# A Feynman integral via higher normal functions

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A recent computation of a Feynman integral (associated to the sunset graph) by S. Bloch and P. Vanhove involved a motive associated to a modular family of elliptic curves. We developed an alternative approach (using our work with C. Doran) via higher Chow cycles and higher normal functions, which seemed promising for integrals in higher dimensions and non-modular settings.

In this talk, we describe joint work with Bloch and Vanhove, which applies this method to the 3-dimensional analogue of the sunset integral.

# §1. The results

These concern the integral

$$I(t) := \int_{(\mathbb{R}_-)^{\times 3}} \frac{\frac{dx}{x} \wedge \frac{dy}{y} \wedge \frac{dz}{z}}{t - \phi(x,y,z)} ,$$

where $\phi(x,y,z) := (1-x-y-z)(1-x^{-1}-y^{-1}-z^{-1})$ and $t \in \mathbb{P}^1 \setminus \{0, 4, 16, \infty\}$.

The inhomogeneous equation and special values are easiest to state. Writing

$$\begin{aligned} D_{PF} = \quad & t^2(t-4)(t-16)\tfrac{d^3}{dt^3} + 6t(t^2-15t+32)\tfrac{d^2}{dt^2} \\ & +(7t^2-68t+64)\tfrac{d}{dt} + (t-4), \end{aligned}$$

and $f^+$ for a certain Hecke character, we have

- $D_{PF} I(t) = -24$,
- $I(0) = 7\zeta(3)$, $I(1) = \frac{12\pi}{\sqrt{15}} L(f^+, 2)$ (="$(2\pi i)^3 \times$period").

To evaluate the integral in terms of elliptic trilogarithms, write $\tau \in \mathfrak{H}$, $q = e^{2\pi i \tau}$, $\eta(\tau) = q^{\frac{1}{24}} \prod_{k\geq 1}(1-q^k)$;
then

$$H(\tau) := -\left(\frac{\eta(\tau)\eta(3\tau)}{\eta(2\tau)\eta(6\tau)}\right)^6$$

is a Hauptmodul for $\Gamma := \Gamma_1(6)^{+3} \leq SL_2(\mathbb{R})$,
while

$$G(\tau) := \frac{(\eta(2\tau)\eta(6\tau))^4}{(\eta(\tau)\eta(3\tau))^2}$$

is a modular form of weight 2 with respect to $\Gamma$.

We shall pull $I(\tau)$ back under $H$.

To wit: writing

$$\widehat{Li}_3(x) := \sum_{k\geq 1} Li_3(x^k) = \sum_{k\geq 1}\sum_{\delta\geq 1}\frac{x^{k\delta}}{\delta^3} = \sum_{m\geq 1} x^m \sum_{\delta|m}\frac{1}{\delta^3},$$

and setting

$$\begin{aligned} Eich(\tau) := \; & -16\left\{2\widehat{Li}_3(q^6) - \widehat{Li}_3(q^3) - 6\widehat{Li}_3(q^2) + 3\widehat{Li}_3(q)\right\} \\ & +16\zeta(3) - 4\log^3 q, \end{aligned}$$

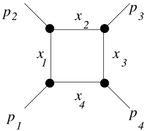
our main result reads

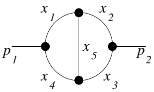
- $I(H(\tau)) = G(\tau) Eich(\tau)$.

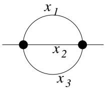
# §2. General context

In QFT, to compute probability amplitudes you have to sum (integrate) over all possible paths for a particle; for multiple particles, we must also integrate over all possible interactions.

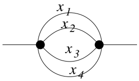
This leads to pictures like the following, where edges (resp. vertices) depict propagating particles (resp. interactions):




*sunset*



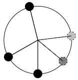

*3−banana*

$x_i$ =*Feynman parameters*

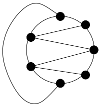
$p_j$ =*external momenta (real D−vectors)*

$\Sigma p = 0$

$m_i$ = *masses*



*wheel with l>2 spokes*



*zigzag graph (l=6, n=12)*

(The inner edges also have momenta, which must sum to 0 at each vertex, but these get “integrated out”.)

# Feynman graphs

The form in which I will present the Feynman integral involves a bit of graph theory, and no physics.

Let $G$ be a connected graph with $n$ internal edges and $r$ vertices. Mapping an edge to the difference of its endpoints yields a homomorphism

$$\mathbb{Z}^n \to \mathbb{Z}^r$$

with kernel $\cong \mathbb{Z}^\ell$ for some $\ell$. This is $H_1(G)$, and $\ell$ is the loop number (or equivalently, the number of independent momenta on the "inside").

