

Lagrangian subbundles of orthogonal vector bundles over a curve

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Segre invariants of vector bundles over curves

Let X be a complex projective smooth curve of genus $g \geq 2$.

Let $V \rightarrow X$ be a vector bundle of rank r .

For $0 < n < r$, the n th Segre invariant $s_n(V)$ is defined as

$$\min\{\deg V \cdot \text{rank } E - \text{rank } V \cdot \deg E : E \subset V \text{ of rank } n\}$$

Segre invariants are a “measure of stability”: A “low” value of s_n indicates a rank n maximal subbundle of “large” degree.

Why Segre invariants?

- Geometry of ruled varieties (Lange 1992)
- Refinement of the notion of (semi)stability
- Coding theory (Johnsen 2003)
- Geometry of moduli spaces (many authors)

Stratifications

Theorem. For $1 \leq n < r$, the function s_n is lower semicontinuous.

So s_n defines a stratification of $U(r, d)$. The strata are the locally closed sets

$$U(r, d; n, s) := \{V \in U(r, d) : s_n(V) = s\}$$

The locus $U(r, d; n, s)$ is nonempty only if $s \geq 0$ and $s \equiv nd \pmod{r}$.

- Lange–Narasimhan (1983) computed the dimensions of the strata $\mathcal{U}(2, d; 1, s)$.
- Hirschowitz (1988) computed the maximum value of s_n , corresponding to a general bundle in $U(r, d)$.
- Brambila-Paz–Lange (1998); Ballico–Brambilla-Paz–Russo (1998); Russo–Teixidor i Bigas (1999) computed the dimensions of the nonempty strata $U(r, d; n, s)$ and proved their irreducibility.
- Lange–Newstead (2003) computed the number of rank n maximal subbundles of a general V in certain cases, when this is finite. Moreover, a general W with $s_n(W)$ lower than the generic value has a unique maximal subbundle of rank n .

Orthogonal bundles

A bundle V over X is called *orthogonal* if it admits

- a nondegenerate symmetric bilinear form $\omega: V \otimes V \rightarrow O_X$; or equivalently,
- a symmetric isomorphism $\omega: V \xrightarrow{\sim} V^*$.

Then $\det V$ is a line bundle of order 2. We restrict our attention to those V of trivial determinant.

A subbundle E of V is called *isotropic* if $\omega(E \otimes E) = 0$.

By linear algebra, an isotropic subbundle has rank at most $\frac{1}{2}\text{rank } V$.

Moduli of orthogonal bundles

Theorem (Ramanathan 1976): There is a coarse moduli space $\mathcal{M}(SO_r\mathbb{C})$ for semistable principal $SO_r\mathbb{C}$ -bundles over X . This is a projective singular variety with two irreducible connected components, each of dimension $\frac{1}{2}r(r - 1)(g - 1)$.

Theorem (Serman 2008): The map

$$\mathcal{M}(SO_r\mathbb{C}) \dashrightarrow \mathcal{M}(SL_r\mathbb{C}) \cong SU(r, O_X),$$

given by extension of structure group, is an embedding if r is odd and has degree 2 if r is even.

We work with the image $\mathcal{MO}(r)$ of $\mathcal{M}(SO_r\mathbb{C})$ in $SU(r, O_X)$.

The components $\mathcal{MO}(r)^\pm$

The universal cover of $SO_r\mathbb{C}$ is the group $Spin_r$.

The component $\mathcal{MO}(r)^+$ consists of those bundles whose $SO_r\mathbb{C}$ -structure is induced by a $Spin_r$ structure.

Another characterisation (Beauvreille 2006): $\vartheta(\mathcal{MO}(r)^\pm) \subseteq |_r\Theta|^\pm$.

Later we will see another characterisation of the components $\mathcal{MO}(r)^\pm$ in terms of the degrees of Lagrangian subbundles.

Lagrangian Segre invariants of orthogonal bundles

We assume $r = 2n$ or $2n + 1$, and focus on *Lagrangian* subbundles; that is, isotropic subbundles of the largest possible rank n .

The *Lagrangian Segre invariant* $t(V)$ is defined by

$$t(V) := \min\{-2 \cdot \deg E : E \text{ a Lagrangian subbundle of } V\}.$$

The function t is again lower semicontinuous, and defines a stratification on any family of orthogonal bundles over X .

A Lagrangian Segre stratification

For an integer e , we write

$$\mathcal{MO}(r; 2e) := \{V \in \mathcal{MO}(r) : t(V) = 2e\}.$$

When nonempty, this is the locus of semistable orthogonal bundles V whose maximal isotropic subbundles of rank n have degree $-e$.

