The homology of the moduli spaces of plane sheaves of multiplicity 4 and 5

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My set-up

$\mathcal{F}$ will be a coherent semi-stable sheaf on $\mathbb{P}^n(\mathbb{C})$ with support an algebraic curve.

The Hilbert polynomial of $\mathcal{F}$ is $P_{\mathcal{F}}(m) = rm + \chi$

$p(\mathcal{F}) = \chi/r$ is *the slope* of $\mathcal{F}$

Gieseker semi-stability (stability) means:

1. $\mathcal{F}$ is pure, i.e. there are no proper subsheaves with support of dimension zero;
2. for any proper subsheaf $\mathcal{F}' \subset \mathcal{F}$ we have $p(\mathcal{F}') \leq (<) p(\mathcal{F})$.

Fix integers $r > 0$ and χ. Denote $\mathrm{M}_{\mathbb{P}^n}(r,\chi) = \mathrm{M}_{\mathbb{P}^n}(rm+\chi)$

Motivation for studying $\mathrm{M}_{\mathbb{P}^2}(r,1)$. Let X be a Calabi-Yau threefold. Fix $\beta \in \mathrm{H}_2(X,\mathbb{Z})$. Let $\mathrm{M}_X(\beta)$ be the moduli space of semi-stable sheaves $\mathcal{F}$ on X supported on a curve of class β and such that $\chi(\mathcal{F}) = 1$. S. Katz defined the genus-zero BPS invariant $n_\beta(X) = \deg[\mathrm{M}_X(\beta)]^{vir}$. J. Choi noted that when X is the local $\mathbb{P}^2$, that is the total space of $\mathcal{O}_{\mathbb{P}^2}(-3)$, then $n_r(X) = (-1)^{r^2+1}\chi_{top}(\mathrm{M}_{\mathbb{P}^2}(r,1))$

If $\gcd(r,\chi)=1$, then $\mathrm{M}_{\mathbb{P}^n}(r,\chi)=\mathrm{M}^s_{\mathbb{P}^n}(r,\chi)$ is a fine moduli space. The map $[\mathcal{F}]\mapsto[\mathcal{F}\otimes\mathcal{O}(1)]$ gives an isomorphism $\mathrm{M}_{\mathbb{P}^n}(r,\chi)\simeq\mathrm{M}_{\mathbb{P}^n}(r,r+\chi)$, hence we may assume $\chi=1,\ldots,r$.

Theorem (J. Le Potier)

The moduli space $\mathrm{M}_{\mathbb{P}^2}(r,\chi)$ is an irreducible projective variety of dimension r^2+1, locally factorial, and smooth at all points given by stable sheaves.

I have classified the sheaves giving points in $\mathrm{M}_{\mathbb{P}^2}(r,\chi)$ in the following cases: $r=4$ (with J.-M. Drézet, 2010), $r=5$ (2011), and $r=6$ (2013).

This makes possible the investigation of the geometry of these moduli spaces. I have computed the Hodge numbers for the following: $\mathrm{M}_{\mathbb{P}^2}(4,1)$ and $\mathrm{M}_{\mathbb{P}^2}(4,3)$ (2014, joint work with J. Choi), $\mathrm{M}_{\mathbb{P}^2}(5,1)$ (2013), and $\mathrm{M}_{\mathbb{P}^2}(5,3)$.

Duality

Theorem

The map $[\mathcal{F}] \mapsto [\mathcal{E}xt^{n-1}(\mathcal{F}, \omega_{\mathbb{P}^n})]$ *gives an isomorphism* $\mathrm{M}_{\mathbb{P}^n}(r,\chi) \simeq \mathrm{M}_{\mathbb{P}^n}(r,-\chi)$.

M. Woolf has computed the nef cones of $\mathrm{M}_{\mathbb{P}^2}(r,\chi)$ and, as a consequence, has shown that $\mathrm{M}_{\mathbb{P}^2}(r,\chi_1)$ is not isomorphic to $\mathrm{M}_{\mathbb{P}^2}(r,\chi_2)$ if $\chi_1 \neq \pm\chi_2$ mod r.

