VBAC coming from VHS and new surfaces with $p_g = q$.

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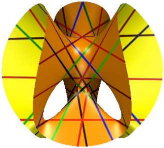

Outline

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Fujita's first theorem

An important progress in classification theory was stimulated by a theorem of Fujita, who showed
(*On Kähler fiber spaces over curves, J. Math. Soc. Japan 30 (1978), no. 4, 779–794*):

Theorem

If X is a compact Kähler manifold and $f : X \to B$ is a fibration onto a projective curve B (i.e., f has connected fibres), then the direct image sheaf

$$V := f_* \omega_{X|B} = f_*(\mathcal{O}_X(K_X - f^* K_B))$$

is a nef vector bundle on B.
'Nef' means that each quotient bundle Q of V has degree $\deg(Q) \geq 0$; sometimes, instead of the word nef, one uses the terminology 'V is numerically semipositive'.

Kawamata's theorem

Soon afterwards, using Griffihts' results on Variation of Hodge Structures, since the fibre of $V := f_* \omega_{X|B}$ over a point $b \in B$ such that $X_b := f^{-1}(b)$ is smooth is the vector space $V_b = H^0(X_b, \Omega^{n-1}_{X_b})$, Kawamata improved on Fujita's result, solving a long standing problem and proving the subadditivity of Kodaira dimension for such fibrations,

$$Kod(X) \geq Kod(B) + Kod(F),$$

(here F is a general fibre) showing the semipositivity also for the direct image of higher powers of the relative dualizing sheaf

$$W_m := f_*(\omega_{X|B}^{\otimes m}) = f_*(\mathcal{O}_X(m(K_X - f^* K_B))).$$

Kawamata also extended his result to the case where the dimension of the base variety B is > 1.

Fujita's second theorem

In the note *The sheaf of relative canonical forms of a Kähler fiber space over a curve* Proc. Japan Acad. Ser. A Math. Sci. 54 (1978), no. 7, 183–184, Fujita announced the following stronger result, sketching the idea of proof, but referring to a forthcoming article concerning the positivity of the so-called local exponents (this article was never written).

Theorem

(Fujita 's second theorem)
Let $f : X \to B$ be a fibration of a compact Kähler manifold X over a projective curve B, and consider the direct image sheaf

$$V := f_* \omega_{X|B} = f_*(\mathcal{O}_X(K_X - f^* K_B)).$$

Then V splits as a direct sum $V = A \oplus Q$, where A is an ample vector bundle and Q is a unitary flat bundle.

Ample, semiample, nef

Let V be a holomorphic vector bundle over a projective curve B.

Definition

Let $p : \mathbb{P} := Proj(V) = \mathbb{P}(V^\vee) \to B$ be the associated projective bundle, and let H be a hyperplane divisor (s.t. $p_*(\mathcal{O}_{\mathbb{P}}(H)) = V$). Then V is said to be:
(NP) numerically semi-positive if and only if every quotient bundle Q of V has degree $deg(Q) \geq 0$,
(NEF) nef if and only if H is nef on $\mathbb{P}$,
(A) ample if and only if H is ample on $\mathbb{P}$
(SA) semi-ample if and only H is semi-ample on $\mathbb{P}$ (there is a positive multiple mH yielding a morphism).
Recall that (A) $\Rightarrow$ (SA) $\Rightarrow$ (NEF) $\Leftrightarrow$(NP).

Flat and unitary flat bundles

Definition

A flat holomorphic vector bundle on a complex manifold M is a holomorphic vector bundle $\mathcal{H} := \mathcal{O}_M \otimes_{\mathbb{C}} \mathbb{H}$, where $\mathbb{H}$ is a local system of complex vector spaces associated to a representation $\rho : \pi_1(M) \to GL(r, \mathbb{C})$,

$$\mathbb{H} := (\tilde{M} \times \mathbb{C}^r)/\pi_1(M),$$

$\tilde{M}$ being the universal cover of M (so that $M = \tilde{M}/\pi_1(M)$). We say that $\mathcal{H}$ is unitary flat if it is associated to a representation $\rho : \pi_1(M) \to U(r, \mathbb{C})$.

