# Minimal K-types for flat $G$-bundles, moduli spaces, and isomonodromy

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# Overview

New approach to the local theory of flat $G$-bundles over curves, i.e. formal flat $G$-bundles, using methods from representation theory: Systematic study of the “leading terms” of the flat structures with respect to Moy-Prasad filtrations

Two main motivations:

- Moduli spaces and the isomonodromy problem for meromorphic flat $G$-bundles with nondiagonalizable irregular singularities
- The wild ramification case of the geometric Langlands program

# Flat $G$-bundles

$X = \mathbb{P}^1(\mathbb{C})$ (for convenience), $\mathcal{O}$ structure sheaf of $\mathbb{P}^1(\mathbb{C})$, $K$ function field (meromorphic functions)
$\Omega^1_{K/\mathbb{C}}$ meromorphic 1-forms
Recall: A flat $\mathrm{GL}_n$-bundle on $\mathbb{P}^1(\mathbb{C})$ is a rank $n$ trivializable vector bundle with a meromorphic connection, i.e., a $\mathbb{C}$-derivation $\nabla : V \to V \otimes_{\mathcal{O}} \Omega^1_{K/\mathbb{C}}$.
If one fixes a trivialization $\phi : V \to V^{\mathrm{triv}}$, then

$$\nabla = d + [\nabla]_\phi, \text{ where } [\nabla]_\phi \in M_n(\Omega^1_{K/\mathbb{C}}).$$

## Definition

A flat $G$-bundle on $X$ is a trivializable principal $G$-bundle $E \to X$ with an abstract meromorphic connection $\nabla$; equivalently, it is a compatible family of flat vector bundles $(E \times_G W, \nabla_W)$, $W$ f.d. rep of $G$, with structure group $G$.

Here, $\nabla = d + [\nabla]_\phi$ with $[\nabla]_\phi \in \Omega^1_{K/\mathbb{C}}(\mathfrak{g})$.

# Localization

$(E, \nabla)$ flat $G$-bundle induces formal flat structures at each $y \in \mathbb{P}^1$
Let $z$ be a parameter at $y$
$\mathfrak{o} = \mathbb{C}[[z]]$ completion of local ring at $y$, $F = \mathbb{C}((z))$ fraction field,
$\Delta_y^\times = \mathrm{Spec}(F)$ is a formal punctured disk at $y$
One obtains an induced formal connection $(\hat{E}_y, \hat{\nabla}_y)$ on $\Delta_y^\times$. Note that $[\hat{\nabla}_y] \in \mathfrak{g}(F)\frac{dz}{z}$.
If the singular points are $y_1, \ldots, y_m$, one gets a localization functor $L : \nabla \mapsto (\hat{\nabla}_{y_i})$.
($\hat{\nabla}_y$ is trivial except at the singularities.)
If $[\hat{\nabla}_{y_i}]_\phi$ has a simple pole for some trivialization $\phi$, then $y_i$ is a regular singular point. Otherwise, it is irregular.

# Gauge and coadjoint actions

Fix a $G$-invariant nondegenerate symm bilinear form $(,)$ on $\mathfrak{g}$
eg for $\mathrm{GL}_n$, $(X, Y) = \mathrm{Tr}(XY)$
There are two natural actions of $\hat{G} := G(F)$ on $\Omega^1_{F/\mathbb{C}}(\mathfrak{g})$.
$[\hat{\nabla}]$ may be viewed as an element of $\mathfrak{g}(F)^\vee$ via

$$X \mapsto \mathrm{Res}(X, [\hat{\nabla}]), \text{ where } X \in \hat{\mathfrak{g}} := \mathfrak{g}(F).$$

Hence, the coadjoint action makes sense.
Change of trivialization gives rise to gauge change on the connection matrix; this gives the coadjoint action with an additional factor.

$$g \cdot [\hat{\nabla}] = \mathrm{Ad}^*(g)([\hat{\nabla}]) - (dg)g^{-1}, \text{ where } g \in \hat{\mathfrak{g}}.$$

Gauge change is the correct notion of equivalence in categories of flat G-bundles.