Summary for $M_{\mathbb{P}^2}(4,1)$

stratum	cohomological conditions	set W_i of morphisms φ	
M_0	$h^0(\mathcal{F}(-1)) = 0$ $h^1(\mathcal{F}) = 0$ $h^0(\mathcal{F} \otimes \Omega^1(1)) = 0$	$0 \to 3\mathcal{O}(-2) \xrightarrow{\varphi} 2\mathcal{O}(-1) \oplus \mathcal{O} \to \mathcal{F} \to 0$ φ_{11} has linearly independent maximal minors	0
M_1	$h^0(\mathcal{F}(-1)) = 0$ $h^1(\mathcal{F}) = 1$ $h^0(\mathcal{F} \otimes \Omega^1(1)) = 1$	$0 \to \mathcal{O}(-3) \oplus \mathcal{O}(-1) \xrightarrow{\varphi} 2\mathcal{O} \to \mathcal{F} \to 0$ φ_{12} has linearly independent entries	2

Geometric quotients: $M_0 = W_0/G_0$, $M_1 = W_1/G_1$

$\zeta_1, \zeta_2, \zeta_3$ the maximal minors of φ_{11}

$U \subset M_0$ the open subset given by: $\zeta_1, \zeta_2, \zeta_3$ have no common factor

This condition is equivalent to: $(\zeta_1, \zeta_2, \zeta_3) = I_Z$ for a zero dimensional subscheme $Z \subset \mathbb{P}^2$ of length 3 that is not contained in a line. Denote $\mathrm{Hilb}^0_{\mathbb{P}^2}(3) \subset \mathrm{Hilb}_{\mathbb{P}^2}(3)$ the corresponding open subset in the Hilbert scheme.

The sheaves in U are precisely the non-split extensions

$$0 \to \mathcal{O}_Q \to \mathcal{F} \to \mathcal{O}_Z \to 0$$

where $Q = \{\det(\varphi) = 0\}$. Thus U is a fiber bundle over $\mathrm{Hilb}^0_{\mathbb{P}^2}(3)$. The fiber over Z is the set of quartics passing through Z.

The sheaves in $M_0 \setminus U$ are precisely the extension sheaves

$$0 \to \mathcal{O}_C \to \mathcal{F} \to \mathcal{O}_L \to 0$$

satisfying $\mathrm{H}^1(\mathcal{F}) = 0$. Here C is a cubic, L is the line $\{l = 0\}$, where $l = \gcd(\zeta_1, \zeta_2, \zeta_3)$. Denote by $M_{L,C} \subset \mathrm{M}_{\mathbb{P}^2}(4,1)$ the subset of such extensions.

The sheaves in M_1 are the twisted ideal sheaves $\mathcal{O}_Q(-P) \otimes \mathcal{O}(1)$ of points P on quartic curves Q. Thus M_1 is isomorphic to the universal quartic.

Torus action on $\mathrm{M}_{\mathbb{P}^2}(4,1)$

Torus $T = (\mathbb{C}^*)^3/\{(c,c,c),\ c \in \mathbb{C}^*\}$

T acts on $\mathbb{P}^2 = \mathbb{P}(V)$ via $(t_0, t_1, t_2)\cdot(x_0, x_1, x_2) = (t_0^{-1}x_0, t_1^{-1}x_1, t_2^{-1}x_2)$

$\mu_t \colon \mathbb{P}^2 \to \mathbb{P}^2$ map of multiplication by $t \in T$

Induced action on $\mathrm{M}_{\mathbb{P}^2}(4,1)$ given by $t[\mathcal{F}] = [\mu_{t^{-1}}^* \mathcal{F}]$

$\{X, Y, Z\}$ basis of V^*

Induced action of T on the symmetric algebra of V^* given by
$t\, X^i Y^j Z^k = t_0^i t_1^j t_2^k X^i Y^j Z^k$

T-fixed points in $\mathrm{M}_{\mathbb{P}^2}(4,1)$

$p_0 = (1,0,0)$, $p_1 = (0,1,0)$, $p_2 = (0,0,1)$ fixed points in $\mathbb{P}^2$
p_{ij} = double point supported on p_i and contained in the line $p_i p_j$
q_i = fixed triple point supported on p_i

There are 10 fixed zero dimensional schemes Z of length 3, Z not contained in a line:
$\{p_0, p_1, p_2\}$,
$\{p_0, p_{12}\}$, $\{p_0, p_{21}\}$, $\{p_1, p_{02}\}$, $\{p_1, p_{20}\}$, $\{p_2, p_{01}\}$, $\{p_2, p_{10}\}$,
$\{q_0\}$, $\{q_1\}$, $\{q_2\}$