Outline

Fujita's question

Recall Fujita's second theorem, for which a complete proof was given in our joint work with Michael Dettweiler (arXiv 1311.3232 and CRAS Ser. I, 352 (2014), 241-244)

Theorem

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Let $f : X \to B$ be a fibration of a compact Kähler manifold X over a projective curve B. Then $V := f_* \omega_{X|B} = f_*(\mathcal{O}_X(K_X - f^* K_B))$ splits as $V = A \oplus Q$, with A an ample vector bundle and Q a unitary flat bundle.

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Fujita posed in 1982 (Proceedings of the 1982 Taniguchi Conference) the following

Question

(Fujita) *Is the direct image $V := f_* \omega_{X|B}$ semi-ample ?*

Fujita's theorem and Fujita's question

The following result is due to Hartshorne:

Proposition

A vector bundle V on a curve is nef if and only it is numerically semi-positive, i.e., if and only if every quotient bundle Q of V has degree $deg(Q) \geq 0$, and V is ample if and only if every quotient bundle Q of V has degree $deg(Q) > 0$.

Then there is a technical result we established, which clarifies how Fujita's question is related to Fujita's II theorem

Theorem

Let $\mathcal{H}$ be a unitary flat vector bundle on a projective manifold M, associated to a representation $\rho : \pi_1(M) \to U(r, \mathbb{C})$. Then $\mathcal{H}$ is nef and moreover $\mathcal{H}$ is semi-ample if and only if $Im(\rho)$ is finite.

Answer to Fujita's question

This is the main new result in our joint work with Dettweiler:

Theorem

There exist surfaces X of general type endowed with a fibration $f : X \to B$ onto a curve B of genus ≥ 3, and with fibres of genus 6, such that $V := f_* \omega_{X|B}$ splits as a direct sum $V = A \oplus Q_1 \oplus Q_2$, where A is an ample rank-2 vector bundle, and the flat unitary rank-2 summands Q_1, Q_2 have infinite monodromy group (i.e., the image of ρ_j is infinite). In particular, V is not semi-ample.

Thus Fujita's question has a negative answer in general.

Cases where V is semiample.

Corollary

Let $f : X \to B$ be a fibration of a compact Kähler manifold X over a projective curve B. Then $V := f_* \omega_{X|B}$ is a direct sum $V = A \bigoplus (\oplus_{i=1}^h Q_i)$, with A ample and each Q_i unitary flat without any nontrivial degree zero quotient. Moreover,
(I) if Q_i has rank equal to 1, then it is a torsion bundle ($\exists$ m such that $Q_i^{\otimes m}$ is trivial) (Deligne)
(II) if the curve B has genus 1, then rank $(Q_i) = 1, \ \forall i$.
(III) In particular, if B has genus at most 1, then V is semi-ample.

(I) This was proven by Deligne (and by Simpson using the theorem of Gelfond-Schneider)
(II) Follows since $\pi_1(B)$ is abelian,if B has genus 1: hence every representation splits as a direct sum of 1-dimensional ones.

Flat versus unitary flat

While a unitary flat bundle is nef, the same does not hold for a flat bundle. This is no surprise by the following
Theorem (Weil-Atiyah) A holomorphic vector bundle V on a curve is flat if and only, given its decomposition $V = \bigoplus V_i$ into indecomposable summands, each V_i has degree equal to zero.

Flat but not nef

In our situation we have
Theorem (C-Dettweiler) *Let $f : X \to B$ be a Kodaira fibration, i.e., X is a surface and all the fibres of f are smooth curves of genus $g \geq 2$ not all isomorphic to each other.*
Then $V := f_*\omega_{X|B}$ has strictly positive degree, hence $\mathcal{H} := R^1 f_*(\mathbb{C}) \otimes \mathcal{O}_B$ is a flat bundle which is not nef (nor semistable).

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Proof 1) Since all the fibres of f are smooth, $V = f_*(\Omega^1_{X|B})$ and we have an exact sequence

$$0 \to V \to \mathcal{H} \to V^\vee \to 0,$$

and it suffices to show that the degree of the quotient bundle $V^\vee$ is strictly negative, or, equivalently, $\deg(V) > 0$.

