Automorphisms of surfaces of general type acting trivially on cohomology

Wenfei Liu

Peking University

August 7, 2015

Cohomology representation of automorphisms

Let X be a compact complex manifold and $\mathrm{Aut}(X)$ the holomorphic automorphism group.

The *cohomology representation* of the automorphism group is

$$\mathrm{Aut}(X) \to \mathrm{Aut}(H^*(X,R))^{op}$$

where $H^*(X,R) =$ cohomology ring with coefficients in a ring R.

In this talk, set $R = \mathbb{C}$.

Problem

Study the cohomology representation.

Denote by $\mathrm{Aut}_0(X)$ the kernel.

First examples

$\mathrm{Aut}(\mathbb{P}^n)$ acts trivially on $H^*(\mathbb{P}^n, \mathbb{C})$, so $\mathrm{Aut}_0(\mathbb{P}^n) = \mathrm{Aut}(\mathbb{P}^n)$.

Let T be a complex torus. Then $\mathrm{Aut}(T) = T \rtimes G$ with G the subgroup of automorphisms fixing the origin.

An automorphism σ acts trivially on $H^*(T, \mathbb{C})$.

$\Leftrightarrow$ it acts trivially on $H^1(T, \mathbb{C})$.

$\Leftrightarrow$ σ is a translation.

So $\mathrm{Aut}_0(T) = T$.

This can be easily explained: both $\mathrm{Aut}(\mathbb{P}^n)$ and T are the identity components of the respective automorphism groups, hence act trivially on cohomology.

Curves of general type

Let C be a curve of genus $g(C) \geq 2$. Then $\mathrm{Aut}(C)$ is finite.

Fact

$\mathrm{Aut}(C)$ acts faithfully on $H^*(C, \mathbb{C})$.

Proof I.

Let $\sigma \in \mathrm{Aut}_0(C)$.

- Then $H^*(C/\sigma, \mathbb{C}) = H^*(C, \mathbb{C})^\sigma = H^*(C, \mathbb{C})$, so $e(C) = e(C/\sigma)$ where $e(\cdot)$ is the topological Euler characteristic.
- On the other hand, by the Riemann–Hurwitz formula,

$$e(C) = |\sigma| e(C/\sigma) - \deg R$$

  where R is the ramification divisor.
- It follows that $|\sigma| = 1$, i.e., $\sigma = \mathrm{id}$.

□

Another proof of faithfulness for curves with $g \geq 2$

Proof II.

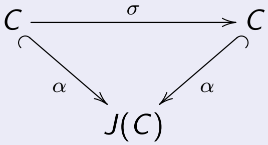
Let $\sigma \in \mathrm{Aut}_0(C)$.

- Let Γ_σ be the graph of σ in $C \times C$ and Δ the diagonal. Then

$$[\Gamma_\sigma] \cdot [\Delta] = [\Delta]^2 = 2 - 2g(C) < 0.$$

  In particular, $\mathrm{Fix}(\sigma) = \Gamma_\sigma \cap \Delta \neq \emptyset$.

- $H^1(C, \mathbb{C}) = H^{1,0}(C) \oplus H^{0,1}(C)$, so σ acts trivially on $H^{0,1}(C)$.
- Therefore σ acts trivially on $J(C)$, the Jacobian of C, and we have



- We infer that $\sigma = \mathrm{id}$.

Broughton's theorem

Let G be a finite group of automorphisms of an algebraic curve C and χ the character of the G-representation on $H^1(C, \mathbb{C})$. Then

$$\chi = 2\chi_0 + [2g(C/G) - 2 + t]\rho - \sum_{i=1}^{t} \rho_i$$

where

- χ_0 is the principle representation, i.e., $\chi_0(\sigma) = 1$ for any $\sigma \in G$,
- ρ is the regular representation of G,
- t is the number of branch points of $C \to C/G$,
- ρ_i is the permutation represenation of G on the orbit over the i-th branch point for $1 \leq i \leq t$.

Known results on surfaces

Theorem (Pjateckiĭ-Šapiro–Šafarevič'71, Peters'80)

Let S be a minimal projective surface. Then $\mathrm{Aut}_0(S) = \mathrm{Aut}^0(S)$, the identity component of $\mathrm{Aut}(S)$, except when S is

1. *a bi-elliptic surface*
2. *an Enriques surface*
3. *a properly elliptic surface with $\chi(\mathcal{O}_S) = 0$ or $p_g(S) = 0$*
4. *a surface of general type*

Moreover, exceptions in 1, 2 and 3 are constructed.