It is clear that $t(V) \geq \frac{2}{r} \cdot s_n(V)$, with equality if and only if at least one rank n maximal subbundle of V is isotropic.

Questions

- (1) What are the possible values of t ?
- (2) What is the dimension of each stratum? Are the strata irreducible?
- (3) How are the strata distributed between the components $\mathcal{MO}(r)^{\pm}$?
- (4) What is the number of maximal Lagrangian subbundles of a general bundle in each stratum?
- (5) How does the isotropic Segre invariant interact with the classical one?

Inspiration: Segre invariant of vector bundles of rank two and secant geometry (Lange–Narasimhan, Math. Ann. 1983)

Let $L \rightarrow X$ be a line bundle of degree $d \geq 2$. Extensions

$$0 \rightarrow \mathcal{O}_X \rightarrow V \rightarrow L \rightarrow 0$$

are parametrised by $\mathbb{P}H^1(L^{-1}) \cong |K_X \otimes L|^*$.

We have the standard map $\phi: X \hookrightarrow |K_X \otimes L|^*$.

Proposition. For each effective divisor D , the subsheaf $L(-D) \subset L$ lifts to an extension V if and only if $[V]$ belongs to the span of $\phi(D)$ in $|K_X \otimes L|^*$.

Corollary. Suppose $g \geq 2$ and $0 < e < d$. Then $s_1(V) \leq 2e - d$ if and only if $[V] \in \text{Sec}^e X$.

Orthogonal extensions

For now, we focus on the **even rank** case.

A bundle $V \in \mathcal{MO}(2n)$ belongs to the stratum $\mathcal{MO}(2n; 2e)$ only if it is an extension

$$0 \rightarrow E \rightarrow V \rightarrow E^* \rightarrow 0$$

where E is Lagrangian of degree $-e$. We obtain a class $[V]$ in $H^1(E \otimes E)$.

Conversely:

Lemma. (Ramanan–H., 2007) Suppose E is a simple vector bundle. Then an extension V as above carries an orthogonal structure with E Lagrangian if and only if $[V]$ belongs to the subspace $H^1(\wedge^2 E)$ of $H^1(E \otimes E)$.

Parameter spaces of orthogonal extensions

Lemma. (1) For each $e \geq 0$, there exist a finite étale cover $\widetilde{U(n, -e)} \rightarrow U(n, -e)$ and a projective bundle \mathbb{A}_e over $\widetilde{U(n, -e)}$ with fibre $\mathbb{P}H^1(\wedge^2 E)$ at each point of $\widetilde{U(n, -e)}$ lying over E .

(2) There is a moduli map $\Phi_e: \mathbb{A}_e \dashrightarrow \mathcal{MO}(2n)$, taking the class of $0 \rightarrow E \rightarrow V \rightarrow E^* \rightarrow 0$ to the moduli point of V , when this exists.

Proof. Idea from Lange–Brambilla-Paz (1998). Follows from constructions by Narasimhan–Ramanan (1975) and Lange (1983), and the previous lemma.

Geometry in $\mathbb{P}H^1(\wedge^2 E)$

A vector bundle W defines a scroll $\pi: \mathbb{P}W \rightarrow X$ with a relative hyperplane bundle $\mathcal{O}_{\mathbb{P}W}(1)$.

Via the identifications

$$H^1(W) \cong H^0(K_X \otimes W)^* = H^0(\mathbb{P}W, \pi^* K_X \otimes \mathcal{O}_{\mathbb{P}W}(1))^*,$$

we have a map $\psi: \mathbb{P}W \dashrightarrow \mathbb{P}H^1(W)$.

If $W = \wedge^2 E$, we compose ψ with the relative Plücker embedding:

$$\text{Gr}(2, E) \hookrightarrow \mathbb{P}(\wedge^2 E) \dashrightarrow \mathbb{P}H^1(\wedge^2 E).$$

Lemma (Geometric criterion for lifting):

Let $0 \rightarrow E \rightarrow V \rightarrow E^* \rightarrow 0$ be an orthogonal extension. Then the following are equivalent:

- The class $[V]$ belongs to $\text{Sec}^k \text{Gr}(2, E)$.
- Some elementary transformation $0 \rightarrow F \rightarrow E^* \rightarrow \tau \rightarrow 0$ with $\deg \tau = 2l \leq 2k$ lifts to a Lagrangian subbundle of V .

In particular, $\deg F \equiv \deg E \pmod{2}$.