$[\mathcal{F}] = [\mathcal{O}_Q(Z)]$ is T-fixed precisely if Q is T-invariant and Z is T-fixed

For each Z there are 12 invariant quartics containing Z

Thus we get 120 isolated fixed points in U

Example: if $Z = \{p_0, p_1, p_2\}$, then $I(Z) = (XY, XZ, YZ)$ and $[\mathcal{F}]$ is represented by the matrices

$$\begin{bmatrix} Y & 0 & X \\ 0 & Z & X \\ X^iY^jZ^k & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} Y & 0 & X \\ 0 & Z & X \\ 0 & X^iY^j & 0 \end{bmatrix}, \quad \begin{bmatrix} Y & 0 & X \\ 0 & Z & X \\ 0 & 0 & Y^jZ^k \end{bmatrix}$$

Likewise, we get 42 fixed points in M_1 of the form $[\mathcal{O}_Q(-P)(1)]$, where P is one of the p_i and Q is a T-invariant quartic passing through P.

Example: if $P = p_2$, then $\mathcal{F}$ is represented by the matrices

$$\begin{bmatrix} X^iY^jZ^k & X \\ 0 & Y \end{bmatrix}, \quad \begin{bmatrix} 0 & X \\ X^iZ^k & Y \end{bmatrix}$$

$M_{L,C}$ contains fixed points only if L and C are T-invariant. We claim that in this case $M_{L,C} \simeq \mathbb{A}^2$. For instance, if $L = \{X = 0\}$ and $C = \{X^i Y^j Z^k = 0\}$, then $M_{L,C}$ has parametrisation

$$\begin{bmatrix} X & 0 & -Y \\ 0 & X & -Z \\ q_1 - (bY + cZ)Z & q_2 + (bY + cZ)Y & q_3 \end{bmatrix}, \quad a, b \in \mathbb{C}$$

where q_1, q_2, q_3 are fixed quadratic forms satisfying $q_1 Y + q_2 Z + q_3 X = X^i Y^j Z^k$.

Action on $M_{L,C}$ given by $t(a,b) = (t_0^{-i} t_1^{2-j} t_2^{1-k} a,\ t_0^{-i} t_1^{1-j} t_2^{2-k} b)$

We get an isolated fixed point $(0,0)$ unless $(i,j,k) = (0,2,1)$ or $(0,1,2)$, in which case we get an affine line of fixed points $(a,0)$, $a \in \mathbb{C}$, resp. $(0,b)$, $b \in \mathbb{C}$.

We get 24 isolated fixed points and 3 affine lines of fixed points in $M_0 \setminus U$

Topology of $M_{\mathbb{P}^2}(4,1)$

Theorem (J. Choi, M. Maican)

The fixed point locus of $M_{\mathbb{P}^2}(4,1)$ *consists of* 180 *isolated points and* 6 *one-dimensional components isomorphic to* $\mathbb{P}^1$*. Furthermore, the integral homology of* $M_{\mathbb{P}^2}(4,1)$ *has no torsion and its Poincaré polynomial is*

$$\begin{aligned} P_{M_{\mathbb{P}^2}(4,1)}(x) = \quad & 1 + 2x^2 + 6x^4 + 10x^6 + 14x^8 + 15x^{10} + \\ & 16x^{12} + 16x^{14} + 16x^{16} + 16x^{18} + 16x^{20} + 16x^{22} + \\ & 15x^{24} + 14x^{26} + 10x^{28} + 6x^{30} + 2x^{32} + x^{34}. \end{aligned}$$

The Hodge numbers h^{pq} *are zero if* $p \neq q$*. The Picard group is* $\mathbb{Z}^2$*. The fundamental group* π_1 *is trivial. Moreover,* $M_{\mathbb{P}^2}(4,1)$ *is rational.*

Białynicki-Birula theory

X = smooth projective variety with a $\mathbb{C}^*$-action. Let $X_1, \ldots, X_r$ be the irreducible components of the fixed locus. They are smooth.

For each i we have a decomposition of the restricted tangent bundle

$$\mathrm{T}_{X|X_i} = \mathrm{T}_i^+ \oplus \mathrm{T}_i^0 \oplus \mathrm{T}_i^-$$

into subbundles on which $\mathbb{C}^*$ acts with positive, zero, negative weights. Denote $p(i) = \mathrm{rank}(\mathrm{T}_i^+)$ and $n(i) = \mathrm{rank}(\mathrm{T}_i^-)$.