Flat versus unitary flat, cont.

We want to show that $\deg(V) > 0$, as proven by Kodaira (this follows also from the results of Kawamata and Arakelov).
We have that

$$12 \deg(V) = K_X^2 - 8(b-1)(g-1),$$

where g is the genus of the fibres of f, and b is the genus of B, since f is a differentiable fibre bundle, and we have for the Euler- Poincaré characteristic of X

$$e(X) = 4(b-1)(g-1).$$

Kodaira proved that for such fibrations the topological index $\sigma(X)$(signature of the intersection form on $H^2(X, \mathbb{R})$) is positive. By the index theorem we have

$$0 < 3\sigma(X) = c_1^2(X) - 2c_2(X) = K_X^2 - 2e(X) = \deg(V).$$

Outline

Curvature decreases in subbundles?

The example of Kodaira fibrations produces subbundles of a flat bundle (they have zero curvature) which are positively curved. Does this contradict the slogan above? Not really, the correct principle is (see the book by Griffiths and Harris): **curvature decreases in Hermitian subbundles.** The above principle is the first ingredient in the proof of the theorem mentioned above.

Theorem

Let $\mathcal{H}$ be a unitary flat vector bundle on a projective manifold M, associated to a representation $\rho : \pi_1(M) \to U(r, \mathbb{C})$. Then $\mathcal{H}$ is nef and moreover $\mathcal{H}$ is semi-ample if and only if $Im(\rho)$ is finite.

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Since $\mathcal{H}$ is unitary flat, $\mathcal{H}$ is a Hermitian holomorphic bundle, and by the principle 'curvature decreases in Hermitian subbundles' each subbundle has degree ≤ 0 and each quotient bundle W of $\mathcal{H}$ has degree ≥ 0, hence $\mathcal{H}$ is nef.

Unitary flat bundles

If $\mathcal{H}$ is unitary flat, $\mathcal{H}$ is nef since ‘curvature decreases in Hermitian subbundles’ each quotient bundle W has degree ≥ 0.

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Assume it is semiample: to show that the representation is finite, by Lefschetz ' theorem, we can reduce to the case where M is a curve.
Let B be a projective curve, let $\rho : \pi_1(B) \to U(r, \mathbb{C})$ be a unitary representation, and let $\mathcal{H}_\rho$ be the associated flat holomorphic bundle. Since ρ is unitary, it is a direct sum of irreducible unitary representations $\rho_j, j = 1, \ldots k$.

Unitary flat bundles

If $\mathcal{H}$ is unitary flat, $\mathcal{H}$ is nef since 'curvature decreases in Hermitian subbundles' each quotient bundle W has degree ≥ 0.
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Let B be a projective curve, let $\rho : \pi_1(B) \to U(r, \mathbb{C})$ be a unitary representation, and let $\mathcal{H}_\rho$ be the associated flat holomorphic bundle. Since ρ is unitary, it is a direct sum of irreducible unitary representations $\rho_j, j = 1, \dots k$.
Accordingly, we have a splitting

$$\mathcal{H}_\rho = \oplus_{j=1}^{k} \mathcal{H}_{\rho_j}.$$

Narasimhan and Seshadri have proven that each $\mathcal{H}_{\rho_j}$ is a stable degree zero holomorphic bundle on B. This result plays another crucial role in the proof of the above theorem.

Curvature and numerical positivity

Definition

Let (E, h) be a Hermitian vector bundle on a complex manifold M. Take the canonical Chern connection associated to the Hermitian metric h, and denote by $\Theta(E, h)$ the associated Hermitian curvature, which gives a Hermitian form on the complex vector bundle bundle $T_M \otimes E$.
Then one says that E is Nakano positive (resp.: semi-positive) if there exists a Hermitian metric h such that the Hermitian form associated to $\Theta(E, h)$ is strictly positive definite (resp.: semi-positive definite).

