant resolution of Y, from one Y we have several X's.

Example. Let $M = \mathbb{Z}^3$. Let
$P :=$ Conv$\{0, (1,0,0), (0,1,0), (1,1,2)\}$.
Then P defines the toric 3-fold $Y = X(\Delta)$, where the fan Δ consists of four 3-simplicial cones and their faces.
A typical cone is

$$\sigma = \mathbb{R}_{\geq 0}\{(2,0,-1), (0,2,-1), (0,0,1)\}.$$

Since σ is singular, we have to make a nonsingular refinement. In the triangle $R(\sigma) :=$ Conv$\{(2, 0, -1), (0, 2, -1), (0, 0, 1)\}$, we have three lattice points $(1, 1, -1), (1, 0, 0)$ and $(0, 1, 0)$. We have 4 triangulations of this triangle $R(\sigma)$ with these lattice points as vertices.

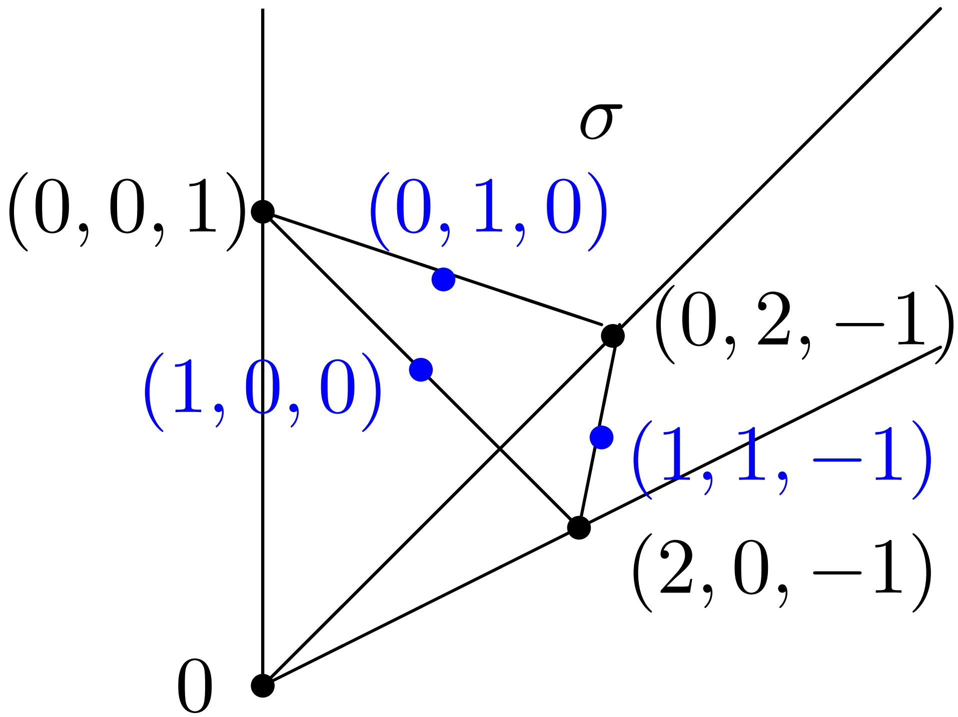

図1　the cone σ and the cross section $R(\sigma)$

7 Main Results

(Key) Proposition Let X be a weak Fano toric 3-fold. If a nef and big line bundle L on X satisfies that $h^0(L+K_X) \neq 0$ and that $h^0(L+2K_X)=0$, then L is normally generated.

Theorem 1 Let X be a weak Fano toric 3-fold. If a nef and big line bundle L on X satisfies that $h^0(L+K_X) \neq 0$, then L is normally generated.

8 Theorem 1 from Key Proposition

We have a useful lemma.

Lemma 1 Let X be a nonsingular projective toric 3-fold and L be a nef and big line bundle on X with $h^0(L + K_X) \neq 0$. If $L + K_X$ is not nef, then its fixed part is a reduced divisor $\sum_i E_i$ with $(E_i, L|_{E_i}) \cong (\mathbb{P}^2, \mathcal{O}(1))$ and $\mathcal{O}_X(E_i)|_{E_i} \cong \mathcal{O}_{\mathbb{P}^2}(-1)$, and $L + K_X - \sum_i E_i$ is nef.

We assume that $h^0(L + rK_X) \neq 0$ and $h^0(L + (r+1)K_X) = 0$ for $r \geq 2$.
Let F_1 be the fixed part of $L + K_X$. Then $L_1 := L + K_X - F_1$ is nef and big and $h^0(L_1 + (r-1)K_X) \neq 0$ and $h^0(L_1 + rK_X) = 0$.