In fact, $\mathbb{Z}^n$ maps onto the kernel of the augmentation, and so

$$\ell = n - r + 1.$$

A **tree** is a graph with $\ell = 0$; a $k$-**forest** is a disjoint union of $k$ trees. A **spanning tree** of $G$ is a subgraph, containing all the vertices of $G$, which is a tree. Let $\mathcal{T}_1$ denote the set of spanning trees, $\mathcal{T}_2$ the set of spanning 2-forests, etc.

## Definition

The **first Symanzik polynomial** of $G$ is

$$\mathcal{U} := \sum_{T \in \mathcal{T}_1} \prod_{e_j \notin T} x_j .$$

Writing $P_T$ for the set of external momenta attached to $T$, the **second Symanzik polynomial** of $G$ is

$$\mathcal{F} := \sum_{(T_1, T_2) \in \mathcal{T}_2} \left( \prod_{e_j \notin (T_1, T_2)} x_i \right) \left( \sum_{p_j \in P_{T_1}} \sum_{p_k \in P_{T_2}} p_j \cdot p_k \right) + \mathcal{U} \cdot \sum_{i=1}^{n} m_i^2 x_i .$$

Consider the differential form

$$\omega := \sum_{j=1}^{n} (-1)^{j-1} x_j dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n,$$

and the simplex

$$\Delta := \left\{ [x_1 : \cdots : x_n] \in \mathbb{P}^{n-1} \,|\, x_i \text{ positive real} \right\}.$$

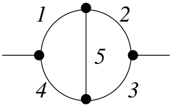
Assume for simplicity $G$ has "no mass insertions".

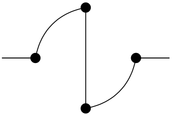
## Definition

The **Feynman integral** (in Feynman reparametrized form) associated to $G$ is

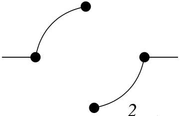
$$I_G := \int_\Delta \frac{\mathcal{U}^{n-(\ell+1)\frac{D}{2}}}{\mathcal{F}^{n-\ell\frac{D}{2}}} \omega.$$

## Example

In the graph 

, the spanning tree 

contributes a term of $x_2 x_4$ to the 1st Symanzik polynomial.

The spanning 2–forest  contributes

a term of $x_2 x_4 x_5 p_1 . p_2 = -x_2 x_4 x_5 p^2$ to the 2nd Symanzik polynomial.

There are two basic cases in which the Feynman integral is finite.

- $D = 4,\ n = 2\ell \implies \left\{ \begin{array}{ccc} n - \ell\frac{D}{2} & = & 0 \\ n - (\ell+1)\frac{D}{2} & = & -2 \end{array} \right\}$

$$\implies I_G = \int_\Delta \frac{\omega}{\mathcal{U}^2}$$

  which is constant. (e.g. wheel, zigzag)

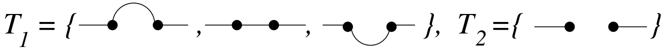
- $D = 2,\ n = \ell + 1 \implies \left\{ \begin{array}{ccc} n - \ell\frac{D}{2} & = & 1 \\ n - (\ell+1)\frac{D}{2} & = & 0 \end{array} \right\}$

$$\implies I_G = \int_\Delta \frac{\omega}{\mathcal{F}}$$

  which depends on $\{p_j\}, \{m_i\}$. (e.g. $n$-banana)

## Example

For the sunset graph,




$$\implies \mathcal{F} = -x_1x_2x_3p^2 + \{x_1x_2 + x_2x_3 + x_1x_3\}\{m_1^2x_1 + m_2^2x_2 + m_3^2x_3\}.$$

Now restrict to $m_1 = m_2 = m_3 =: m$; set $t = \frac{m^2}{p^2}$, and pass to the chart $(x, y) \mapsto [x : y : 1]$. Then

$$-p^2 I_{sunset} = \int_\Delta \frac{\frac{dx}{x} \wedge \frac{dy}{y}}{1 - t\phi(x, y)}$$

where $\phi = (1 + x + y)(1 + x^{-1} + y^{-1})$.

The sunset and 3-banana graphs are of particular interest with regard to *what classes of functions* arise in evaluating $I_G$, as first instances of "something new" beyond multiple polylogs.