# Some problems on moduli spaces of flat $G$-bundles

$$\begin{array}{ccc} \left\{\begin{matrix}\text{flat } G\text{-bundles with} \\ \text{singularities at } y_1, \ldots, y_m\end{matrix}\right\} & \xrightarrow{M} & \left\{\begin{matrix}\text{enhanced monodromy} \\ \text{data}\end{matrix}\right\} \\ \big\downarrow {\scriptstyle L=\prod L_i} & & \\ \prod_i \left\{\begin{matrix}\text{formal flat} \\ G\text{-bundles on } \Delta_{y_i}^{\times}\end{matrix}\right\} & & \end{array}$$

Want to study these categories via the geometry of the moduli spaces. In general, these moduli spaces are stacks; to understand, look for better-behaved subcategories of flat $G$-bundles.

1. Find classes of formal isomorphism types for which $L^{-1}((\hat{\nabla}_i))$ are well-behaved moduli spaces.
2. When are such moduli spaces nonempty (Deligne-Simpson problem)? Reduced to a singleton (a version of rigidity)?
3. Investigate the fibers of the monodromy map restricted to reasonable moduli spaces.

# Nonresonant case for $\mathrm{GL}_n$ (reg semisimple leading term)

$[\hat{\nabla}_y] = (M_{-r}z^{-r} + M_{1-r}z^{1-r} + \dots)\frac{dz}{z}$, $M_i \in \mathfrak{gl}_n(\mathbb{C})$, $M_{-r} \neq 0$.

If $M_{-r}$ is regular semisimple, then $[\hat{\nabla}_y]$ is gauge equivalent to an element of $\mathcal{A}(r)\frac{dz}{z} = \{D_{-r}z^{-r} + \dots + D_0 \mid D_i \text{ diag}, D_{-r} \text{ reg}\}\frac{dz}{z}$.
($\mathcal{A}(r)$ is the set of "formal types").

Consider connections with only nonresonant singularities.

$$\begin{array}{ccc} \widetilde{\mathcal{M}}^{\mathrm{nr}}(\mathbf{r}) & \xrightarrow{M} & \widetilde{\mathcal{S}}^{\mathrm{nr}}(\mathbf{r}) \\ \downarrow {\scriptstyle L=\prod L_i} & & \\ \prod_i \mathcal{A}(r_i) & & \end{array}$$

## Results of Boalch (2001) building on Jimbo-Miwa-Ueno (1981)

- The moduli space $\widetilde{\mathcal{M}}^{\mathrm{nr}}(\mathbf{r})$ is a Poisson manifold; its symplectic leaves are the connected components of the fibers of $L$.
- The fibers of $M$ form an integrable system (solutions of the isomonodromy equations).
- These two foliations are "orthogonal".

# Generalized Airy connections

For $s \geq 1$, consider the nondiagonalizable connection

$$d + \begin{pmatrix} 0 & z^{-s} \\ z^{-s+1} & 0 \end{pmatrix} \frac{dz}{z} = d + \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) z^{-s} \frac{dz}{z} + \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) z^{-s+1} \frac{dz}{z}.$$

Irregular singular at 0, regular singular at $\infty$

$s = 2$ classical Airy, $s = 1$ $GL_2$-version of Frenkel-Gross rigid connection

Leading term is nilpotent; classical techniques do not apply.

However, there are filtrations of $\mathfrak{gl}_2(F)$ better adapted to it than $(z^k \, \mathfrak{gl}_2(\mathfrak{o}))_{k \in \mathbb{Z}}$.

Take filtration wrt complete lattice chain

$\supset L_{-1} = z^{-1} L_1 \supset L_0 = \mathfrak{o}^2 \supset L_1 = \mathfrak{o} e_1 \oplus z \mathfrak{o} e_2 \supset L_2 = z L_0 \supset$

$$\mathfrak{I}^k = \{x \in \mathfrak{gl}_2(F) \mid x(L_i) \subset L_{i+k} \forall i\}$$

Eg $\mathfrak{I}^0 \subset \mathfrak{gl}_2(\mathfrak{o})$ are upper triang matrices mod $z$, Iwahori subalg.