Theorem (Homology basis formula)

For any integer m with $0 \leq m \leq 2\dim(X)$, we have a decomposition

$$\mathrm{H}_m(X, \mathbb{Z}) \simeq \bigoplus_{1 \leq i \leq r} \mathrm{H}_{m-2p(i)}(X_i, \mathbb{Z}) \simeq \bigoplus_{1 \leq i \leq r} \mathrm{H}_{m-2n(i)}(X_i, \mathbb{Z}).$$

Reason: we have a decomposition of X into *plus cells*

$$X = X_1^+ \cup \ldots \cup X_r^+, \quad X_i^+ = \{x \in X \mid \lim_{t \to 0} tx \in X_i\}$$

and a decomposition into *minus cells*

$$X = X_1^- \cup \ldots \cup X_r^-, \quad X_i^- = \{x \in X \mid \lim_{t \to \infty} tx \in X_i\}$$

X_i^+ and X_i^- are topological fiber bundles over X_i with fiber $\mathbb{C}^{p(i)}$ and $\mathbb{C}^{n(i)}$

Thus, the Poincaré polynomial of X is

$$P_X(x) = \sum_{i=1}^{r} P_{X_i}(x) x^{2p(i)} = \sum_{i=1}^{r} P_{X_i}(x) x^{2n(i)}$$

The Homology Basis Formula respects the Hodge decomposition:

$$\mathrm{H}^p(X,\Omega^q) = \bigoplus_{1\le i\le r} \mathrm{H}^{p-p(i)}(X_i,\Omega^{q-p(i)}_{X_i}) = \bigoplus_{1\le i\le r} \mathrm{H}^{p-n(i)}(X_i,\Omega^{q-n(i)}_{X_i})$$

If $\mathrm{h}^{pq}(X_i)=0$ for all $1\le i\le r$ and $p\ne q$, then $\mathrm{h}^{pq}(X)=0$ for $p\ne q$.

The $\mathbb{C}^*$-action on $\mathrm{M}_{\mathbb{P}^2}(4,1)$ will be induced by a one-parameter subgroup $\lambda\colon \mathbb{C}^*\to T$. We will choose λ such that the sets of fixed points coincide: $\mathrm{M}_{\mathbb{P}^2}(4,1)^\lambda = \mathrm{M}_{\mathbb{P}^2}(4,1)^T$. This is equivalent to

$$\langle \chi,\lambda\rangle \ne 0$$

for all $\chi\in\chi^*(T)$ that appear in the weight-decomposition of T_i^+ and T_i^- for all $1\le i\le r$.

Action of T on the tangent spaces at the fixed points

$[\mathcal{F}] \in M_0$ is T-fixed, $\mathcal{F} = \mathcal{C}oker(\varphi)$, $\varphi \in W_0$.
Assume that there are morphisms of groups

$$u\colon (\mathbb{C}^*)^3 \to \mathrm{Aut}(3\mathcal{O}(-2)), \quad v\colon (\mathbb{C}^*)^3 \to \mathrm{Aut}(2\mathcal{O}(-1) \oplus \mathcal{O})$$

such that $t\varphi = v(t)\varphi u(t)$ for all $t \in (\mathbb{C}^*)^3$. Let $\rho\colon W_0 \to M_0$ be the quotient map, $W = \mathrm{T}_\varphi W_0 = \mathrm{Hom}(3\mathcal{O}(-2), 2\mathcal{O}(-1) \oplus \mathcal{O})$, $\theta\colon W \to W$ the map $\psi \mapsto v(t)\psi u(t)$. Then:

$$\begin{aligned}
d(\mu_t)_{[\mathcal{F}]}(d\rho_\varphi(w)) &= d\rho_{(t\varphi)}(d(\mu_t)_\varphi(w)) && \text{because } \mu_t \circ \rho = \rho \circ \mu_t \\
&= d\rho_{(\theta\varphi)}(tw) && \text{because } \mu_t\colon W \to W \text{ is linear} \\
&= d\rho_\varphi(d(\theta^{-1})_{(\theta\varphi)}(tw)) && \text{because } \rho \circ \theta = \rho \\
&= d\rho_\varphi(\theta^{-1}(tw)) && \text{because } \theta^{-1}\colon W \to W \text{ is linear}
\end{aligned}$$