# §3. Naive perspective

Recall the 3-banana integral

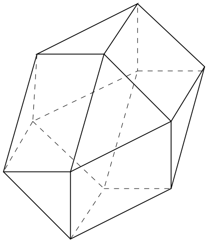
$$I(t) := \int_{(\mathbb{R}_-)^{\times 3}} \frac{\frac{dx}{x} \wedge \frac{dy}{y} \wedge \frac{dz}{z}}{t - \phi(x, y, z)}, \quad \phi = (1-x-y-z)(1-\frac{1}{x}-\frac{1}{y}-\frac{1}{z}).$$

Let $\Delta$ be the Newton polytope $\Delta_\phi$ of $\phi$, i.e.

$$\Delta = \text{Convex hull}\left\{\{\pm e_i\}_{i=1}^3, \{\pm(e_i - e_j)\}_{1\le i<j\le 3}\right\} \subset \mathbb{R}^3,$$

and $\mathbb{P} := \widetilde{\mathbb{P}_\Delta}$. That is, $\mathbb{P}$ is obtained by blowing up $\mathbb{P}^3$ first at the "vertices", then on the "edges", then blowing down twelve $-1$-curves:

$\mathbb{P}$



Let $Y = \mathbb{P}\backslash(\mathbb{C}^*)^3$,

$$X = X_t = \overline{\{t - \phi(x, y, x) = 0\}} \subset \mathbb{P},$$

and $Z = X \cap Y$. Write $T_3 := \overline{(\mathbb{R}_-)^{\times 3}}$.
Consider the relative-cohomology long-exact sequence

$$H^2(\mathbb{P}\backslash Y) \to H^2(X\backslash Z) \to H^3(\mathbb{P}\backslash Y, X\backslash Z) \to H^3(\mathbb{P}\backslash Y) \to 0;$$

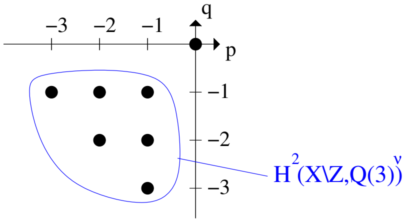
twisting by $\mathbb{Q}(3) = (2\pi\sqrt{-1})^3\mathbb{Q}$, this becomes

$$0 \to H^2(X\backslash Z, \mathbb{Q}(3))^\nu \to H^3(\mathbb{P}\backslash Y, X\backslash Z; \mathbb{Q}(3)) \to \mathbb{Q}(0) \to 0.$$

Two lifts of $1 \in \mathbb{Q}(0)$ to the middle term are given by

- $\Omega_3 := \frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2} \wedge \frac{dx_3}{x_3}$, which is in $F^0$ hence compatible with the Hodge filtration; and
- $(2\pi i)^3 T_3$, which is compatible with the $\mathbb{Q}$-structure.

The mixed Hodge structure on the middle term takes the form




The corresponding extension class

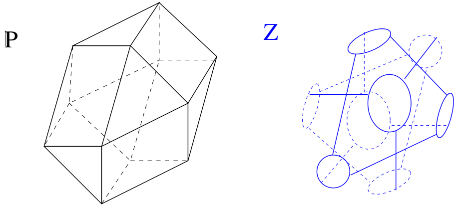
$$\varepsilon \in H^2(X \backslash Z, \mathbb{C}/\mathbb{Q}(3))^\nu \cong Ext^1_{\mathsf{MHS}}(\mathbb{Q}(0), H^2(X \backslash Z, \mathbb{Q}(3))^\nu)$$

maps to the difference $\Omega_3 - (2\pi\sqrt{-1})^3 T_3$ in the middle term.

Now the Laurent polynomials $\phi'$ with $\Delta_{\phi'} \subseteq \Delta_\phi$ yield sections of the sheaf $\mathcal{O}(1) \to \mathbb{P}$.

Since $\Delta_\phi$ is *reflexive* (i.e. the polar $\Delta_\phi^\circ$ has integral vertices), $\mathcal{O}(1)$ is the anticanonical bundle $K_{\mathbb{P}}^{-1}$. As $X$ consists of the zeroes of a section, it is an anticanonical hypersurface. By adjunction, $K_X \cong \mathcal{O}_X$ and $X$ is $K3$ (in fact, of Picard rank 19).