Now, $\begin{pmatrix} 0 & z^{-s} \\ z^{-s+1} & 0 \end{pmatrix} \in \mathfrak{I}^{1-2s} \setminus \mathfrak{I}^{2-2s}$; can be viewed as its own leading term wrt this filtration (so nonnilp; in fact, reg semisimple)

# Moy-Prasad filtrations

$\mathfrak{B}$ the Bruhat-Tits building for $G$–cell complex with cells parameterized by "parahoric subgroups"
$x \in \mathfrak{B}$, $\hat{\mathfrak{g}}_x$ corresponding parahoric subalgebra. There is a decreasing filtration $(\hat{\mathfrak{g}}_{x,r})_{r\in\mathbb{R}}$ of the loop algebra $\hat{\mathfrak{g}} = \mathfrak{g}(F)$ by $\mathfrak{o}$-lattices with $\hat{\mathfrak{g}}_{x,0} = \hat{\mathfrak{g}}_x$, $\hat{\mathfrak{g}}_{x,r+1} = z\hat{\mathfrak{g}}_{x,r}$, only a finite number of jumps in $[0, 1]$. Compatible filtration of parahoric subgroup $\hat{G}_x$.

## Examples

- The degree filtration $(z^k\mathfrak{g}(\mathfrak{o}))$ is the MP filtration associated to the "origin" $o$ of $\mathfrak{B}$–the vertex associated to the maximal parahoric $G(\mathfrak{o})$.
- The standard Iwahori filtration for $\mathrm{GL}_2$ is the MP filtration coming from the barycenter $x_I$ of the edge corresponding to the standard Iwahori subgroup (upper triangular mod $z$), except the jumps here are at $\frac{1}{2}\mathbb{Z}$.

# Fundamental strata

In $p$-adic representation theory, fundamental strata (or minimal $K$-types) were introduced by Bushnell and Kutzko ($GL_n$) and Moy and Prasad.

## Definition

- A stratum $(x, r, \beta)$ consists of $x \in \mathfrak{B}$, a real number $r \geq 0$, and a functional $\beta \in (\hat{\mathfrak{g}}_{x,r}/\hat{\mathfrak{g}}_{x,r+})^\vee$.
- $(x, r, \beta)$ is fundamental if $\beta$ is a semistable point in the $\hat{G}_x/\hat{G}_{x+}$ representation $(\hat{\mathfrak{g}}_{x,r}/\hat{\mathfrak{g}}_{x,r+})^\vee$ (non-nilpotency condition)

Moy-Prasad: Every irreducible admissible representation $W$ of a $p$-adic group contains a minimal $K$-type. Any such has the same depth, allowing one to define the depth of $W$.
We have a geometric analogue of their result.

One can define what it means for a flat $G$-bundle to contain a stratum; the stratum should be viewed as the leading term of the connection wrt the given filtration.
Fundamental strata give the most useful leading terms of a formal flat $G$-bundle.

## Examples

- $[\hat{\nabla}] = \left(z^{-r} M_{-r} + z^{-r+1} M_{1-r} + \mathrm{h.o.t.}\right) \frac{dz}{z}$ with $M_i \in \mathfrak{g}$.
  $\hat{\nabla}$ contains the $G$-stratum $(o, r, \beta)$, where $\beta \in (z^r \mathfrak{g}(\mathfrak{o}) / z^{r+1} \mathfrak{g}(\mathfrak{o}))^\vee$ is induced by $z^{-r} M_{-r} \frac{dz}{z}$, fundamental if $M_{-r}$ is non-nilpotent.
- $\hat{V} = F^2$, $[\hat{\nabla}] = \begin{pmatrix} 0 & z^{-s} \\ z^{-s+1} & 0 \end{pmatrix} \frac{dz}{z}$.
  Here, $(\hat{V}, \hat{\nabla})$ contains a nonfundamental stratum of depth $s$ at $o$ and the fundamental stratum $(x_I, s - \frac{1}{2}, \beta)$, where $I \subset \mathrm{GL}_2(\mathfrak{o})$ is the standard Iwahori subgroup, $\beta \in (\mathfrak{I}^{2s-1} / \mathfrak{I}^{2s-2})^\vee$.