Remark. If E is stable of slope < -1 then ψ is an embedding $\text{Gr}(2, E) \hookrightarrow \mathbb{P}H^1(\wedge^2 E)$, but this is not essential for the statement.

First application of the lifting criterion:

Lemma. A general point of \mathbb{A}_e represents a stable orthogonal bundle.

Idea of proof. Consider orthogonal extensions

$$0 \rightarrow E \rightarrow V \rightarrow E^* \rightarrow 0$$

where E is general of degree -1 . It is easy to show that a general such V is a stable orthogonal bundle.

Unpacking the geometric criterion for lifting, one can find stable V as above also admitting rank n isotropic subbundles lifting from E^* , of degree $1 - 2k$ for any $k \geq 2$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & E & \longrightarrow & V & \longrightarrow & E^* \longrightarrow 0 \\ & & \downarrow & & \swarrow & & \downarrow \\ & & & & F & & \end{array}$$

The subbundle F is not maximal, but shows that V also occurs in \mathbb{A}_{2k-1} . Thus for all $k \geq 1$, the parameter space \mathbb{A}_{2k-1} contains a stable point.

The same argument applied to a general E of degree -2 shows that there exist stable points in A_{2k-2} for all $k \geq 2$. \square

Corollary. For each e , the moduli map $\Phi_e: \mathbb{A}_e \dashrightarrow \mathcal{MO}(2n)$ is defined on an open subset.

Theorem (Topological classification). An orthogonal bundle V belongs to the component $\mathcal{MO}(2n)^+$ (resp., $\mathcal{MO}(2n)^-$) if and only if its Lagrangian subbundles have even degree (resp., odd degree).

Proof. Since each \mathbb{A}_e is connected, its image is entirely contained in either $\mathcal{MO}(2n)^+$ or $\mathcal{MO}(2n)^-$. Moreover, by an argument similar to Russo–Teixidor i Bigas (1999), the image $\Phi_e(\mathbb{A}_e)$ is dense in $\mathcal{MO}(2n; 2e)$.

We have seen that the images of \mathbb{A}_e and \mathbb{A}_{e+2k} intersect for all k . The trivial bundle belongs to $\mathcal{MO}(2n)^+$, and hence \mathbb{A}_{2k} maps to $\mathcal{MO}(2n)^+$ for all k .

But any orthogonal bundle of rank $2n$ has a Lagrangian subbundle. Therefore some and hence all \mathbb{A}_{2k+1} map to $\mathcal{MO}(2n)^-$.

Corollary. Suppose E_1 and E_2 are Lagrangian subbundles of an orthogonal vector bundle of rank $2n$. Then $\deg E_1$ and $\deg E_2$ have the same parity.

Corollary. For each $k \geq 0$, the stratum

$$\mathcal{MO}(2n; 2 \cdot (2k))$$

belongs to $\mathcal{MO}(2n)^+$, and the stratum

$$\mathcal{MO}(2n; 2 \cdot (2k + 1))$$

belongs to $\mathcal{MO}(2n)^-$.

The nongeneric strata

Theorem. Suppose $1 \leq e < \frac{1}{2}n(g - 1)$.

(1) The stratum $\mathcal{MO}(2n; 2e)$ is irreducible of dimension

$$\dim \mathbb{A}_e = (g - 1) \frac{n(3n - 1)}{2} + (n - 1)e.$$

(2) A general V in the stratum $\mathcal{MO}(2n; 2e)$ has a unique maximal Lagrangian subbundle of degree $-e$.

Idea of proof.

As mentioned before, $\Phi_e(\mathbb{A}_e)$ is dense in $\mathcal{MO}(2n; 2e)$.

Let E be a general bundle in $U(n, -e)$, and consider orthogonal extensions

$$0 \rightarrow E \rightarrow V \rightarrow E^* \rightarrow 0.$$

We will show that if E and $[V]$ are general, then V admits no Lagrangian subbundle of degree $\geq -e$ apart from E .

For this we require a generalisation of the lifting criterion.