Since $\Delta_\phi^\circ \cap \mathbb{Z}^3 = \text{vertices} \cup \{0\}$, $\mathbb{P}$ is smooth except at "vertices", and so (with a bit of work) $X$ is smooth. The structure at infinity (i.e. of $Z = X \cap Y$) is given by



The unique (up to scale) holomorphic 2-form on $X_t$ is given by $\omega_t := \frac{-1}{(2\pi\sqrt{-1})^2} Res_{X_t}\Omega_{X_t}$, where

$$\Omega_{X_t} := \frac{dx/x \wedge dy/y \wedge dz/z}{t - \phi(x,y,z)} \in \Omega^3(\mathbb{P}\backslash X_t).$$

Under the Poincaré pairing $\langle\, , \,\rangle : H^3(\mathbb{P}\backslash Y, X\backslash Z; \mathbb{C}) \times H^3(\mathbb{P}\backslash X, Y\backslash Z; \mathbb{C}) \to \mathbb{C}$, we have

$$I(t) = \langle T_3, \Omega_{X_t}\rangle = \langle T_3 - \frac{1}{(2\pi\sqrt{-1})^3}\Omega_3, \Omega_{X_t}\rangle.$$

In terms of the pairing $H^2(X, Z; \mathbb{C}) \times H^2(X\backslash Z, \mathbb{C})$, this translates to

$$I(t) \equiv \langle \varepsilon, \omega_t \rangle,$$

modulo periods of $\omega_t$ over (relative) cycles $\gamma \in H_2(X, Z; \mathbb{Q})$. To get the inhomogeneous equation, we need a more precise identification, modulo periods over cycles on $X$.

# §4. The higher normal function

Bloch's higher Chow groups give a geometric representation of the graded pieces of algebraic $K$-theory of a smooth projective variety $X$, and come with cycle-class ("$AJ$"=Abel-Jacobi) maps to complex semi-tori

$$
\begin{gathered}
CH^p(X,n) \cong CH^p\left(X \times (\mathbb{P}^1 \setminus \{1\}, \{0,\infty\})^{\times n}\right) \underset{\otimes\mathbb{Q}}{\cong} Gr_\gamma^p K_n^{alg}(X) \\
AJ \downarrow \\
H^{2p-1}(X,\mathbb{C}) / \left\{ F^p H^{2p-n-1}(X,\mathbb{C}) + H^{2p-n-1}(X,(2\pi\sqrt{-1})^p\mathbb{Z}) \right\}
\end{gathered}
$$

computed in terms of cohomology classes of currents on $X$.

- If $X$ is a Riemann surface, $p=1$, $n=0$, then this recovers Abel's map $CH^1(X)_{hom} \to J(X)$. (I have left out "hom" above because we are interested in $n>0$.)

- The key case is $X = K3$ surface, $p=n=3$.

Define Milnor $K$-theory of a field $k$ by

$$K_n^M(k) := \bigwedge^n k^* \Big/ \langle \alpha \wedge (1-\alpha) \rangle.$$

The Tame symbol

$$Tame: \ K_3^M(\mathbb{C}(X)) \to \oplus_{y \in X^1} K_2^M(\mathbb{C}(y))$$

is computed on a Milnor symbol $\{f, g, h\}$ by using relations in $K^M$ to arrange that $ord_y(g) = 0 = ord_y(h)$, then sending

$$\{f, g, h\} \mapsto ord_y(f) \cdot \{g|_y, h|_y\}.$$

The kernel $K_3^M(X)$ of *Tame* is identified with $CH^3(X, 3)$ (modulo decomposables) by taking “graphs”

$$\{f, g, h\} \rightsquigarrow \{(x, f(x), g(x), h(x)) | \, x \in X(\mathbb{C})\}.$$

So (up to decomposables) an element of $CH^3(X,3)$ looks like $\xi = \sum m_i\{f_i, g_i, h_i\}$, and (up to $H^2_{alg}$)

$$AJ: \ CH^3(X,3) \to Hom\left(H_2(X,\mathbb{Z}), \mathbb{C}/(2\pi\sqrt{-1})^3\mathbb{Z}\right)$$

is computed by $\xi \mapsto \int_{(\cdot)} R_\xi$ where (writing $\log(f)$ for the discontinuous branch with cut $T_f := \overline{f^{-1}(\mathbb{R}_-)}$)

$$\begin{aligned} R_{\{f,g,h\}} \ := \ & \log(f)\tfrac{dg}{g} \wedge \tfrac{dh}{h} + 2\pi\sqrt{-1}\log(g)\tfrac{dh}{h}\delta_{T_f} \\ & +(2\pi\sqrt{-1})^2 \log(h)\delta_{T_f \cap T_g} \quad \in D^2(X). \end{aligned}$$

If $\xi = \sum m_i\{f_i, g_i, h_i\}$ (with appropriate constraints), then

$$d[R_\xi] = -(2\pi\sqrt{-1})^3\delta_{T_\xi} \ \text{ where } \ T_\xi := \sum m_i(T_{f_i} \cap T_{g_i} \cap T_{h_i}).$$

Taking $\Gamma$ to be a 2-chain with $\partial\Gamma = T_\xi$, the 2-current

$$\tilde{R}_\xi := R_\xi + (2\pi\sqrt{-1})^3 \delta_\Gamma$$

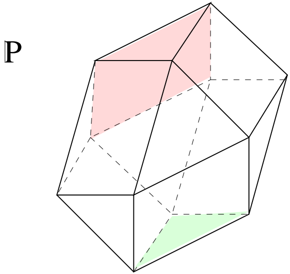
is closed, hence gives a cycle-class $[\tilde{R}_\xi] \in H^2(X, \mathbb{C})$.

