Test „Algebraic Geometry I“

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Proposed solutions

Problem 1 (The five lemma; 7+2+1 points).
Let $\underline{A}$ be an abelian category and

$$\begin{array}{ccccccccc} M_1 & \xrightarrow{\varphi_1} & M_2 & \xrightarrow{\varphi_2} & M_3 & \xrightarrow{\varphi_3} & M_4 & \xrightarrow{\varphi_4} & M_5 \\ \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4 & & \downarrow f_5 \\ M_1' & \xrightarrow{\varphi_1'} & M_2' & \xrightarrow{\varphi_2'} & M_3' & \xrightarrow{\varphi_3'} & M_4' & \xrightarrow{\varphi_4'} & M_5' \end{array}$$

a commutative diagram in $\underline{A}$ with exact rows.
a) Assume that f_1, f_2, f_4, and f_5 are isomorphisms. Show that f_3 is an isomorphism, too.
b) Which conditions on f_1, f_2, f_4, and f_5 do you have to assume in order to grant that f_3 is an epimorphism (monomorphism)?
c) Can you relax the conditions on f_1, f_2, f_4, and f_5, so that f_3 is still an isomorphism?

a) We may assume that $\underline{A}$ is the category of left R-modules for some (not necessarily commutative) ring R and do diagram chasing.
First, we show that f_3 is surjective. Let $m' \in M_3'$. Since f_4 is surjective, there is an element $n \in M_4$ with $f_4(n) = \varphi_3'(m')$. Now, the commutativity of the right hand square shows $f_5(\varphi_4(n)) = \varphi_4'(f_4(n)) = (\varphi_4' \circ \varphi_3')(m') = 0$. Since f_5 is injective, we have $\varphi_4(n) = 0$. By the exactness of the top row, there is an element $\widetilde{m} \in M_3$ with $\varphi_3(\widetilde{m}) = n$. By construction, $m' - f_3(\widetilde{m})$ is an element of the kernel of φ_3'. By the exactness of the bottom row, there is an element $n' \in M_2'$ with $\varphi_2'(n') = m' - f_3(\widetilde{m})$. Since f_2 is surjective, there exists an element $\widetilde{n}$ with $f_2(\widetilde{n}) = n'$. Set $m := \widetilde{m} + \varphi_2(\widetilde{n})$. We check

$$f_3(m) = f_3(\widetilde{m}) + f_3(\varphi_2(\widetilde{n})) = f_3(\widetilde{m}) + \varphi_2'(f_2(\widetilde{n})) = f_3(\widetilde{m}) + \varphi_2'(n') = m'.$$

Note that we have used that f_2 and f_4 are surjective and that f_5 is injective.
In order to show that f_3 is injective, we simply pass to the opposite category and apply what we have already proved.
In this step, we need that f_2 and f_4 are injective and that f_1 is surjective.

b) As we have already mentioned, we need that f_2 and f_4 are epimorphisms and that f_5 is a monomorphism in order to prove that f_3 is an epimorphism. Similarly, we need that f_2 and f_4 are monomorphisms and that f_1 is an epimorphism to establish that f_3 is a monomorphism.

c) As is apparent from a) and b), we have just used that f_2 and f_4 are isomorphisms, that f_1 is an epimorphism, and that f_5 is a monomorphism.

Problem 2 (Direct limits of sheaves; 7+3 points).

Let X be a topological space, $(I, \leq)$ a directed set[1], and $((\mathcal{F}_i)_{i\in I}, (f_{ij})_{i\leq j\in I})$ a direct system of sheaves of abelian groups on X.[2]

a) Prove that the direct limit[3]

$$\varinjlim_{i\in I} \mathcal{F}_i$$

of this directed system exists in the category of sheaves of abelian groups on X.

b) Let $x \in X$. What is the stalk of $\varinjlim_{i\in I} \mathcal{F}_i$ at x? (A short justification of your conclusion is sufficient.)

a) For an open subset $U \subset X$, we set

$$\mathcal{L}(U) := \varinjlim_{i\in I} \mathcal{F}_i(U).$$

Since, for an open subset $V \subset U$ and $i \leq j \in I$, the diagram

$$\begin{array}{ccc} \mathcal{F}_i(U) & \xrightarrow{f_{ij}(U)} & \mathcal{F}_j(U) \\ {\scriptstyle \rho^{\mathcal{F}_i}_{UV}}\downarrow & & \downarrow{\scriptstyle \rho^{\mathcal{F}_j}_{UV}} \\ \mathcal{F}_i(V) & \xrightarrow{f_{ij}(V)} & \mathcal{F}_j(V) \end{array}$$

