Cohomological Hall algebras and Higgs bundles

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The big picture

- The starting point for this project is an attempt to get a deeper understanding, and eventually prove, the conjectures of Hausel, de Cataldo, Migliorini, Villegas et al on the cohomology of twisted character varieties $\mathcal{M}^{tw}_{g,n} :=$

  $$\{A_1, \ldots, A_g, A'_1, \ldots, A'_g | \prod(A_i, A'_i) = \exp(2\pi\sqrt{-1}/n)\mathrm{Id}_{n\times n}\}/\mathrm{PGL}_n(\mathbb{C})$$

- In fact these spaces seem to be a little off centre from "the big picture" which involves instead $\mathcal{M}_{g,n}$, the stack of representations of $\pi_1(\Sigma_g)$.
- The relation between the cohomology of $\mathcal{M}^{tw}_{g,n}$ and the cohomology of $\mathcal{M}_{g,n}$ turns out to be precisely formulated in terms of BPS state counting in string theory/Donaldson-Thomas theory for 3-Calabi-Yau categories.
- In more detail, the conjecture is that $\mathcal{H}_{B_g,W_g} := \bigoplus \mathrm{H}_c(\mathcal{M}_{g,n})^*$ is an associative algebra satisfying a PBW type theorem, establishing that

  $$\mathcal{H}_{B_g,W_g} = \mathrm{Sym}(\bigoplus_n \mathrm{H}(\mathcal{M}^{tw}_{g,n}) \otimes \mathrm{H}(B\,\mathbb{C}^*))$$

The 3d theory: Smooth algebras with superpotential

The construction of a 'critical' cohomological Hall algebra (CoHA) will always start with the same input, and the associated 3-Calabi-Yau category.

Jacobi algebra starting data

1. B will be a nc smooth algebra. One way to express nc smoothness is via the nc analogue of Grothendieck's criterion for smoothness in the commutative world: for every map of algebras $f : B \to A/I$ where I is a two sided nilpotent ideal in A, there is a lift to a map $\tilde{f} : B \to A$. **Examples: 1)** $B = \mathbb{C}Q$ for Q a quiver, or **2)** $B = \mathbb{C}\langle x_1^{\pm 1}, \ldots, x_a^{\pm 1}\rangle$.
2. $W \in HH_0(B) = B/[B,B]$ will be a 'noncommutative function' on B.

- If $B = \mathbb{C}\langle x_1, \ldots, x_i | r_1, \ldots, r_j\rangle$ is smooth let $\mathcal{R}ep_n(B) = [\mathrm{Rep}_n(B)/\mathrm{GL}_n(\mathbb{C})] = [Z(r_1, \ldots, r_j)/\mathrm{GL}_n(\mathbb{C})] \subset [\mathrm{Mat}_{n\times n}(\mathbb{C})^{\times i}/\mathrm{GL}_n(\mathbb{C})]$ be the stack of n-dimensional B-modules – this stack is *always smooth*.
- $\mathrm{Tr}(W)$ defines a function on $\mathcal{R}ep_n(B)$.

The (categorically) 3d theory: the Jacobi algebra

The nc derivatives of W generate a 2-sided ideal I_W in B, and we define $\mathrm{Jac}(B, W) = B/I_W$.

Example

In the quiver case, think of W as a linear combination of cyclic words in Q, then

$$I_W = \langle \partial W/\partial a | a \in Q_1 \rangle$$

where $\partial W/\partial a$ is obtained by cyclically permuting each instance of a to the front then deleting it.

Example

$B = \mathbb{C}\langle x^{\pm 1}, y^{\pm 1}, z^{\pm 1}\rangle$, $W = xyz - xzy$, then
$\mathrm{Jac}(B, W) = \mathbb{C}\langle x^{\pm 1}, y^{\pm 1}, z^{\pm 1}\rangle / \langle yz - zy, zx - xz, xy - yx\rangle \cong \mathbb{C}[\pi_1(\Sigma_1 \times S^1)]$.
$\mathcal{R}ep_n(\mathrm{Jac}(B, W))$ can be identified with the stack of length n zero-dimensional sheaves on $(\mathbb{C}^*)^3$.

Vanishing cycles from scratch

The lesson from DT theory: Vanishing cycles provide the right coefficient system for counting objects in 3-Calabi-Yau categories.Let $f : X \to \mathbb{C}$ be a regular function on a smooth variety X. Let $X_{<0} := f^{-1}(\mathbb{R}_{<0})$, $X_0 = f^{-1}(0)$.

Definition

The sheaf of nearby cycles $\psi_f \in D^b(X)$ is defined by
$\psi_f := (X_0 \to X)_*(X_0 \to X)^*(X_{<0} \to X)_*\mathbb{Q}_{X_{<0}}$.
$\phi_f := \mathrm{cone}(\mathbb{Q}_{X_0} \to \psi_f)[-1]$ (i.e. $(X_0 \to X)_*(X_0 \to X)^*$ applied to the adjunction $\mathbb{Q}_X \to (X_{<0} \to X)_*(X_{<0} \to X)^*\mathbb{Q}_X$).