### Theorem (Bremer-S. 2014)

*Every formal flat $G$-bundle $\hat{\nabla}$ contains a fundamental stratum $(x, r, \beta)$ with $r \in \mathbb{Q}$; the depth $r$ is positive iff $\hat{\nabla}$ is irregular singular. Moreover,*

- *If $\hat{\nabla}$ contains a stratum $(x', r', \beta')$, then $r' \geq r$.*
- *If $r > 0$, $(x', r', \beta')$ is fundamental if and only if $r' = r$.*

We can now define the slope of $\hat{\nabla}$ as this minimal depth.

### Theorem (Bremer-S, 2014)

*The slope of the formal flat $G$-bundle $(\hat{E}, \hat{\nabla})$ is a nonnegative rational number. It is positive if and only if $(\hat{E}, \hat{\nabla})$ is irregular singular. The slope may also be characterized as*

1. *the maximum slope of the associated flat connections; or*
2. *the maximum slope of the flat connections associated to the adjoint representations and the characters.*

Other defs of slope by Frenkel-Gross and Chen-Kamgarpour.

# Regular strata

One needs a stronger condition on strata to get nice moduli spaces. Let $S \subset G(F)$ be a (possibly non-split) maximal torus. There is a unique Moy-Prasad filtration $\{\mathfrak{s}_r\}$ on $\mathfrak{s} = \mathrm{Lie}(S)$.

## Definition

A point $x \in \mathfrak{B}$ is compatible with $\mathfrak{s}$ if $\mathfrak{s}_r = \hat{\mathfrak{g}}_{x,r} \cap \mathfrak{s}$ for all $r$.

A stratum $(x, r, \beta)$ is *S-regular* if $x$ is compatible with $\mathfrak{s}$ and it satisfies a graded version of regular semisimplicity.

## Examples

- If $M_{-r}$ is reg. semisimple, then $(o, r, z^{-r} M_{-r} \frac{dz}{z})$ is $T = Z_{G(F)}(M_{-r})$-regular (split torus).
- Let $\omega = \left(\begin{smallmatrix} 0 & z \\ 1 & 0 \end{smallmatrix}\right)$, so $S = \mathbb{C}((\omega))^*$ is a non-split maximal torus. $(x_I, s - \frac{1}{2}, \begin{pmatrix} 0 & z^{-s} \\ z^{-s+1} & 0 \end{pmatrix} \frac{dz}{z})$ is $S$-regular.

# Toral connections and formal types

If $\hat{\nabla}$ contains an $S$-regular stratum, it can be "diagonalized" into $S$ (and we call it $S$-toral). More precisely, one can define an affine variety $\mathcal{A}(S, r) \subset \mathfrak{s}_{-r} \frac{dz}{z}$ of $S$-formal types of depth $r$.

## Examples

- $T$ diagonal,
  $\mathcal{A}(T, r) = \{D_{-r} z_{-r} + \cdots + D_0 \mid D_i \text{ diag}, D_{-r} \text{ reg}\} \frac{dz}{z}$.
- $S = \mathbb{C}((\omega))$, $\mathcal{A}(S, s - 1/2) = \{\text{deg } 2s - 1 \text{ polys in } \omega^{-1}\} \frac{dz}{z}$.

## Theorem (Bremer-S. 2015)

*If $\hat{\nabla}$ contains the $S$-regular stratum $(x, r, \beta)$, then $[\hat{\nabla}]$ is $\hat{G}_{x+}$-gauge equivalent to a unique elt of $\mathcal{A}(S, r)$ with "leading term" $\beta$.*

The set of formal types is a $W_S^{\mathrm{aff}}$-torsor over the set of formal isomorphism classes, with $W_S^{\mathrm{aff}}$ the relative affine Weyl group.

# Framable connections ($G = \mathrm{GL}_n$)

$(V, \nabla)$ global connection; fix a trivialization $\phi$.
Assume $\hat{\nabla}_y$ has formal type $A_y$.

**Definition**

$g \in \mathrm{GL}_n(\mathbb{C})$ is a compatible framing for $\nabla$ at $y$ if $g \cdot [\hat{\nabla}_y]$ has the same leading term as $A_y \frac{dz}{z}$. If such a $g$ exists, $\nabla$ is framable at $y$.