commutes, we obtain a restriction homomorphism

$$\rho^{\mathcal{L}}_{UV}: \mathcal{L}(U) \longrightarrow \mathcal{L}(V).$$

It is clear that $\mathcal{L}$ is a presheaf[4] of abelian groups and that there are homomorphisms $\widetilde{\varphi}_i: \mathcal{F}_i \longrightarrow \mathcal{L}$, $i \in I$, of presheaves of abelian groups with $\widetilde{\varphi}_i = \widetilde{\varphi}_j \circ f_{ij}$, $i \leq j \in I$. It is apparent from the construction that $(\mathcal{L}, \widetilde{\varphi}_i, i \in I)$ is the direct limit of the $\mathcal{F}_i$, $i \in I$, in the category of **presheaves** of abelian groups on X.

We introduce the sheaf

$$\varinjlim_{i\in I} \mathcal{F}_i := \mathcal{L}^\#$$

as the sheaf associated to the presheaf $\mathcal{L}$ and the homomorphisms

$$\varphi_i: \mathcal{F}_i \xrightarrow{\widetilde{\varphi}_i} \mathcal{L} \longrightarrow \mathcal{L}^\#, \quad i \in I.$$

[1]Recall that this is means that "$\leq$" is a partial ordering on I, such that for all $i, j \in I$ there exists a $k \in I$ with $i \leq k$ and $j \leq k$.

[2]This is a family $(\mathcal{F}_i)_{i\in I}$ of sheaves together with a family $f_{ij}: \mathcal{F}_i \longrightarrow \mathcal{F}_j$, $i \leq j$, of homomorphisms, such that $f_{ii} = \mathrm{id}_{\mathcal{F}_i}$, $i \in I$, and $f_{ik} = f_{jk} \circ f_{ij}$, $i \leq j \leq k$.

[3]This is a sheaf $\mathcal{F}$ together with sheaf homomorphisms $\varphi_i: \mathcal{F}_i \longrightarrow \mathcal{F}$ with $\varphi_i = \varphi_j \circ f_{ij}$, $i \leq j$, such that for every sheaf $\mathcal{G}$ and every family $\psi_i: \mathcal{F}_i \longrightarrow \mathcal{G}$, $i \in I$, of homomorphisms with $\psi_i = \psi_j \circ f_{ij}$, $i \leq j$, there is a unique homomorphism $\psi: \varinjlim_{i\in I} \mathcal{F}_i \longrightarrow \mathcal{G}$ with $\psi_i = \psi \circ \varphi_i$, $i \in I$.

[4]In general, one may not expect that this is a sheaf: E.g., the gluing of sections on open coverings will require that every open subset is quasi-compact, i.e., that X is a Noetherian topological space. (See Hartshorne, Exercise II.1.11 and II.2.13)

In order to show that this sheaf is the direct limit, let a sheaf $\mathcal{G}$ and homomorphisms $\psi_i: \mathcal{F}_i \longrightarrow \mathcal{G}$, $i \in I$, with $\psi_i = \psi_j \circ f_{ij}$, $i \leq j$, be given. Then, we obtain a homomorphism

$$\widetilde{\psi}: \mathcal{L} \longrightarrow \mathcal{G}$$

of presheaves with $\psi_i = \widetilde{\psi} \circ \widetilde{\varphi}_i$, $i \in I$.
Now, $\widetilde{\psi}$ factorizes over a homomorphism $\psi: \varinjlim_{i \in I} \mathcal{F}_i \longrightarrow \mathcal{G}$, because $\mathcal{G}$ is a sheaf. Clearly,

$$\psi_i = \widetilde{\psi} \circ \widetilde{\varphi}_i = \psi \circ \varphi_i, \quad i \in I.$$

b) We have

$$(\varinjlim_{i \in I} \mathcal{F}_i)_x = \varinjlim_{i \in I} \mathcal{F}_{i,x}, \quad x \in X.$$

Indeed, for $x \in X$,

$$\begin{aligned} (\varinjlim_{i \in I} \mathcal{F}_i)_x &= \varinjlim_{x \in U} (\varinjlim_{i \in I} \mathcal{F}_i(U)) \\ &= \varinjlim_{i \in I} (\varinjlim_{x \in U} \mathcal{F}_i(U)) \\ &= \varinjlim_{i \in I} \mathcal{F}_{i,x}. \end{aligned}$$

Problem 3 (Quasi-coherent sheaves; 6+2+2 points).
a) Let $(X, \mathcal{O}_X)$ be a scheme and