Example

Let $X = \mathbb{A}^1$, let $f = x^d$. Then $\psi_f = \mathbb{Q}_0^{\oplus d}$, and $\phi_f = \mathbb{Q}_0^{\oplus d-1}[-1]$.

Example

Let $X = \mathbb{A}^2$, let $f = xy$. Then $\mathrm{H}_c^t(\phi_f) = 0$ if $t \neq 2$, and $\mathrm{H}_c^2(\phi_f) \cong \mathbb{Q}$.

Hall algebra structure

Let $\mathcal{R}ep_{n,m}(B) \subset \left[\begin{pmatrix} \mathrm{Mat}_{n\times n}(\mathbb{C}) & \mathrm{Mat}_{n\times m}(\mathbb{C}) \\ 0 & \mathrm{Mat}_{m\times m}(\mathbb{C}) \end{pmatrix}^{\times i} / \mathrm{GL}_{n,m}(\mathbb{C}) \right]$ be the stack of 2-step flags of representations of B (recall that we assumed that B has i generators). There is a diagram of stacks

$$\begin{tikzcd} \mathcal{R}ep_{n,m}(B) \arrow[r, "\pi_1\times\pi_3"'] \arrow[d, "\pi_2"] & \mathcal{R}ep_n(B) \times \mathcal{R}ep_m(B) \\ \mathcal{R}ep_{n+m}(B) & \end{tikzcd}$$

and via $(\pi_2)_*(\pi_1 \times \pi_3)^*$ we get an associative multiplication $\mathbf{m}$ on $\mathcal{H}_{B,W} := \bigoplus_n \mathcal{H}_{B,W,n} := \bigoplus_n \mathrm{H}_c(\mathcal{R}ep_n(B), \phi_{\mathrm{Tr}(W)_n})^*$.

Definition

$(\mathcal{H}_{B,W}, \mathbf{m})$ is the the critical/3d *CoHA* associated to (B, W), it is a $\mathbb{Z}^2$-graded algebra in the category of mixed Hodge structures by results of Steenbrink, Navarro Aznar, Kontsevich and Soibelman.

Extra structure on $\mathcal{H}_{B,W}$.

Question: How to push forward compactly supported cohomology with coefficients in the complex of vanishing cycles?? **Answer:** use Verdier duality on the *smooth* $\mathcal{R}ep_n(B)$.

Hidden smoothness

A feature that is exploited in the definition of the multiplication, as well as all the structural results that follow, is that the cohomologies we are using can be treated as the cohomologies of smooth spaces, due to the shifted Verdier self-duality of the vanishing cycle complex.

We can use Verdier duality to define appropriate umkehr (wrong way) maps in compactly supported cohomology for both $\pi_1 \times \pi_3$ and π_2, producing a (localized) coproduct $\Delta : \mathcal{H}_{B,W} \to \mathcal{H}_{B,W} \otimes \mathcal{H}_{B,W}$.

Theorem:

The structure $(\mathcal{H}_{B,W}, m, \Delta)$ forms a Hopf algebra, i.e. Δ is an algebra homomorphism.

How to calculate $\mathcal{H}_{B,W}$ part 1:) PBW basis

The Hermitian inner product on $\mathbb{C}^n$ induces a U_n-equivariant isomorphism $f : \mathrm{Rep}_n(B) \to \mathrm{Rep}_n(B^{op})$.

Definition

Say we have an isomorphism $g : B \to B^{op}$ inducing a map $g^* : \mathrm{Rep}_n(B^{op}) \to \mathrm{Rep}_n(B)$ such that $\mathrm{Tr}(W)_n = \lambda\overline{\mathrm{Tr}}(W)_n g^* f$ for a fixed $\lambda \in \mathbb{C}^*$. Then we say that g is a *self-duality structure* for (B, W).

If $B = \mathbb{C}Q$ this is a natural condition: a self-duality is an isomorphism of quivers $\theta : Q \to Q^{op}$, fixing the vertices, such that $\theta^* W^{op} = \lambda W$. g induces a Hopf algebra anti-automorphism $g^* f : \mathcal{H}_{B,W} \to \mathcal{H}_{B,W}$.

Theorem:

If (B, W) admits a self-duality structure, with $\mathrm{H}_c(g^* f) = \mathrm{id}$, then $\mathcal{H}_{B,W}$ is supercommutative, and so is *free* supercommutative. Under weaker hypotheses we still have that $\mathcal{H}_{B,W}$ is a PBW algebra.