T-action on W given by $t \star w = v(t)^{-1}(tw)u(t)^{-1}$

T-action on $\mathrm{T}_{[\varphi]}\,\mathrm{M}_{\mathbb{P}^2}(4,1) = W/\,\mathrm{T}_{\varphi}(G_0\varphi)$ given by $t[w] = [t \star w]$

$G_0 \to G_0\varphi$, $(g,h) \mapsto h\varphi g^{-1}$ is an isomorphism because $\mathrm{Stab}_{G_0}(\varphi) = \{e\}$

Differential of this map at e is $(A,B) \mapsto B\varphi - \varphi A$

$\star$ induces the action $t(A,B) = (u(t)(tA)u(t)^{-1}, v(t)^{-1}(tB)v(t))$

For all fixed $\mathcal{F}$ we can choose φ such that u, v exist and are diagonal:

$$u(t) = \begin{bmatrix} u_1 & 0 & 0 \\ 0 & u_2 & 0 \\ 0 & 0 & u_3 \end{bmatrix}, \quad v(t) = \begin{bmatrix} v_1 & 0 & 0 \\ 0 & v_2 & 0 \\ 0 & 0 & v_3 \end{bmatrix}$$

$\{x, y, z\}$ standard basis for $\chi^*((\mathbb{C}^*)^3)$

$s^l = \{ix + jy + kz,\ i, j, k \in \mathbb{Z},\ i + j + k = l,\ i, j, k \geq 0\}$

Tables of weights for the T-action on W, resp. on $\mathrm{T}_\varphi(G_0\varphi)$

$$\begin{array}{lll} -v_1 - u_1 + s^1 & -v_1 - u_2 + s^1 & -v_1 - u_3 + s^1 \\ -v_2 - u_1 + s^1 & -v_2 - u_2 + s^1 & -v_2 - u_3 + s^1 \\ -v_3 - u_1 + s^2 & -v_3 - u_2 + s^2 & -v_3 - u_3 + s^2 \end{array}$$

$$\begin{array}{lllll} 0 & u_1 - u_2 & u_1 - u_3 & 0 & -v_1 + v_2 \\ u_2 - u_1 & 0 & u_2 - u_3 & -v_2 + v_1 & 0 \\ u_3 - u_1 & u_3 - u_2 & 0 & -v_3 + v_1 + s^1 & -v_3 + v_2 + s^1 \end{array}$$

Example

$$t\begin{bmatrix} Y & 0 & X \\ 0 & Z & X \\ X^iY^jZ^k & 0 & 0 \end{bmatrix} = \begin{bmatrix} t_1Y & 0 & t_0X \\ 0 & t_2Z & t_0X \\ t_0^it_1^jt_2^kX^iY^jZ^k & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} t_0 & 0 & 0 \\ 0 & t_0 & 0 \\ 0 & 0 & t_0^{i+1}t_1^{j-1}t_2^k \end{bmatrix} \varphi \begin{bmatrix} t_0^{-1}t_1 & 0 & 0 \\ 0 & t_0^{-1}t_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$v_1 = v_2 = x,\ v_3 = (i+1)x + (j-1)y + kz,\ u_1 = y - x,\ u_2 = z - x,\ u_3 = 0$

Subtracting the second list of weights from the first list we get the list

$$\{-(1+i)x - jy - (1+k)z + \{3x+y,\ 2x+2y,\ x+3y,\ 3x+z,\ 2x+2z,\ x+3z,$$
$$3y+z,\ 2y+2z,\ y+3z,\ 2x+y+z,\ x+2y+z,\ x+y+2z\},$$
$$x-y,\ x-z,\ y-x,\ y-z,\ z-x,\ z-y\} \setminus \{0\}$$

$[\mathcal{F}] \in M_1$ is T-fixed, $\mathcal{F} = \mathcal{C}oker(\varphi)$, $\varphi \in W_1$.
We can choose φ such that there are diagonal morphisms of groups

$$u\colon (\mathbb{C}^*)^3 \to \mathrm{Aut}(\mathcal{O}(-3) \oplus \mathcal{O}(-1)), \quad v\colon (\mathbb{C}^*)^3 \to \mathrm{Aut}(2\mathcal{O})$$

with $t\varphi = v(t)\varphi u(t)$ for all t. Weights for the action of T on $\mathrm{T}_\varphi W_1$

$$\begin{array}{ll} -v_1 - u_1 + s^3 & -v_1 - u_2 + s^1 \\ -v_2 - u_1 + s^3 & -v_2 - u_2 + s^1 \end{array}$$

Weights for the action of T on $\mathrm{T}_\varphi(G_1\varphi)$

$$\begin{array}{lll} 0 & -v_1 + v_2 & 0 \\ -v_2 + v_1 & 0 & u_2 - u_1 + s^2 \end{array}$$

The normal space to M_1 at φ is isomorphic to $\mathrm{H}^0(\mathcal{F})^* \otimes \mathrm{H}^1(\mathcal{F})$ and has weights $u_1 + v_1 - x - y - z$, $u_1 + v_2 - x - y - z$.