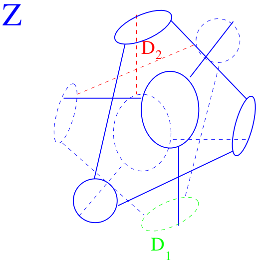
Now let $\mathcal{X} \xrightarrow{\pi} \mathbb{P}^1 \backslash \mathcal{L}$ be a family of $K3$ surfaces. Given
(1) a family $\xi_t \in CH^3(X_t, 3)$ (and associated $\tilde{R}_t$) descended from $\Xi = \sum m_i \{F_i, G_i, H_i\} \in CH^3(\mathcal{X}, 3)$, and
(2) a family of holomorphic 2-forms $\omega_t \in \Gamma(\mathbb{P}^1 \backslash \mathcal{L}, \Omega^2_\pi)$:
we may define the **higher normal function**

$$V_\xi(t) := \langle [\tilde{R}_t], [\omega_t] \rangle$$

over $\mathbb{P}^1$, where $\langle \cdot, \cdot \rangle : H^2 \times H^2 \to H^4 \xrightarrow{\int} \mathbb{C}$ is the Poincaré pairing. Note that $V$ is multivalued with log, $\log^2$, and $\log^3$ poles.

Turning to the 3-banana $K3$ family, recall the structure at $\infty$:




Put $\xi_t := \{x, y, z\}|_{X_t} \in CH^3(X \backslash Z, 3)$, and write $Z = \cup D_i$. To lift $\xi_t$ to $CH^3(X_t, 3)$, check the vanishing of the $Tame_{D_i}\xi_t$: e.g.

- $\phi_{D_1}(x, y) = 1 - x - y$ (and $z = 0$) $\Longrightarrow$ $Tame_{D_1}\{x, y, z\} = \{x, y\}|_{D_1} = \{x, 1 - x\} = 0$

- $\phi_{D_2}(u, z) = (1 - u)(1 - z)$ $(u = x/y,\ x = 0 = y)$ $\Longrightarrow$ $Tame_{D_2}\{x, y, z\} = Tame_{D_2}\{-u, y, z\} = \{-u, z\}|_{D_2} = \{-1, z\}$ or $\{-u, 1\} = 0$

Bloch's moving lemma produces a quasi-isomorphism

$$Z^3(X\backslash Z, \bullet) \xleftarrow[\jmath^*]{\simeq} \frac{Z^3(X, \bullet)}{\imath_*^Z Z^2(Z, \bullet)}.$$

Since $Tame(\xi) = 0$, there exist $\mu \in Z^3(X\backslash Z, 4)$ and $\hat{\xi} \in \ker(\partial) \subset Z^3(X, 3)$ s.t.

$$\xi + \partial\mu = \jmath^*\hat{\xi}.$$

In fact, $\xi$ extends across six $\{D_i\}$ (where $x$, $y$, or $z$ is $\equiv 1$). Let $D$ be the union of the remaining $\{D_i\}$; replace $X\backslash Z$ by $U = X\backslash D$, and $\xi$ by its extension to $U$. We have:

- $T_3 \cap X = \emptyset \implies T_\xi = 0 \implies R_\xi$ closed on $U \implies R_{\hat{\xi}}$ closed on $U$.

- $H_1(D) = \{0\} \implies d[R_{\hat{\xi}}]$ (supported on $D$) is closed on $D$ $\implies \langle \tilde{R}_{\hat{\xi}}, \omega\rangle = \int \tilde{R}_{\hat{\xi}} \wedge \omega = \int R_\xi \wedge \omega$.

Now we may *recognize the Feynman integral as a higher normal function:*

$$I(t) = \int_{T_3} \Omega_{X_t} = \int_{\mathbb{P}} \delta_{T_3} \wedge \Omega_{X_t}$$

$$= \int_{\mathbb{P}} \frac{-d[R_3]}{(2\pi\sqrt{-1})^3} \wedge \Omega_{X_t} = \frac{-1}{(2\pi\sqrt{-1})^3} \int_{\mathbb{P}} R_3 \wedge d[\Omega_{X_t}]$$

$$= \int_{X_t} R_3|_{X_t} \wedge \omega_t = \int R_{\xi_t} \wedge \omega_t$$

$$= \int_{X_t} R_{\hat{\xi}_t} \wedge \omega_t = \langle \underbrace{\widetilde{AJ(\hat{\xi}_t)}}_{R_t}, \omega_t \rangle.$$

What has this accomplished in the context of the "naive perspective"?