Suppose F is a Lagrangian subbundle of V . We have a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H & \longrightarrow & F & \longrightarrow & F/H \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & E & \longrightarrow & V & \longrightarrow & E^* \longrightarrow 0 \end{array}$$

where H is a subbundle of E and F . The subbundle generated by E and F equals H^\perp , and H^\perp/H is an orthogonal extension

$$0 \rightarrow \frac{E}{H} \rightarrow \frac{H^\perp}{H} \rightarrow \left(\frac{E}{H}\right)^* \rightarrow 0.$$

Generalised lifting criterion: Fix $H \subset E$ of degree $-h$, and write $q: E \rightarrow E/H$. Then there exists F as above with $\deg F \geq -e$ if and only if the class $[H^\perp/H] = q_*^t q^*[V]$ belongs to

$$\mathrm{Sec}^{(e-h)} \mathrm{Gr}(2, E/H).$$

By the generalised lifting criterion, the dimension of the locus in $\mathbb{P}H^1(\wedge^2 E)$ of bundles admitting a Lagrangian subbundle of degree $\geq -e$ different from E is (abusing notation)

$$\dim\{H\} + \dim \text{Ker}(q_*^t q^*) + \dim \text{Sec}^{(e-h)} \text{Gr}(2, E/H)$$

(note that q varies with H).

One computes that for $e < \frac{1}{2}n(g-1)$, and general E , this is strictly smaller than $\dim \mathbb{P}H^1(\wedge^2 E)$.

Thus a general V represented in $H^1(\wedge^2 E)$ has the unique maximal Lagrangian subbundle E . \square

The two dense strata

The positions of the dense strata in $\mathcal{MO}(2n)^\pm$ depend on the congruence class of $n(g - 1)$ modulo 4. We consider just one case:

Theorem. Suppose $n(g - 1) \equiv 0 \pmod{4}$.

- (1) A generic bundle in $\mathcal{MO}(2n)^+$ has a finite number of maximal Lagrangian subbundles, each of degree $-\frac{1}{2}n(g - 1)$.
- (2) A generic bundle in $\mathcal{MO}(2n)^-$ has an $(n - 1)$ -dimensional family of maximal Lagrangian subbundles, each of degree
$$-\left(\frac{1}{2}n(g - 1) + 1\right).$$
(Analogous statements hold for $n(g - 1) \equiv 1, 2, 3 \pmod{4}$.)

Proof of (1). For $e = \frac{1}{2}n(g-1)$, we have $\dim \mathbb{A}_e = \dim \mathcal{MO}(2n)$. Thus we must show that the classifying map

$$\Phi_e: \mathbb{A}_e \dashrightarrow \mathcal{MO}(2n)^+$$

is dominant.

Let V be a general bundle in $\mathcal{MO}(2n)^+$, Then V is an extension $0 \rightarrow F \rightarrow V \rightarrow F^* \rightarrow 0$ for some Lagrangian subbundle $F \subset V$.

We have $\deg F = -f \leq -\frac{1}{2}n(g-1)$ since V is general. Furthermore, $\deg F$ is even.

To show that V in fact has a subbundle of degree $-\frac{1}{2}n(g-1)$, we need to prove that

$$\text{Sec}_{\frac{1}{2}(f+n(g-1)/2)} \text{Gr}(2, F) \text{ fills up } \mathbb{P}H^1(\wedge^2 F).$$

A dimension count shows this is the case if the secant variety is nondefective.

To show the nondefectivity of $\text{Gr}(2, F)$, we use:

Lemma (Terracini): For any variety Y in projective space,

$$\dim \text{Sec}^k Y = \dim \text{Span}(\mathbb{T}_{y_1} Y \cup \dots \cup \mathbb{T}_{y_k} Y)$$

where the y_i are general points of Y and $\mathbb{T}_{y_i} Y$ is the embedded tangent space of Y at y_i .

One proves that the span of k embedded tangent spaces of $\text{Gr}(2, F)$ is given by

$$\mathbb{P}\text{Ker} \left(H^1(\wedge^2 F) \rightarrow H^1(\wedge^2 \overline{F}) \right)$$

where \overline{F} is a certain elementary transformation of F . An adaptation of Hirschowitz's lemma then shows that $h^0(\wedge^2 \overline{F})$ is minimal if F and the points are general. In this case the above span has the required dimension.

How do the classical and Lagrangian Segre invariants interact?

Hirschowitz (1988) showed that any rank $2n$ vector bundle has a rank n subbundle of degree at least $-\lceil \frac{1}{2}n(g-1) \rceil$.

However, by what we have seen, a general orthogonal bundle V in the top (“extra”) stratum of $MO(2n)$ has no *Lagrangian* subbundles of this degree or greater.

Thus $t(V) > \frac{2}{\text{rank } V} \cdot s_n(V)$ for such V .

What happens in the nongeneric strata is still quite open.

Remark

All maximal vector subbundles E of a general V in the top stratum are nonisotropic. Thus they satisfy $h^0(\text{Sym}^2 E^*) \geq 1$.

When $n(g-1)$ is even, this is a condition of positive codimension on E .

Thus in this case, none of the maximal vector subbundles of such a V are general.