Remark

Umemura proved that a vector bundle V over a curve B is positive (i.e., Griffiths positive, or equivalently Nakano positive) if and only if V is ample.

Outline

Idea of proof in the case of no singular fibres

V is a holomorphic subbundle of the holomorphic vector bundle $\mathcal{H}$ associated to the local system
$\mathbb{H} := \mathcal{R}^m f_*(\mathbb{Z}_X),\ m = \dim(X) - 1$ (i.e., $\mathcal{H} = \mathbb{H} \otimes_{\mathbb{Z}} \mathcal{O}_B$).
The bundle $\mathcal{H}$ is flat, hence the curvature $\Theta_{\mathcal{H}}$ associated to the flat connection satisfies $\Theta_{\mathcal{H}} \equiv 0$.
We view V as a holomorphic subbundle of $\mathcal{H}$, while

$$V^\vee \cong R^m f_* \mathcal{O}_X,\ m = \dim(X) - 1$$

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$$V^\vee \cong R^m f_* \mathcal{O}_X, \; m = \dim(X) - 1$$

is a holomorphic quotient bundle of $\mathcal{H}$. The curvature formula for subbundles gives (σ is the II fundamental form)

$$\Theta_V = \Theta_{\mathcal{H}}|_V + \bar{\sigma}\ {}^t\sigma = \bar{\sigma}\ {}^t\sigma,$$

and Griffiths proves that the curvature of $V^\vee$ is semi-negative, since its local expression is of the form $ih'(z) d\bar{z} \wedge dz$, where $h'(z)$ is a semi-positive definite Hermitian matrix.

The case of no singular fibres

In particular we have that the curvature Θ_V of V is semipositive and, moreover, that the curvature vanishes identically if and only if the second fundamental form σ vanishes identically, i.e., if and only if V is a flat subbundle.
However, by semi-positivity, we get that the curvature vanishes identically if and only its integral, the degree of V, equals zero.
Hence V is a flat bundle if and only if it has degree 0.
The same result then holds true, by a similar reasoning, for each holomorphic quotient bundle Q.

The general case

In the general case we use:
1) The semistable reduction theorem (a base change $B' \to B$ such that all fibres of the pull-back $X' \to B'$ are reduced with normal crossings)
2) A comparison of the pull-back of V with the analogously defined V'
3) Some crucial estimates given by Zucker (using Schmid's asymptotics for Hodge structures) for the growth of the norm of sections of the L^2-extension of Hodge bundles, and

The general case, cont.

4) A lemma by Kawamata

Lemma

Let L be a holomorphic line bundle over a projective curve B, and assume that L admits a singular metric h which is regular outside of a finite set S and has at most logarithmic growth at the points $p \in S$.
Then the first Chern form $c_1(L, h) := \Theta_h$ is integrable on B, and its integral equals $deg(L)$.

This shows that in the semistable case singularities are ininfluent, and the argument runs as in the case of no singular fibres.

Outline

Symmetry by a cyclic group of order 7

Proposition

Let $f : X \to B$ be a semistable fibration of a surface X onto a projective curve, such that the group $G = \mu_7 \cong \mathbb{Z}/7$ acts on this fibration inducing the identity on B. Assume that the general fibre F has genus 6 and that G has exactly 4 fixed points on F, with tangential characters $(1, 1, 1, 4)$.
Then if we split $V = f_*(\omega_{X|B})$ into eigensheaves, then the eigensheaves V_1, V_2 are unitary flat rank 2 bundles.

Symmetry by a cyclic group of order 7, cont.

Idea of proof:

- we show that V_1, V_2 have rank 2, V_3, V_4 have rank 1, $V_5 = V_6 = 0$
- Let $\mathcal{H}_j := \mathbb{H}_j \otimes \mathcal{O}_B$: for $j = 1, 2$ we have that $(\bar{V})_j = \overline{V_{7-j}} = 0$, hence $V_j = \mathcal{H}_j$ over $B^* = B \setminus S$, S being the set of critical values of f.
- We saw that the norm of a local frame of V_j has at most logarithmic grow at the points $p \in S$. This shows that V_j is a subsheaf of $\mathcal{H}_j$: by semipositivity we conclude that we have equality $V_j = \mathcal{H}_j$.