Theorem (Mukai–Namikawa'84)

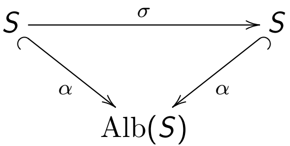
Let S be an Enriques surface. Then $|\mathrm{Aut}_0(S)| \leq 4$ with equality if and only if the surface is as contructed in an example.

A non-example of surface of general type

Let $S = C \times D$, where C and D are curves of genera ≥ 2. Then

$$\mathrm{Aut}_0(S) = \{\mathrm{id}\}.$$

Look at the Albanese map $S \to \mathrm{Alb}(S) = J(C) \times J(D)$, which is an embedding. For any $\sigma \in \mathrm{Aut}_0(S)$ we have a commutative diagram



from which follows $\sigma = \mathrm{id}$.

Surfaces isogenous to a product

Definition

A surface of general type S is *isogenous to a product* if there exists an étale cover $C \times D \to S$, where C and D are curves of genera ≥ 2.

Lemma (Catanese)

The étale cover $C \times D \to S$ can be chosen to be Galois so that $S = (C \times D)/G$ where $G < \mathrm{Aut}(C \times D)$ acts freely.

If $G < \mathrm{Aut}(C) \times \mathrm{Aut}(D)$ then the second cohomology of S decomposes:

$$H^2(S,\mathbb{C}) = H^2(C \times D,\mathbb{C})^G = W \bigoplus [\oplus_\chi H^1(C,\mathbb{C})^\chi \otimes H^1(D,\mathbb{C})^{\bar{\chi}}]^G$$

where

- $W = [H^2(C,\mathbb{C}) \otimes H^0(D,\mathbb{C})] \bigoplus [H^0(C,\mathbb{C}) \otimes H^2(D,\mathbb{C})]$,
- χ runs through all irreducible characters of G.

Surfaces of general type: the main results

Theorem (Cai–Liu–Zhang'13, Cai–Liu'13, Liu'15)

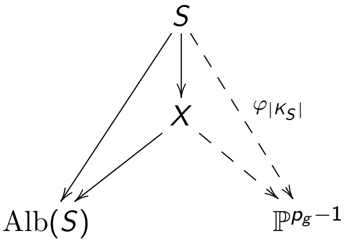
Let S be a minimal surface of general type.

- *If $q(S) \geq 3$ then $\mathrm{Aut}_0(S)$ is trivial.*
- *If $q(S) = 2$ then $|\mathrm{Aut}_0(S)| \leq 2$ with equality if and only if S is certain explicitly constructed surface isogenous to a product.*
- *If $q(S) = 1$ then $|\mathrm{Aut}_0(S)| \leq 4$ with equality only if S is a surface isogenous to a product.*

Remember $q(S) := \dim_{\mathbb{C}} H^1(S, \mathcal{O}_S)$ is the *irregularity* of the surface.

Idea of the proof

- Find reference spaces for the action of $\mathrm{Aut}_0(S)$:



where

  - $X = S/\mathrm{Aut}_0(S)$ is the quotient surface,
  - $\varphi_{|K_S|}: S \dashrightarrow \mathbb{P}^{p_g-1}$ is the canonical map.

- Analyze(!) these maps, using Lefschetz fixed point formula, generic vanishing theory, BMY inequality, Severi inequality etc.

Explanation for the non-trivial $\mathrm{Aut}_0(S)$?

Let S be a minimal surface of general type.

Fact

The identity component of $\mathrm{Aut}(S)$ is trivial.

Question

Does $\mathrm{Aut}_0(S)$ live in $\mathrm{Diff}^0(S)$, the identity component of the diffeomorphism group?

Surfaces isogenous to a product are rigidified

Definition (Catanese'13)

A compact complex manifold X is *rigidified* if $\mathrm{Diff}^0(X)$ does not contain any non-trivial (holomorphic) automorphisms.

Proposition (Cai–Liu–Zhang'13)

Let S be smooth projective surface. Assume that the universal cover of S is a bounded domain in $\mathbb{C}^2$. Then S is rigidified. In particular, surfaces isogenous to a product are rigidified.

Corollary

For those surfaces with

- $q(S) = 2$ *and* $|\mathrm{Aut}_0(S)| = 2$, *or*
- $q(S) = 1$ *and* $|\mathrm{Aut}_0(S)| = 4$

the group $\mathrm{Aut}_0(S)$ is not contained in $\mathrm{Diff}^0(S)$.

Thank you!