Orthogonal bundles of odd rank

Recall from linear algebra that there is a canonical 2:1 map

$$\begin{aligned} \{ \text{Lagrangian subspaces of } \mathbb{C}^{2n+2} \} \\ \rightarrow \{ \text{Lagrangian subspaces of } \mathbb{C}^{2n+1} \} \end{aligned}$$

given by $\Gamma \mapsto \Gamma \cap P$ where $P = \mathbb{C}^{2n+1}$ is an orthogonal subspace of \mathbb{C}^{2n+2} .

Analogously:

Lemma. Let V be an orthogonal bundle of rank $2n+1$. Then there is a canonical 2:1 map

$$\begin{aligned} \{ \text{Lagrangian subbundles of } V \perp O_X \} \\ \rightarrow \{ \text{Lagrangian subbundles of } V \} \end{aligned}$$

given by $\tilde{E} \mapsto \tilde{E} \cap V$. Moreover, $\det \tilde{E} = \det (\tilde{E} \cap V)$.

Segre loci in the odd rank case

For $e \geq 0$, we consider the loci $\mathcal{MO}(2n+1; 2e)$ of bundles admitting a Lagrangian subbundle of degree $-e$.

Proposition. Let E_1 and E_2 be Lagrangian subbundles of an orthogonal bundle V of rank $2n+1$. Then $\deg E_1 \equiv \deg E_2 \pmod{2}$.

Proof. For each i , we have $E_i = \tilde{E}_i \cap V$ for some Lagrangian subbundle $\tilde{E}_i \subset V^\perp O_X$. Then

$$\deg E_1 = \deg \tilde{E}_1 = \deg \tilde{E}_2 = \deg E_2$$

by our results on orthogonal bundles of even rank, together with the last lemma.

In a similar way, one can deduce the location of the strata from the even rank case:

Proposition. (1) The stratum $\mathcal{MO}(2n+1; 2 \cdot (2k))$ is contained in $\mathcal{MO}(2n+1)^+$, when it is nonempty.

(2) The stratum $\mathcal{MO}(2n+1; 2 \cdot (2k+1))$ is contained in $\mathcal{MO}(2n+1)^-$, when it is nonempty.

Results on the stratification

Theorem (Nongeneric strata): Suppose $1 \leq e < \frac{1}{2}(n+1)(g-1)$.

(1) The stratum $\mathcal{MO}(2n+1; 2e)$ is irreducible of dimension

$$\frac{1}{2}n(3n+1)(g-1) + ne.$$

(2) A general V in the stratum $\mathcal{MO}(2n+1; 2e)$ has a unique maximal Lagrangian subbundle of degree $-e$.

Orthogonal bundles of odd rank as iterated extensions

Suppose V is an orthogonal bundle of odd rank and E a Lagrangian subbundle. Write $F := E^\perp$. There is a diagram

$$\begin{array}{ccccccc}
 & & O_X & & & & \\
 & & \downarrow & & & & \\
 0 & \longrightarrow & E & \longrightarrow & V & \longrightarrow & F^* \longrightarrow 0 \\
 & & \downarrow j & & \downarrow \sim & & \downarrow \\
 0 & \longrightarrow & F & \longrightarrow & V^* & \longrightarrow & E^* \longrightarrow 0 \\
 & & \downarrow & & & & \\
 & & O_X & & & &
 \end{array}$$

Lemma. A diagram as above is induced by an orthogonal structure on V if and only if $j_*[V]$ belongs to the subspace $H^1(\wedge^2 F)$ of $H^1(\mathrm{Hom}(F^*, F))$.

The generic strata

Theorem. For any $V \in \mathcal{MO}(2n+1)$, we have

$$t(V) \leq (n+1)(g-1) + 3.$$

More precisely, the two even numbers t with

$$(n+1)(g-1) \leq t \leq (n+1)(g-1) + 3$$

correspond to the values of $t(V)$ for a general V in the components $\mathcal{MO}(2n+1)^{\pm}$.

As in the even rank case, the locations of the top strata and the dimensions of the spaces of maximal Lagrangian subbundles of a generic V depend on the class of $(n+1)(g-1)$ modulo 4.

Joint projects with Insong Choe (Konkuk University, Seoul)

- *A stratification on the moduli spaces of symplectic and orthogonal bundles over a curve*

Int. J. Math. **25** (2014) – arxiv:1204.0834

- *Lagrangian subbundles of orthogonal bundles of odd rank over an algebraic curve*

arxiv:1402.2816

- *Nondefectivity of Grassmann bundles*
(in progress)