Consider the diagram with exact rows and coefficients in $\mathbb{C}$:

$$\begin{array}{ccccc} H^2(X) & \to & H^3(\mathbb{P}, X) & \to & \overbrace{H^3(\mathbb{P})}^{0} \\ \downarrow & & \downarrow & & \downarrow \\ H^2(X \backslash Z) & \to & H^3(\mathbb{P} \backslash Y, X \backslash Z) & \to & H^3(\mathbb{P} \backslash Y). \end{array}$$

Modulo *classes pairing with* $\Omega_X$ *(or* $\omega$*) to* 0, we have lifted $[T_3] \in H^3(\mathbb{P} \backslash Y)$ all the way to $H^2(X)$.

- to $T_3$ in $H^3(\mathbb{P} \backslash Y, X \backslash Z)$ (since $T_3$ avoids $X$), which $\equiv T_3 - \frac{1}{(2\pi\sqrt{-1})^3}\Omega_3$
- which comes from $H^3(\mathbb{P}, X)$, hence $H^2(X)$: indeed, since

  $$d[R_3] = \Omega_3 - (2\pi\sqrt{-1})^3 \delta_{T_3} + \text{terms supported on } Y,$$

  it is precisely $\frac{-R_3|_X}{(2\pi\sqrt{-1})^3} \in H^2(X)$ (modulo stuff supported on $Z$) that it lifts to.

# §5. The inhomogeneous equation

Recall $\mathcal{X} \xrightarrow{\pi} \mathbb{P}^1 \backslash \mathcal{L}$ ($\mathcal{L} = \{0, 4, 16, \infty\}$) is our family of $K3$ surfaces. Given a family of cohomology classes $\eta \in \Gamma(\mathbb{P}^1 \backslash \mathcal{L}, \mathcal{H}^2_\pi)$, the Gauss-Manin connection produces a family

$$\nabla_{\partial_t} \eta \in \Gamma(\mathbb{P}^1 \backslash \mathcal{L}, \mathcal{H}^2_\pi)$$

which has periods given by $\frac{d}{dt}$ of the periods of $\eta$.

One way to compute it is to lift $\eta$ to $\tilde{\eta} \in A^2(\mathcal{X})$ ($C^\infty$ 2-form) and contract $d\tilde{\eta}$ with a lift $\widetilde{d/dt}$ (=vector field on $\mathcal{X}$). For $[\tilde{R}_t]$, this lift is $R_\Xi$, and (with $\Omega_\Xi := \sum m_i \frac{dF_i}{F_i} \wedge \frac{dG_i}{G_i} \wedge \frac{dH_i}{H_i}$)

$$dR_\Xi = \Omega_\Xi \underbrace{- (2\pi\sqrt{-1})^3 \delta_{T_\Xi} + \text{“Residue terms”}}_{\text{non-generic support}}$$

$$\implies \quad \nabla_{\partial_t}[\tilde{R}_t] =: \tilde{\omega}_t \text{ is a family of holomorphic 2-forms.}$$

Such a family $\tilde{\omega}_t$ satisfies a Picard-Fuchs equation

$$\nabla_{\mathrm{PF}}[\tilde{\omega}_t] := \left(g_0(t)\nabla^3_{\partial_t} + g_1(t)\nabla^2_{\partial_t} + g_2(t)\nabla_{\partial_t} + g_3(t)\right)[\tilde{\omega}_t] = 0,$$

meaning that the periods $\int_{\gamma_t} \tilde{\omega}_t$ $(\gamma_t \in H_2(X_t, \mathbb{Z}))$ satisfy

$$D_{\mathrm{PF}}(\cdot) := \left(g_0(t)\partial^3_t + g_1(t)\partial^2_t + g_2(t)\partial_t + g_3(t)\right)(\cdot) = 0.$$