We can choose $\lambda\colon \mathbb{C}^* \to T$, $\lambda(c) = (1, c, c^5)$

$p(i)$ = number of characters χ appearing in the weight decomposition of $\mathrm{T}_i^+ \oplus \mathrm{T}_i^-$ such that $\langle \chi, \lambda \rangle > 0$

$$P_{M_{\mathbb{P}^2}(4,1)}(x) = \sum_{X_i = \text{point}} x^{2p(i)} + \sum_{X_i = \text{line}} (1 + x^2) x^{2p(i)}$$

It turns out that the source S, i.e. the X_i for which $n(i) = 0$, is a point, hence the moduli space contains an open subset isomorphic to an affine space, so it is rational. Also, $\pi_1 \simeq \pi_1(S)$, so $\pi_1 = \{1\}$. We have the exact sequence

$$0 \to \mathbb{Z}^{b_2} \to \mathrm{Pic}(M_{\mathbb{P}^2}(4,1)) \to \mathrm{Pic}(S) \to 0$$

hence $\mathrm{Pic}(M_{\mathbb{P}^2}(4,1)) \simeq \mathbb{Z}^2$.

Summary for $M_{\mathbb{P}^2}(5,1)$

stratum	cohomological conditions	subset W_i of morphisms φ	
M_0	$h^0(\mathcal{F}(-1)) = 0$ $h^1(\mathcal{F}) = 0$ $h^0(\mathcal{F} \otimes \Omega^1(1)) = 0$	$4\mathcal{O}(-2) \xrightarrow{\varphi} 3\mathcal{O}(-1) \oplus \mathcal{O}$ φ is injective φ_{11} is semi-stable	0
M_1	$h^0(\mathcal{F}(-1)) = 0$ $h^1(\mathcal{F}) = 1$ $h^0(\mathcal{F} \otimes \Omega^1(1)) = 0$	$\mathcal{O}(-3) \oplus \mathcal{O}(-2) \xrightarrow{\varphi} 2\mathcal{O}$ φ is injective φ_{12} and φ_{22} are linearly independent two-forms	2
M_2	$h^0(\mathcal{F}(-1)) = 0$ $h^1(\mathcal{F}) = 1$ $h^0(\mathcal{F} \otimes \Omega^1(1)) = 1$	$\mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-1) \xrightarrow{\varphi} \mathcal{O}(-1) \oplus 2\mathcal{O}$ φ is injective $\varphi_{13} = 0$ $\varphi_{12} \neq 0$, $\varphi_{12} \nmid \varphi_{11}$ φ_{23} has linearly independent entries	3
M_3	$h^0(\mathcal{F}(-1)) = 1$ $h^1(\mathcal{F}) = 2$ $h^0(\mathcal{F} \otimes \Omega^1(1)) = 3$	$2\mathcal{O}(-3) \xrightarrow{\varphi} \mathcal{O}(-2) \oplus \mathcal{O}(1)$ φ is injective φ_{11} has linearly independent entries	5

Kronecker moduli space

$$N(m,p,q) = \mathrm{Hom}(\mathbb{C}^p, \mathbb{C}^m \otimes \mathbb{C}^q)^{ss} /\!/ \mathrm{GL}(p,\mathbb{C}) \times \mathrm{GL}(q,\mathbb{C})$$

M_0 is a proper open subset inside a fibre bundle with base $N(3,4,3)$ and fibre $\mathbb{P}^{14}$

M_1 is a proper open subset inside a fibre bundle with base $\mathrm{Grass}(2,\mathbb{C}^6)$ and fibre $\mathbb{P}^{16}$

M_2 is a proper open subset inside a fibre bundle with fibre $\mathbb{P}^{17}$ and base $\mathrm{Hilb}_{\mathbb{P}^2}(2) \times \mathbb{P}^2$,