Symmetry by a cyclic group of order 7, cont.

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Since the fibration is semistable, the local monodromies are unipotent: on the other hand, they are unitary, hence they must be trivial. This implies that the local systems $\mathbb{H}_1^*$ and $\mathbb{H}_2^*$ have respective flat extensions to local systems $\mathbb{H}_1$ and $\mathbb{H}_2$ on the whole curve B.

The examples

The equation

$$z_1^7 = y_1 y_0 (y_1 - y_0)(x_0 y_1 - x_1 y_0)^4 x_0^3.$$

describes a singular surface Σ' which is a cyclic covering of $\mathbb{P}^1 \times \mathbb{P}^1$ with group $G := \mathbb{Z}/7$.
Let Y be a minimal resolution of singularities of Σ: Y admits a fibration $\varphi : Y \to \mathbb{P}^1$ with fibres curves of genus 6.
We let X be the minimal resolution of the fibre product of $\varphi : Y \to \mathbb{P}^1$ with $\psi : B \to \mathbb{P}^1$, where ψ is the G-Galois cover branched on $\infty = \{x_0 = 0\}, 0 = \{x_1 = 0\}, 1 = \{x_1 = x_0\}$, and with local characters $(1, 1, -2)$. In particular B has genus 3 by Hurwitz' formula.

Properties of the example

Theorem

The above surface X is a surface of general type endowed with a fibration $f : X \to B$ onto a curve B of genus 3, and with fibres of genus 6, such that $V := f_* \omega_{X|B}$ splits as a direct sum $V = A \oplus Q_1 \oplus Q_2$, where A is an ample rank-2 vector bundle, and the unitary flat rank-2 summands Q_1, Q_2 have infinite monodromy.

The last assertion is a consequence of the classification by Schwarz of the cases where the monodromy of hypergeometric integrals is finite, as we now see.

Hypergeometric integrals

Another example is given by the equation

$$z_1^7 = y_1 y_0^4 (y_1 - y_0)(y_1 - x y_0), \quad x \in \mathbb{C} \setminus \{0, 1\}$$

which gives another family of curves. It is similar to the previous family, except that here V_1 is generated by

$$\eta := y^{-\frac{6}{7}}(y-1)^{-\frac{6}{7}}(y-x)^{-\frac{6}{7}} dy, \text{ and by } y \cdot \eta.$$

Varying x, we obtain a rank-2 local system over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, which is equivalent, in view of the Riemann-Hilbert correspondence, to a second order differential equation with regular singular points. Indeed, using results of Deligne-Mostow and Kohno, we see that we have a Gauss hypergeometric equation, and we can see that the local monodromies have order 7, hence we are not in the Schwarz list and the monodromy is infinite (and irreducible).

Outline

Surfaces fibred over elliptic curves

Let now $X = S$ be a surface and $B = E$ be an elliptic curve, and assume we have a fibration $f : S \to E$. Then, by Atiyah's classification of vector bundles on elliptic curves

$$V = (\oplus_j A_j) \bigoplus (\mathcal{O}_E^{q-1}) \bigoplus (\oplus_i Q_i),$$

where A_j is ample and indecomposable, Q_i is a nontrivial torsion line bundle, and $q := q(S) = h^1(\mathcal{O}_S)$.

Surfaces with $q = 1$

Let $X = S$ be a surface with $q := q(S) = 1$ so that the Albanese map yields a fibration onto an elliptic curve E. Then

$$V = (\oplus_j A_j) \bigoplus (\oplus_i Q_i),$$

where A_j is ample and indecomposable, Q_i is a nontrivial torsion line bundle, and $p_g := p_g(S) = h^0(V) = \sum_j h^0(A_j)$. If moreover $p_g = 1$, then

$$V = A_1 \bigoplus (\oplus_i Q_i),$$

where A_1 is indecomposable of degree 1 and uniquely determined by its rank, up to tensoring with a line bundle. With Ciliberto we proved many years ago:

Surfaces with $p_g = q = 1$

Theorem

(C. - Ciliberto) Let S be a surface with $p_g = q = 1$, let $f : S \to E$ the Albanese map and set $V = A_1 \bigoplus (\oplus_i^\lambda Q_i)$. Then the projectivization of the first summand is a symmetric product of the elliptic curve

$$\mathbb{P}(A_1^\vee) = E^{(\iota)}, \ \iota := g - \lambda$$

and the natural (rational map)

$$S \to E^{(\iota)}$$

is the paracanonical map associating to $x \in S$ the $\{t \in B\}$ such that $x \in C_t$ (here C_t is, for general t, the unique curve in $|K + t|$, and is called a paracanonical curve).

Status of the classification of surfaces with $p_g = q = 1$

For these surfaces $K^2 \in \{2, 3, \ldots, 9\}$ (Miyaoka-Yau inequality).

1. $K^2 = 2$: an irreducible moduli space (C. 1977, Horikawa 1978), S is a double cover of $E^{(2)}$, g=2
2. $K^2 = 3$: $g \leq 3$, there are exactly 5 irreducible connected components of the moduli space, one with $g = 3$ (C.- Ciliberto, 1989), 4 with $g = 2$ (C.- Pignatelli, 2004)
3. $K^2 = 4, 5$ exist, as I showed (98) with examples having $g = 2$; later Pignatelli showed: for $K^2 = 4, g = 2$ get more than 8 connected components !
4. $K^2 = 4, 6, 8$ exist by Polizzi, Rito, Frapporti-Pignatelli (quotients of products of curves) ($g = 3, 4, 5, 7$)
5. $K^2 = 7$ exist by Lei Zhang, Rito
6. $K^2 = 9$ existence shown by Cartwright and Steger

Outline

New surfaces with $p_g = q$

Theorem (Bauer, C., Frapporti)

There are 16 irreducible families of generalized Burniat type surfaces with $K_S^2 = 6$, $0 \leq p_g(S) = q(S) \leq 3$. Those with $p_g(S) = q(S) = 1$ are summarized in the following table, 5)-10) form 6 connected components of the moduli space, 11) and 12) are contained in a unique irreducible connected component of the moduli space.

Table: $p_g = q = 1$

	p_g	$H_1(S,\mathbb{Z})$	dim	
5)	1	$(\mathbb{Z}/2)^3 \times \mathbb{Z}^2$	3	
6)	1	$(\mathbb{Z}/2)^2 \times \mathbb{Z}^2$	3	
7)	1	$\mathbb{Z}/4 \times \mathbb{Z}^2$	3	
8)	1	$(\mathbb{Z}/2)^2 \times \mathbb{Z}^2$	3	
9)	1	$(\mathbb{Z}/2 \times \mathbb{Z}/4) \times \mathbb{Z}^2$	3	
10)	1	$(\mathbb{Z}/2)^2 \times \mathbb{Z}^2$	3	
11)	1	$(\mathbb{Z}/2)^3 \times \mathbb{Z}^2$	3	
12)	1	$(\mathbb{Z}/2)^3 \times \mathbb{Z}^2$	3	

Sicilian surfaces

Definition

A *Sicilian surface* is a minimal surface S of general type with $K_S^2 = 6$, $p_g = q = 1$ such that

- there exists an unramified double cover $\hat{S} \to S$ with $q(\hat{S}) = 3$, and
- such that the Albanese morphism $\hat{\alpha} \colon \hat{S} \to A$ is birational onto its image Z, which is a divisor in A with $Z^3 = 12$.

Remark

A generalized Burniat type surface S is a Sicilian surface if and only if S is in one of the families 11 or 12.

Theorem (Bauer, C. , Frapporti)

Sicilian surfaces have an irreducible four dimensional moduli space, and the general fibre of their Albanese map $\alpha\colon S \to A_1$ *is a non hyperelliptic curve of genus* $g = 3$.
Moreover, any surface homotopically equivalent to a Sicilian surface is a Sicilian surface.

Method used: the theory of Inoue type varieties and Bagnera de Franchis varieties, introduced in joint work with Ingrid Bauer.