$g \circ \phi$ is a global trivialization which makes the leading term of $[\hat{\nabla}_y]$ match the leading term of $A_y \frac{dz}{z}$.

**Example**

$P = \mathrm{GL}_n(\mathfrak{o})$, $A_y = (D_{-r} z^{-r} + \cdots + D_0)\frac{dz}{z}$
$g \cdot [\hat{\nabla}_y] = (D_{-r} z^{-r} + M_{1-r} z^{1-r} + \mathrm{h.o.t.})\frac{dz}{z}$.

# Moduli spaces ($G = \mathrm{GL}_n$)

## Starting data

- $\{y_i\}$ *irregular singular points*
- $\mathbf{A} = (A_i)$ *collection of* $S_i$*-formal types at* $y_i$ *(which determine* $S_i$*-regular strata* $(x_i, r_i, \beta_i)$ *at each* $y_i$*).*

Let $\mathcal{C}(\mathbf{A})$ be the category of framable connections $(V, \nabla)$ with formal types $\mathbf{A}$:

- $V$ is a trivializable rank $n$ vector bundle on $\mathbb{P}^1$;
- $\nabla$ is a mero. connection on $V$ with sing. points only at $\{y_i\}$;
- $\nabla$ is framable and has formal type $A_i$ at $y_i$.

The morphisms are vector bundle maps compatible with the connections.

$\mathcal{M}(\mathbf{A})$ is the corresponding moduli space.

Note that if two framable connections are isomorphic as meromorphic connections (i.e. as $D$-modules), then they are isomorphic as framable connections. Thus, $\mathcal{M}(\mathbf{A})$ is a subspace of the moduli stack of meromorphic connections.

# Variants

There are also moduli spaces $\widetilde{\mathcal{M}}(\mathbf{A})$ (resp. $\widetilde{\mathcal{M}}(\mathbf{S}, \mathbf{r})$) of framed connections with fixed formal types (resp. fixed regular combinatorics), which include data of compatible framings.

One can also allow additional regular singular points $\{q_j\}$; formal isomorphism classes are given by coadjoint orbit of the residue $\mathrm{res}_{q_j}([\hat{\nabla}]) := [\hat{\nabla}]|_{\mathfrak{gl}_n(\mathbb{C})}$.
If $\mathbf{B} = (\mathcal{O}^j)$ collection of nonresonant coadjoint orbits in $\mathfrak{gl}_n(\mathbb{C})^\vee$, can construct $\mathcal{M}(\mathbf{A}, \mathbf{B})$ etc; here, $\nabla$ has residue at $q_j$ in $\mathcal{O}^j$.

# Symplectic and Poisson reduction

We will construct these moduli spaces via symplectic (or Poisson) reduction of a symplectic (Poisson) manifold which is a direct product of local pieces. This is a result of Boalch (2001) in the case of regular diagonalizable leading terms.

## Setup

- *$X$ symplectic mfld with Hamiltonian action of the group $G$*
- *$\mu : X \to \mathfrak{g}^\vee$ the moment map*
- *$\alpha \in \mathfrak{g}^\vee$ is a singleton coadjoint orbit.*

## Definition

The symplectic reduction $X /\!\!/_\alpha G$ is defined to be the quotient $\mu^{-1}(\alpha)/G$.

## Fact

*If $\mu^{-1}(\alpha)/G$ is smooth, then the symplectic structure on $X$ descends to $X /\!\!/_\alpha G$.*

Poisson reduction is analogous.

# Local pieces

$A$ a formal type with parahoric $P$. $A$ can be viewed as an elt of $\mathfrak{P}^\vee$; let $\mathcal{O}_A$ be the $P$-coadjoint orbit.
Associated parabolic to $P$: $P/z\,\mathrm{GL}_n(\mathfrak{o}) \cong Q \subset \mathrm{GL}_n(\mathbb{C})$

Let $\mathcal{M}(A) \subset (Q\backslash \mathrm{GL}_n(\mathbb{C})) \times \mathfrak{gl}_n(\mathfrak{o})^\vee$ be the subvariety

$$\mathcal{M}(A) = \{(Qg, \alpha) \mid (\mathrm{Ad}^*(g)(\alpha))|_{\mathfrak{P}} \in \mathcal{O}_A)\}.$$

$GL_n(\mathbb{C})$ acts on $\mathcal{M}(A)$ via $h(Qg, \alpha) = (Qgh^{-1}, Ad^*(h)\alpha)$.