If X is weak Fano, since $-K_X$ is nef, then $L_0 := L - F_1 = L_1 + (-K_X)$ is also nef and big.

We have another useful lemma.

Lemma 2 Let S be a nonsingular toric surface. Let A and B be nef line bundles on S. Then the multiplication map

$$\Gamma(A) \otimes \Gamma(A + B) \to \Gamma(2A + B)$$

is surjective.

Let $D := T \setminus T$ be the boundary divisor of X.
Then $D \sim -K_X$.

From an exact sequence

$$0 \to L_1 \to L_0 \to L_0|_D \to 0,$$

we have a commutative diagram

$$\begin{array}{ccccc}
\Gamma(L_1) \otimes \Gamma(L_1) & \to & \Gamma(L_1) \otimes \Gamma(L_0) & \to & \Gamma(L_1) \otimes \Gamma(L_0|_D) \\
\downarrow & & \downarrow & & \downarrow \\
\Gamma(2L_1) & \to & \Gamma(L_1 + L_0) & \to & \Gamma((L_1 + L_0)|_D)
\end{array}$$

The vertical arrow in the right is surjective from Lemma 2.
If L_1 is normally generated, then the arrow in the middle is surjective.
We also have an exact sequence

$$0 \to \mathcal{O}_X \to -K_X \to (-K_X)|_D \to 0.$$

From this we can see the subjectivity of

$$\Gamma(-K_X) \otimes \Gamma(L_1 + L_0) \to \Gamma(2L_0).$$

From a commutative diagram

$$\begin{array}{ccc} \Gamma(-K_X) \otimes \Gamma(L_1) \otimes \Gamma(L_0) & \to & \Gamma(L_0) \otimes \Gamma(L_0) \\ \downarrow & & \downarrow \\ \Gamma(-K_X) \otimes (L_1 + L_0) & \to & \Gamma(2L_0), \end{array}$$

we see that the projective normality of L_1 induces that of L_0.

Next consider the exact sequence

$$0 \to L_0 \to L \to L|_{F_1} \to 0.$$

From Lemma 1, we have $F_1 = \sum_i E_i, E_i \cong \mathbb{P}^2$ and $L|_{E_i} \cong \mathcal{O}(1)$.

After taking global sections, by tensoring with $\Gamma(L_0)$ and $\Gamma(L)$, the projective normality of L_0 implies the subjectivity of the multiplication maps of $\Gamma(L_0) \otimes \Gamma(L) \to \Gamma(L_0 + L)$ and $\Gamma(L)^{\otimes 2} \to \Gamma(2L)$.

By induction on r, we have a proof of Theorem.

9 Applications

Theorem 2 Let X be a toric weak Fano 3-fold. All ample line bundles on X are normally generated.

Theorem 3 Let Z be a nonsingular toric Fano 4-fold and Y be a general member of the anti-canonical system $|-K_Z|$. Then all ample line bundles on Y are very ample, moreover, normally generated.

As a direct corollary of Theorem 1, we have

Theorem 4 Let X be a Gorenstein toric Fano 3-fold. If an ample line bundle L on X satisfies that $h^0(L + K_X) \neq 0$, then L is normally generated.

Remark 3 We have an example of a polarized Gorenstein toric Fano 3-folds (X, L) with not very ample L. The corresponding polytope is

$$\mathrm{Conv}\{0, (1,0,0), (0,1,0), (1,1,2)\}.$$

10 References

1. Ewald-Wessels, On the ampleness of line bundles in complete toric varieties, 1991.
2. Bruns et al., Semigroup algebras and discrete geometry, 2002.
3. Bruns et al., Polytopes, Rings, and K-Theory, Springer, 2009.
4. Kobayashi-Ochiai, Characterizations of complex projective spaces and hyper quadrics, 1973.
5. Koelman, Generators for the ideal of a projectively embedded toric surface, 1993.
6. Nakagawa, Generators for the ideal of a projectively embedded toric varieties, 1994.

7. O, Projective normality of toric 3-folds with non-big adjoint hyperplane sections, II, 2014.
8. O, Very ample but not normal lattice polytopes, 2013.
9. O, Nef and big divisors on toric weak Fano 3-folds, 2014.
10. O, Ample line bundles on certain toric fibered 3-folds, arXiv.AG/1104.5573.
11. O.-Zhao, A characterization of toric Fano varieties with higher index, arXiv.AG/ 1404.6870.