Now compute

- $\partial_t\langle \tilde{R}, \tilde{\omega}\rangle = \underbrace{\langle \tilde{\omega}, \tilde{\omega}\rangle}_{0} + \langle \tilde{R}, \nabla_{\partial_t}\tilde{\omega}\rangle$
- $\partial^2_t\langle \tilde{R}, \tilde{\omega}\rangle = \underbrace{\langle \tilde{\omega}, \nabla_{\partial_t}\tilde{\omega}\rangle}_{0} + \langle \tilde{R}, \nabla^2_{\partial_t}\tilde{\omega}\rangle$
- $\partial^3_t\langle \tilde{R}, \tilde{\omega}\rangle = \underbrace{\langle \tilde{\omega}, \nabla^2_{\partial_t}\tilde{\omega}\rangle}_{=:\mathcal{Y}(t)} + \langle \tilde{R}, \nabla^3_{\partial_t}\tilde{\omega}\rangle$

where $\mathcal{Y}(t) \in \mathbb{C}(t)^*$ is the **Yukawa coupling**.

In our case, $\tilde{\omega}_t = (2\pi\sqrt{-1})^2 \omega_t$, and we conclude that

$$(2\pi\sqrt{-1})^2 D_{\mathsf{PF}} I(t) = D_{\mathsf{PF}} \langle \tilde{R}, \tilde{\omega} \rangle$$

$$= g_0(t)\mathcal{Y}(t) + \langle \tilde{R}, \underbrace{\nabla_{\mathsf{PF}} \tilde{\omega}}_{0} \rangle$$

$$= g_0(t)\mathcal{Y}(t).$$

With a modest effort, it can be shown that

$$\frac{1}{(2\pi\sqrt{-1})^2} g_0(t)\mathcal{Y}(t) = -24,$$

so that

$$D_{\mathsf{PF}} I(t) = -24.$$

# §6. Modular interpretation

We conclude with a brief summary of how the main result

$$I(H(\tau)) = G(\tau)Eich(\tau)$$

is obtained. This begins with a modular description of the family of $K3$ surfaces.

Denote by $\Gamma_1(6) \leq SL_2(\mathbb{Z})$ the subgroup consisting of matrices congruent to $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ mod 6, and let

$$\mathcal{E}_1(6) \to Y_1(6) := \Gamma_1(6)\backslash\mathfrak{H}$$

be the universal elliptic curve with a marked 6-torsion point. The Fricke involution $\alpha_3 := \sqrt{3}\begin{pmatrix} 1 & \frac{2}{3} \\ -2 & -1 \end{pmatrix}$ ($\alpha_3^2 = -\mathrm{id}$) acts on this family. Put

$$\Gamma_1(6)^{+3} := \langle \Gamma_1(6), \alpha_3 \rangle \leq SL_2(\mathbb{R}).$$

Quotienting the fiber product

$$\mathcal{E}_1(6) \underset{Y_1(6)}{\times} \alpha_3^* \mathcal{E}_1(6) \to Y_1(6)$$

by the involution $(\tau, z_1, z_2) \mapsto (\alpha_3(\tau), z_2, z_1)$ defines $\mathcal{E}_1^{(2)}(6)^{+3} \to Y_1(6)^{+3}$, a family of abelian surfaces.
Also let $\mathcal{E}^{(2)}(6)$ denote the self-fiber product of the universal curve $\mathcal{E}(6) \to Y(6)$ with marked 6-torsion lattice.

Applying the Shioda-Inose construction fiberwise to $\mathcal{E}_1^{(2)}(6)^{+3}$ yields the middle column of

$$\begin{array}{ccccc} \mathcal{E}^{(2)}(6) & \underset{\Theta}{\dashrightarrow} & \mathcal{X}_1(6)^{+3} & \underset{\cong}{\xrightarrow{\mathscr{H}}} & \mathcal{X} \\ & & \downarrow & & \downarrow \\ & & Y_1(6)^{+3} & \underset{\cong}{\xrightarrow{H}} & \mathbb{P}^1 \backslash \mathcal{L} \end{array}$$

where $H$ is the Hauptmodul. (Here $\Theta$ is a rational map.)

Now associated to any function $\varphi : (\mathbb{Z}/6\mathbb{Z})^2 \to \mathbb{C}$ is an Eisenstein series

$$E_\varphi(\tau) := \sum{}' \frac{\hat{\varphi}(m,n)}{(m\tau+n)^4} \in M_4(\Gamma(6)),$$

where $\hat{\varphi}$ is the finite Fourier transform.