## Proposition

*$\mathcal{M}(A)$ is a symplectic manifold, and the $\mathrm{GL}_n(\mathbb{C})$-action is Hamiltonian with moment map $(Qg, \alpha) \mapsto \mathrm{res}(\alpha) := \alpha|_{\mathfrak{gl}_n(\mathbb{C})}$.*

$\mathcal{M}(A_i)$ encodes the local data of $\nabla \in \mathcal{M}(\mathbf{A})$ at $y_i$.
There are similar local manifolds $\widetilde{\mathcal{M}}(A)$ and $\widetilde{\mathcal{M}}(P, r)$ (symplectic and Poisson respectively) corresponding to the other moduli spaces.

# Structure of the moduli spaces

## Theorem (Bremer-S. 2013a)

1. *The moduli space* $\widetilde{\mathcal{M}}(\mathbf{A}, \mathbf{B})$ *is a symplectic manifold obtained as a symplectic reduction of the product of local data:*

$$\widetilde{\mathcal{M}}(\mathbf{A}, \mathbf{B}) \cong \left[ \left( \prod_i \widetilde{\mathcal{M}}(A_i) \right) \times \left( \prod_j \mathcal{O}^j \right) \right] /\!\!/_0 \, \mathrm{GL}_n(\mathbb{C}).$$

2. *The moduli space* $\mathcal{M}(\mathbf{A}, \mathbf{B})$ *may be constructed in a similar way. Moreover, it is the symplectic reduction of* $\widetilde{\mathcal{M}}(\mathbf{A}, \mathbf{B})$ *via a torus action.*

The condition that the moment map take value 0 just says that the sum of the residues over all singular points is 0.

These results and those on the next slide are due to Boalch (2001) in the case where all irregular formal types have regular semisimple leading term.

## Theorem (Bremer-S. 2012)

1. *The space* $\widetilde{\mathcal{M}}(\mathbf{P}, \mathbf{r})$ *is a Poisson manifold obtained by Poisson reduction of the product of local pieces.*
2. *The fibers of the localization map* $L$ *are the* $\widetilde{\mathcal{M}}(\mathbf{A})$*.*
3. *The symplectic leaves are the connected components of the* $\widetilde{\mathcal{M}}(\mathbf{A})$*'s.*

## Theorem (Bremer-S. 2012)

*There is an explicitly defined, Frobenius integrable Pfaffian system* $\mathcal{I}$ *on* $\widetilde{\mathcal{M}}(\mathbf{P}, \mathbf{r})$ *such that the solution leaves of* $\mathcal{I}$ *correspond to the fibers of the monodromy map* $M$*. The independent variables of this system are the coefficients of the formal types.*

# Some rigid connections

Let $x = 0$, $y = \infty$. Let the formal type at 0 be the simplest possible Iwahori type $A = \omega^{-1}$. Let $\mathcal{O}$ be any nonresonant adjoint orbit at $\infty$.

## Proposition (Bremer-S)

*$\mathcal{M}(A, \mathcal{O})$ is a singleton when $\mathcal{O}$ is regular and empty otherwise. Thus, one obtains a family of rigid connections including the Frenkel-Gross example.*

## Idea of proof when $\mathcal{O}$ irregular ($n = 3$)

- Let $X = \{ \left(\begin{smallmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & y & 0 \end{smallmatrix}\right) + b \mid x, y \in \mathbb{C}^*, b \in \mathfrak{b} \cap \mathfrak{sl}_3(\mathbb{C})\}$.
- The moment map conditions imply that $\mathcal{M}(A, \mathcal{O})$ is the set of $B$ orbits in the set $X \cap \mathcal{O}$.
- All elements of $X$ are regular, so if $\mathcal{O}$ is not regular, the moduli space is empty.