– Chow stability and the Projectivisation of Stable Bundles –

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Joint work with Julius Ross (University of Cambridge)
Set up - CSCK problem

$M$ smooth complex projective manifold, $L \to M$ ample line bundle
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\[ \exists h \in \text{Met}(L) \text{ such that } c_1(h) > 0 \]
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**Yau-Tian-Donaldson’s conjecture**
\( \exists \omega \in c_1(L) \) Constant Scalar \( \iff \) \((M, L)\) is “stable”
Curvature Kähler metric on \( M \)
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\( \iff \) Problem : what is the right definition of stability?
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Curvature Kähler metric on \( M \)

\( \iff \) Problem : what is the right definition of stability ?
Many candidates : Hilbert stability, Chow stability, asymptotic Hilbert/Chow stability, slope stability, \( K \)-stability, \( \overline{K} \)-stability, \( b \)-stability ....
Correspondence for vector bundles

$E$ irreducible holomorphic hermitian vector bundle on $(M, L)$, \( \omega \in c_1(L) > 0 \) Kähler. Then

\[ \exists h \in Met(E) \text{ such that } \sqrt{-1} \Lambda \omega F_h = Const \times Id_E \iff E \text{ is Mumford-Takemoto stable wrt } L \]
Correspondence for vector bundles

$E$ irreducible holomorphic hermitian vector bundle on $(M, L)$, $\omega \in c_1(L) > 0$ Kähler. Then

$\exists h \in \text{Met}(E)$ such that $\Leftrightarrow E$ is Mumford-Takemoto stable wrt $L$

$\sqrt{-1} \Lambda_\omega F_h = \text{Const} \times \text{Id}_E$

Definition

$E$ Mumford-Takemoto (semi-)stable if $\forall$ proper subsheaf $\mathcal{F} \subset E,$

$\mu(\mathcal{F}) < \mu(E)$ \quad ($\leq$)

Here $\text{Const} = \mu(E) = \frac{\deg_L(E)}{\text{rank}(E)}$

$\leftrightarrow$ Many applications
A few words about Chow stability

$L \to M^n$ very ample line bundle. Embed $i : M \to \mathbb{P}^N$, image of degree $d = (L^n)/n!$. Consider the Chow variety

$$Z_M = \{ L \in Gr(N - n - 1, \mathbb{P}^N) \text{ s. t. } L \cap i(M) \neq \emptyset \}$$

Set $Gr' = Gr(N - n - 1, \mathbb{P}^N)$. $Z_M$ has degree $d$ and $Z_M = \{ f_M = 0 \}$ with $f_M \in H^0(Gr', \mathcal{O}_{Gr'}(d))$. Chow form of $M$ is $[f_M] \in \mathbb{P}H^0(Gr', \mathcal{O}_{Gr'}(d))$
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Definition

$(M, L)$ Chow stable if the Chow form G.I.T stable wrt $SL(N + 1, \mathbb{C})$. 
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Definition

$(M, L)$ Chow stable if the Chow form G.I.T stable wrt $SL(N+1, \mathbb{C})$.

Denote $L^k = L \otimes^k$. $N = N_k = h^0(M, L^k)$.

Definition

$(M, L)$ Asymptotically Chow stable (A-Chow stable) if $\forall k \gg 0$,

$(M, L^k)$ Chow stable.

$\rightarrow$ Difficult to check / Few examples
A few words about $K$-stability

**Definition**
A test-configuration $(\mathcal{M}, \mathcal{L})$ is an equivariant flat family $\mathcal{M} \to \mathbb{C}$ of schemes with a fibrewise ample line bundle $\mathcal{L}$ and a $\mathbb{C}^*$ action on $\mathcal{M}$ such that for $\forall t \neq 0$, $(\mathcal{M}_t, \mathcal{L}_t) \simeq (M, L)$.

$\mathcal{L}_0 \to \mathcal{M}_0$.

$\mathbb{C}^*$ action on the central fiber $\mathcal{L}_0 \to \mathcal{M}_0$. 
A few words about $K$-stability

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$\mapsto \mathbb{C}^*$ action on $\tilde{V}_k = H^0(\mathcal{M}_0, \mathcal{L}_0^k)$ that has dimension $d_k$. Let $w_k$ the induced weight on $\Lambda^{d_k} \tilde{V}_k$.

$\mapsto$ Equivariant Riemann-Roch gives $w_k, d_k$.

One can write: $\frac{w_k}{kd_k} = F_0 + k^{-1} F_1 + k^{-2} F_2 + ...$

**Definition (Tian, Donaldson, Stoppa)**
$(M, L)$ $K$-stable if $\forall$ non trivial test-configuration $(\mathcal{M}, \mathcal{L})$, the Futaki-Donaldson invariant $F_1$ is $> 0$. 
A few words about $K$-stability

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Definition
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Theorem (Donaldson -2005)

$$\inf_{\omega \in \mathfrak{c}_1(L)} \| \text{scal}(\omega) - \overline{\text{scal}} \|_{L^2} \geq \sup_{\text{test-configs}} (-F_1)$$
A few words about $K$-stability

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$\leftrightarrow$ Equivariant Riemann-Roch gives $w_k, d_k$.
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**Definition**
$(\mathcal{M}, \mathcal{L})$ $K$-stable if $\forall$ non trivial test-configuration $(\mathcal{M}, \mathcal{L})$, the Futaki-Donaldson invariant $F_1$ is $> 0$.