How to calculate $\mathcal{H}_{B,W}$ part 2:) Dimensional reduction

(*A 2d algebra*): Say $B = \langle x_1, \ldots, x_i | r_1, \ldots, r_j \rangle$ carries a $\mathbb{C}^*$ action that scales the $x_1, \ldots, x_{i'}$ for $i' < i$, and acts trivially on the remaining generators, and that W has weight one. Then define $\mathrm{Jac}_{2d}(B, W) = \mathbb{C}\langle x_{i'+1}, \ldots, x_i | r_1, \ldots, r_j, \partial W/\partial x_1, \ldots, \partial W/\partial x_{i'} \rangle$

Dimensional reduction theorem:

$\mathrm{H}_c(\mathcal{R}ep_n(\mathrm{Jac}(B, W)), \phi_{\mathrm{Tr}_W}) \cong \mathrm{H}_c(\mathcal{R}ep_n(\mathrm{Jac}_{2d}(B, W)))$

Example (The original Behrend-Bryan-Szendrői example)

Let $B = \mathbb{C}\langle x, y, z \rangle$, $W = xyz - xzy$, then if we let $\mathbb{C}^*$ scale x, we get $\mathrm{Jac}_{2d}(B, W) \cong \mathbb{C}[y, z]$, and

$$\mathrm{H}_c(\mathcal{R}ep_n(B), \phi_{\mathrm{Tr}_W}) \cong \mathrm{H}_c(\mathcal{R}ep_n(\mathbb{C}[y, z])).$$

(and so $\chi_q(\mathrm{H}_c(\mathcal{R}ep_n(B), \phi_{\mathrm{Tr}_W}), q) = \chi_q(\mathrm{H}_c(\mathcal{R}ep_n(\mathbb{C}[y, z])), q)$).

$\mathbb{C}[\pi_1(\Sigma_g)]$ as a 2d Jacobi algebra

Coming back to Σ_g, we'll use the following:

Proposition:

1. There exists a function W_g on the smooth algebra $B_g := \mathbb{C}\langle x_1, \ldots, x_{g+1}, y_1^{\pm 1}, \ldots, y_{3g+3}^{\pm 1}\rangle$ such that $\mathrm{Jac}(B_g, W_g) \cong \mathbb{C}[\pi_1(\Sigma_g)][z]$.
2. The pair (B_g, W_g) carries a self duality structure.
3. W_g is homogeneous of weight one, after giving B_g the action that scales the x variables and leaves invariant the y variables. There is an isomorphism

$$\begin{aligned}\mathrm{Jac}_{2d}(B_g, W_g) &:= \mathbb{C}\langle y_1^{\pm 1}, \ldots, y_{3g}^{\pm 1} | \partial W/\partial x_1, \ldots, \partial W/\partial x_{g+1}\rangle \\ &\cong \mathbb{C}[\pi_1(\Sigma_g)]\end{aligned}$$

The CoHA $\mathcal{H}_{B_g,W_g}$

Corollary

1. There is an associative algebra structure on the mixed Hodge structure

$$\begin{aligned}\mathcal{H}_{B_g,W_g} &= \bigoplus_n \mathrm{H}_c(\mathcal{R}ep_n(B_g, \phi_{\mathrm{Tr}(W_g)})^* \\ &\cong \bigoplus_n \mathrm{H}_c(\mathcal{R}ep_n(\mathbb{C}[\pi_1(\Sigma_g)][z]), \phi_{\mathrm{Tr}(W_g)})^*\end{aligned}$$

2. This algebra is (conjecturally) **free supercommutative** with respect to the cohomological grading (or at least admits a PBW theorem).
3. There is an isomorphism of mixed Hodge structures

$$\mathrm{H}_c(\mathcal{R}ep_n(B_g), \phi_{\mathrm{Tr}(W_g)}) \cong \mathrm{H}_c(\mathcal{R}ep_n(\mathbb{C}[\pi_1(\Sigma_g)])) \text{ (dimensional reduction)}$$

and so $\bigoplus_n \mathrm{H}_c(\mathcal{R}ep_n(\mathbb{C}[\pi_1(\Sigma_g)]))^*$ is a (conjecturally) free supercommutative algebra in the category of $\mathbb{Z}^2$-graded mixed Hodge structures.

A conjectural space of generators

Define

$$M^{tw}_{g,n} := \{A_1, \dots, A_g, A'_1, \dots, A'_g \mid \prod (A_i, A'_i) = \exp(2\pi\sqrt{-1}/n)\mathrm{Id}_{n\times n}\},$$
$$\mathcal{M}^{tw}_{g,n} := M^{tw}_{g,n}/\,\mathrm{PGL}_n(\mathbb{C}) \ \leftarrow \text{a smooth variety!}$$

Conjecture:

There is an isomorphism of mixed Hodge structures

$$\bigoplus_n \mathrm{H}_c(\mathcal{R}ep_n(\mathbb{C}[\pi_1(\Sigma_g)]))^* \cong \mathbf{Sym}\left(\mathrm{H}([M^{tw}_{g,n}/\,\mathrm{GL}_n(\mathbb{C})])\right)$$
$$\cong \mathbf{Sym}\left(\mathrm{H}(\mathcal{M}^{tw}_{g,n}) \otimes \mathrm{H}(B\,\mathbb{C}^*)\right).$$

Assuming the conjecture, all the HRV conjectures on the mixed Hodge structures of twisted character varieties have twin equivalent conjectures regarding the cohomology of untwisted character varieties. In addition, I'll conjecture that there is a Hall algebra structure on the cohomology of the stack of semistable degree zero Higgs bundles preserving the perverse

The end!