Assuming $\sum_{m,n} \varphi(m,n) = 0$, there is also an explicit higher Chow cycle $\mathfrak{Z}_\varphi \in CH^3(\mathcal{E}^{(2)}(6), 3)$ with

$$\Omega_{\mathfrak{Z}_\varphi} = (2\pi\sqrt{-1})^3 E_\varphi(\tau)\, dz_1 \wedge dz_2 \wedge d\tau.$$

If $\mathcal{R}_\tau$ is the fiberwise Abel-Jacobi class of $\mathfrak{Z}_\varphi$, then $\nabla \mathcal{R} = \Omega_{\mathfrak{Z}_\varphi}$. Put

$$\mathcal{V}_\varphi(\tau) := \langle \mathcal{R}_\tau, [dz_1 \wedge dz_2] \rangle.$$

We claim that $\mathcal{V}_\varphi$ is an Eichler (triple) integral of $E_\varphi$:

$$\frac{d^3}{d\tau^3}\mathcal{V}_\varphi = \frac{d^3}{d\tau^3}\langle \mathcal{R}, [dz_1 \wedge dz_2]\rangle = \frac{d^2}{d\tau^2}\langle \mathcal{R}, \nabla_{\partial_\tau}[dz_1 \wedge dz_2]\rangle$$

$$= \frac{d}{d\tau}\langle \mathcal{R}, \nabla^2_{\partial_\tau}[dz_1 \wedge dz_2]\rangle = \frac{d}{d\tau}\langle \mathcal{R}, 2[\alpha_1 \times \alpha_2]\rangle$$

$$= \langle (2\pi\sqrt{-1})^3 E_\varphi(\tau)\, dz_1 \wedge dz_2, 2[\alpha_1 \times \alpha_2]\rangle = -2(2\pi\sqrt{-1})^3 E_\varphi(\tau).$$

This combined with identifying $\varphi$ such that

$$\Theta^*\Xi = \mathfrak{Z}_\varphi,$$

and $G(\tau)$ such that $\Theta^*\omega = G(\tau)\, dz_1 \wedge dz_2$, yields the main result up to constant, log and $\log^2$ terms. These require a precise calculation of $\mathcal{R}$, done in joint work with C. Doran.

To give a flavor of why the triple integral produces $\widehat{Li}_3$, suppose $\hat{\varphi} = \sum_{\alpha,\beta|6} \mu_{\alpha\beta}\psi_{\alpha,\beta}$ where

$$\psi_{\alpha,\beta}(m,n) := \begin{cases} 1, & \text{if } \alpha|m \text{ and } \beta|n \\ 0, & \text{otherwise} \end{cases} .$$

Basic theory of modular forms tells us that the Eisenstein series

$$-E_\varphi(\tau) = C + \frac{1}{6^4}\sum_{M\geq 1} q^{\frac{M}{6}} \sum_{r|M} r^3 \sum_{a\in\mathbb{Z}/6\mathbb{Z}} \zeta_6^{ar}\hat{\varphi}(\frac{M}{r}, a)$$

Say $C = 0$; then basic (but tricky) manipulations yield

$$-E_\varphi(\tau) = \sum_{\alpha,\beta} \frac{\mu_{\alpha\beta}}{\beta^4} \sum_{m\geq 1} m^3 (q^{\frac{\alpha}{\beta}})^m \sum_{\delta|m} \frac{1}{\delta^3}$$

whereupon applying $\int_0 \frac{dq}{q}$ thrice gives

$$\sum_{\alpha,\beta} \frac{\mu_{\alpha\beta}}{\beta\alpha^3} \sum_{m\geq 1} (q^{\frac{\alpha}{\beta}})^m \sum_{\delta|m} \frac{1}{\delta^3} = \sum_{\alpha,\beta} \frac{\mu_{\alpha\beta}}{\beta\alpha^3} \widehat{Li}_3(q^{\frac{\alpha}{\beta}}).$$

# §7. Concluding remarks

The approach via higher Chow cycles and Abel-Jacobi currents is expected to yield insights into several other integrals of the form we have considered here.

(1) For the higher-dimensional integrals associated to $n$-banana graphs, the identification with a higher normal function should at least work up to $n = 5$. (Beyond this case, "higher" Tame symbols might intervene.)

However, for $n > 3$ the Calabi-Yau toric hypersurface families are no longer modular, so the approach in §6 will not generalize.

(2) Even for $n = 2$ (sunset integral) the unequal masses case is still of interest. The Abel-Jacobi (extension) class is still computed by the non-closed current $R_t$, and $I(t)$ by $\int_{E_t} R_t \wedge \omega_t$. We can identify the inhomogeneous term in the Picard-Fuchs equation.

The joint work with Doran contains formulas for the periods of $R_t$ in this setting. The periods with log poles at $t = 0$ are computed directly.

To get the $\log^2(t)$ period, one makes use of local mirror symmetry: it is written as a generating function of instanton numbers of $K_{\mathbb{P}_\Delta}$. Whether this is a physically natural connection remains to be seen.