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

an exact sequence of $\mathcal{O}_X$-modules. Show that $\mathcal{F}$ is quasi-coherent, if $\mathcal{F}'$ and $\mathcal{F}''$ are so.
b) Let $(X, \mathcal{O}_X)$ be a scheme and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module. Verify that, for every point of $x \in X$, there are an affine open neighborhood U and $r > 0$, and a surjection

$$\mathcal{O}_{X|U}^{\oplus r} \longrightarrow \mathcal{F}_{|U}.$$

c) Assume in a) that $\mathcal{F}'$ and $\mathcal{F}''$ are coherent. Check that $\mathcal{F}$ is also coherent.

a) We may assume without loss of generality that $(X, \mathcal{O}_X)$ is affine. Set $M' := \Gamma(X, \mathcal{F}')$, $M := \Gamma(X, \mathcal{F})$, and $M'' := \Gamma(X, \mathcal{F}'')$. There is the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{M'} & \longrightarrow & \widetilde{M} & \longrightarrow & \widetilde{M''} & \longrightarrow & 0 \\ \| & & \downarrow & & \downarrow & & \downarrow & & \| \\ 0 & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}'' & \longrightarrow & 0 \end{array}$$

with exact rows.
By our assumption, the homomorphisms $\widetilde{M'} \longrightarrow \mathcal{F}'$ and $\widetilde{M''} \longrightarrow \mathcal{F}''$ are isomorphisms. By the snake or the five lemma, it follows that $\widetilde{M} \longrightarrow \mathcal{F}$ is also an isomorphism.

b) By definition, we may assume that $(X, \mathcal{O}_X)$ is affine and that $M := \Gamma(X, \mathcal{F})$ is a finitely generated R-module, $R := \Gamma(X, \mathcal{O}_X)$. So, there are an $r > 0$ and a surjection

$$\varphi \colon R^{\oplus r} \longrightarrow M$$

of R-modules. It gives the surjection

$$\widetilde{\varphi} \colon \mathcal{O}_X^{\oplus r} \cong \widetilde{R^{\oplus r}} \longrightarrow \widetilde{M} \cong \mathcal{F}.$$

c) Let $x \in X$. We may find affine open neighborhoods V and W of x, such that $\Gamma(V, \mathcal{F}')$ is a finitely generated $\Gamma(V, \mathcal{O}_X)$-module and $\Gamma(W, \mathcal{F}'')$ is a finitely generated $\Gamma(W, \mathcal{O}_X)$-module. Let $U \subset V \cap W$ be a third open affine neighborhood of x. By b), $\Gamma(U, \mathcal{F}')$ and $\Gamma(U, \mathcal{F}'')$ are both finitely generated $\Gamma(U, \mathcal{O}_X)$-modules. Furthermore, the sequence

$$0 \longrightarrow \Gamma(U, \mathcal{F}') \longrightarrow \Gamma(U, \mathcal{F}) \longrightarrow \Gamma(U, \mathcal{F}'') \longrightarrow 0$$

is exact. So, $\Gamma(U, \mathcal{F})$ is also a finitely generated $\Gamma(U, \mathcal{O}_X)$-module. Since

$$\mathcal{F}_{|U} \cong \widetilde{\Gamma(U, \mathcal{F})},$$

we are done.

Problem 4 (Fibers of a morphism; 3+7 points).
Let $(X, \mathcal{O}_X)$ be the spectrum of the ring

$$R := \mathbb{C}[t, x, y]/\langle t - xy \rangle.$$

a) Define a ring homomorphism $\varphi\colon \mathbb{C}[t] \longrightarrow R$, such that the associated morphism $\pi\colon X \longrightarrow \mathbb{A}^1_{\mathbb{C}}$ is given on closed points by

$$(a, b, c) \longmapsto a.$$

b) Describe the fiber of π over a closed point $a \in \mathbb{A}^1_{\mathbb{C}}$. Draw the picture of the real points in $\mathbb{R}^2$ of the fiber, if a is real.

a) The homomorphism is

$$\begin{aligned} \varphi\colon \mathbb{C}[t] &\longrightarrow R \\ t &\longmapsto t. \end{aligned}$$

b) The scheme theoretic fiber over $a \in \mathbb{C} \subset \mathbb{A}^1_k$ is the spectrum of the ring

$$\big(\mathbb{C}[t]/\langle t - a \rangle\big) \otimes_k R \cong \mathbb{C}[t, x, y]/\langle t - a, t - xy \rangle \cong \mathbb{C}[x, y]/\langle xy - a \rangle.$$

If $a \in \mathbb{R} \setminus \{0\}$, the equation $xy - a = 0$ defines a hyperbola in $\mathbb{R}^2$. For $a = 0$, the equation $xy = 0$ describes the union of the two coordinate axes.