M_3 is the universal quintic in $\mathbb{P}^2 \times \mathbb{P}(S^5\, V^*)$

Geometric quotients: $M_i = W_i / G_i$

Theorem

The T-fixed locus of $M_{\mathbb{P}^2}(5,1)$ *consists of* 1407 *isolated points,* 132 *projective lines and six irreducible components of dimension two that are isomorphic to* $\mathbb{P}^1 \times \mathbb{P}^1$. *The integral homology groups of* $M_{\mathbb{P}^2}(5,1)$ *have no torsion and its Poincaré polynomial is*

$$\begin{aligned} P(x) = & x^{52} + 2x^{50} + 6x^{48} + 13x^{46} + 26x^{44} + 45x^{42} + 68x^{40} + 87x^{38} + \\ & 100x^{36} + 107x^{34} + 111x^{32} + 112x^{30} + \\ & 113x^{28} + 113x^{26} + 113x^{24} + \\ & 112x^{22} + 111x^{20} + 107x^{18} + 100x^{16} + \\ & 87x^{14} + 68x^{12} + 45x^{10} + 26x^{8} + 13x^{6} + 6x^{4} + 2x^{2} + 1. \end{aligned}$$

The Euler characteristic of $M_{\mathbb{P}^2}(5,1)$ *is* 1695 *and its Hodge numbers satisfy the relation* $h^{pq} = 0$ *if* $p \neq q$. *The Picard group is* $\mathbb{Z}^2$. *The fundamental group* π_1 *is trivial. Moreover,* $M_{\mathbb{P}^2}(5,1)$ *is rational.*

Summary for $M_{\mathbb{P}^2}(5,3)$

stratum	cohomological conditions	subset W_i of morphisms φ	codim.
M_0	$h^0(\mathcal{F}(-1))=0$ $h^1(\mathcal{F})=0$ $h^0(\mathcal{F}\otimes\Omega^1(1))=1$	$2\mathcal{O}(-2)\oplus\mathcal{O}(-1)\xrightarrow{\varphi}3\mathcal{O}$ φ is injective φ is not equivalent to $\begin{bmatrix}\star&\star&\star\\\star&\star&0\\\star&\star&0\end{bmatrix}$ or $\begin{bmatrix}\star&\star&\star\\\star&\star&\star\\\star&0&0\end{bmatrix}$	0
M_1	$h^0(\mathcal{F}(-1))=0$ $h^1(\mathcal{F})=0$ $h^0(\mathcal{F}\otimes\Omega^1(1))=2$	$2\mathcal{O}(-2)\oplus2\mathcal{O}(-1)\xrightarrow{\varphi}\mathcal{O}(-1)\oplus3\mathcal{O}$ φ is injective, $\varphi_{12}=0$, φ_{11} has linearly independent entries, φ_{22} has linearly independent maximal minors	2
M_2	$h^0(\mathcal{F}(-1))=1$ $h^1(\mathcal{F})=0$ $h^0(\mathcal{F}\otimes\Omega^1(1))=3$	$3\mathcal{O}(-2)\xrightarrow{\varphi}2\mathcal{O}(-1)\oplus\mathcal{O}(1)$ φ is injective, φ_{11} has linearly independent maximal minors	3
M_3	$h^0(\mathcal{F}(-1))=1$ $h^1(\mathcal{F})=1$ $h^0(\mathcal{F}\otimes\Omega^1(1))=4$	$\mathcal{O}(-3)\oplus\mathcal{O}(-1)\xrightarrow{\varphi}\mathcal{O}\oplus\mathcal{O}(1)$ φ is injective, $\varphi_{12}\neq0$, $\varphi_{12}\nmid\varphi_{22}$	4

M_1 is a proper open subset inside a fibre bundle over $\mathbb{P}^2 \times \mathrm{N}(3,2,3)$ with fibre $\mathbb{P}^{16}$

M_2 is a proper open subset inside a fibre bundle over $\mathrm{N}(3,3,2)$ with fibre $\mathbb{P}^{17}$

M_3 is isomorphic to the Hilbert flag scheme of quintic curves in $\mathbb{P}^2$ containing zero-dimensional subschemes of length 2

Geometric quotients: $M_i = W_i/G_i$

Theorem

The T-fixed point locus of $\mathrm{M}_{\mathbb{P}^2}(5,3)$ *consists of* 1329 *isolated points,* 174 *projective lines, and* 3 *irreducible components of dimension two that are isomorphic to the surface obtained by blowing up* $\mathbb{P}^1 \times \mathbb{P}^1$ *at three points on the diagonal, then blowing down the strict transform of the diagonal. The integral homology groups of* $\mathrm{M}_{\mathbb{P}^2}(5,3)$ *have no torsion and its Poincaré polynomial is*

$$\begin{aligned} P(x) = & x^{52} + 2x^{50} + 6x^{48} + 13x^{46} + 26x^{44} + 45x^{42} + 68x^{40} + 87x^{38} + \\ & 100x^{36} + 107x^{34} + 111x^{32} + 112x^{30} + \\ & 113x^{28} + 113x^{26} + 113x^{24} + \\ & 112x^{22} + 111x^{20} + 107x^{18} + 100x^{16} + \\ & 87x^{14} + 68x^{12} + 45x^{10} + 26x^{8} + 13x^{6} + 6x^{4} + 2x^{2} + 1. \end{aligned}$$

The Euler characteristic of $\mathrm{M}_{\mathbb{P}^2}(5,3)$ *is* 1695 *and its Hodge numbers satisfy the relation* $\mathrm{h}^{pq} = 0$ *if* $p \neq q$*. The Picard group is* $\mathbb{Z}^2$*. The fundamental group* π_1 *is trivial. Moreover,* $\mathrm{M}_{\mathbb{P}^2}(5,3)$ *is rational.*

The open stratum $M_0 \subset \mathrm{M}_{\mathbb{P}^2}(r, r-1)$ looks as in the examples above: cokernels of

$$\varphi\colon \mathcal{O}(-2) \oplus (r-2)\mathcal{O}(-1) \to (r-1)\mathcal{O}, \quad \varphi \text{ injective, } \varphi_{12} \text{ semistable.}$$

$\mathrm{Hilb}_{\mathbb{P}^2}(l, r) =$ relative Hilbert scheme of l points on a curve of degree r

$l = (r-2)(r-1)/2$

$\mathrm{Hilb}^0_{\mathbb{P}^2}(l, r) =$ open subset given by the condition that the points do not lie on a curve of degree $r-3$. It is a projective bundle over $\mathrm{Hilb}^0_{\mathbb{P}^2}(l)$ (subset defined by the same condition), so it is rational.

The open subset of M_0 given by the condition that the maximal minors of φ_{11} have no common factor is isomorphic to $\mathrm{Hilb}^0_{\mathbb{P}^2}(l, r)$

Proposition

$\mathrm{M}_{\mathbb{P}^2}(r, r-1)$ *and* $\mathrm{Hilb}_{\mathbb{P}^2}(l, r)$ *are birational so* $\mathrm{M}_{\mathbb{P}^2}(r, r-1)$ *is rational.*

Assume $2 \leq r \leq n$. $M_0 \subset \mathrm{M}(n+r, n)$ is the open subset of sheaves $\mathcal{F}$ that have smooth schematic support and that satisfy the conditions

$$\mathrm{H}^0(\mathcal{F}(-1)) = 0, \qquad \mathrm{H}^1(\mathcal{F} \otimes \Omega^1(1)) = 0, \qquad \mathrm{H}^1(\mathcal{F}) = 0$$

$$l = \frac{1}{2}(n+r)(n+r-1) - n$$

$H_0 \subset \mathrm{Hilb}(l, n+r)$ is the set of pairs (Z, C) such that C is smooth and $\mathcal{I}_Z \subset \mathcal{O}_{\mathbb{P}^2}$ satisfies the cohomological conditions

$$\mathrm{H}^0(\mathcal{I}_Z(n+r-3)) = 0, \quad \mathrm{H}^1(\mathcal{I}_Z(n+r-1) \otimes \Omega^1) = 0, \quad \mathrm{H}^1(\mathcal{I}_Z(n+r-2)) = 0.$$

$\mathcal{J}_Z =$ the ideal sheaf of Z in C

Proposition

Assume that $2 \leq r < \dfrac{n(\sqrt{5}-1)}{2}$. *Then* $\mathrm{M}_{\mathbb{P}^2}(n+r, n)$ *is stably rational.*

Proof.

We have a surjective morphism $H_0 \to M_0$ given by $(Z, C) \mapsto [\mathcal{J}_Z \otimes \mathcal{O}(n+r-2)]$. Its generic fiber is $\mathbb{P}^{r-1}$. □

Thank you!