**Conjecture (Donaldson -2005)**

$$
\inf_{\omega \in c_1(L)} \| \text{scal}(\omega) - \overline{\text{scal}} \|_{L^2} = \sup_{\text{test-configs}} (-F_1)
$$
A few words about $K$-stability

Some references about stability notions:


*K*-stability is VERY difficult to check
Examples / stability notions

- Curves (Chow stable by Mumford, K-stable by Odaka)
- Surfaces. Tian’s classification of Fano Kähler-Einstein surfaces ($L =$anti-canonical polarization).
- Homogeneous spaces, abelian varieties, toric manifolds, or more generally manifolds with symmetries (Wang & Zhu, Batyrev & Selivanova, Song, Donaldson, Ono, Podestá & Spiro, Dancer & Wang, Guan, Koiso & Sakane, Kempf, Wang,...)
- Hypersurfaces (Mumford, Lu, Tian,...)
- Projective bundles
Relations between stability notions (general case)

\[
\begin{align*}
\text{K-stable} & \Rightarrow \text{K-semistable} \\
\text{A-Chow stable} & \Rightarrow \text{A-Chow semistable} \\
\downarrow & \\
\text{A-Hilbert stable} & \Rightarrow \text{A-Hilbert semistable}
\end{align*}
\]
Relations between stability notions (general case)

\[ \text{K-stable} \Rightarrow \text{K-semistable} \]

\[ \text{A-Chow stable} \Rightarrow \text{A-Chow semistable} \]

\[ \text{A-Hilbert stable} \Rightarrow \text{A-Hilbert semistable} \]

\[ \text{Mabuchi} \]

For singular varieties : more complicated !
Relations between stability notions (general case)

Kähler-Einstein metric

\[ \cap \]

CSCK metric

K-stable \implies K-semistable

\[ \Downarrow \]

A-Chow

A-Chow stable \implies semistable

\[ \Leftrightarrow \]

A-Hilbert

A-Hilbert stable \implies semistable

\[ \Downarrow \]

Donaldson (only true for Aut(M, L) discrete)
Relations between stability notions (general case)

Kähler-Einstein metric

\[ \cap \]

CSCK metric \( \Rightarrow \) K-stable \( \Rightarrow \) K-semistable

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A-Chow stable \( \Rightarrow \) semistable

\[ \Uparrow \]

A-Hilbert stable \( \Rightarrow \) semistable

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\[ \Rightarrow \]

Donaldson, Stoppa, Mabuchi
Relations between stability notions (general case)

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\( \Rightarrow \) Donaldson, Stoppa, Mabuchi

\( \Downarrow \) Ono & Sano & Yotsutani, Della Vedova & Zuddas
Relations between stability notions (general case)

Kähler-Einstein metric

\[
\cap \quad \leftrightarrow
\]

CSCK metric \implies K-stable \implies K-semistable

\[
\downarrow \quad \downarrow \quad \uparrow \quad \uparrow
\]

A-Chow

A-Chow stable \implies semistable

\[
\uparrow \quad \uparrow
\]

A-Hilbert

A-Hilbert stable \implies semistable

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\downarrow \quad \downarrow
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Donaldson (only true for Aut(M, L) discrete)

\[
\implies
\]

Donaldson, Stoppa, Mabuchi

\[
\downarrow \quad \downarrow
\]

Ono & Sano & Yotsutani, Della Vedova & Zuddas

\[
\leftrightarrow
\]

Chen & Donaldson & Sun (L = -K_M)
Projectivisation of bundles

\(B\) is a base manifold of dimension \(n\), polarized by \(L\) \(\pi : E \to B\) holomorphic vector bundle of rank \(r\). Consider the projectivisation \(M = \mathbb{P}(E)\). Define

\[ \mathcal{L}_m := \mathcal{O}_M(1) \otimes \pi^* L^m \]

\(\rightarrow\) Problem: Relate existence of metrics with special curvature properties for \(B\) and \(E\) to existence of metrics with special curvature properties on \((M, \mathcal{L}_m)\).

\(\rightarrow\) Problem: Do the same but for stability notions.
**Projectivisation of bundles**

$B$ is a base manifold of dimension $n$, polarized by $L \pi : E \to B$ holomorphic vector bundle of rank $r$. Consider the projectivisation $M = \mathbb{P}(E)$. Define

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↔ Problem: Relate existence of metrics with special curvature properties for $B$ and $E$ to existence of metrics with special curvature properties on $(M, \mathcal{L}_m)$.

↔ Problem: Do the same but for stability notions.

**Objective**

- Classify unstable/stable ruled manifolds,
- Find polarizations with Kähler-Einstein metrics on 3-Fano bundles (Szurek & Wísniewski),
- Obtain a cartography of different stability notions.
Projectivisation of bundles (over a curve, genus \( \geq 2 \))

\( B \) is a curve.

**Theorem (Morrison -1980)**

\( E \) Mumford stable/semistable/unstable of rank 2. Then \( (M, \mathcal{L}_m) \) is Chow stable/semistable/unstable for \( m \gg 0 \).

**Theorem (Burns & de Bartolomeis -1988)**

There exists \( E \) Mumford semistable, non stable of rank 2 such that \( M \) has no CSCK/extremal metric.

\( \rightarrow \) Provides an example of an A-Chow semistable object, not K-stable.
Projectivisation of bundles (over a curve, genus $\geq 2$)

$B$ is a curve.
Building on the work of Narashimhan & Seshadri, Fujiki, Lebrun & Simanca, ...

Theorem (Apostolov & Calderbank & Gauduchon & Tønnesen-Friedman -2008)

$M$ admits a CSCK metric in any Kähler class $\iff E$ is Mumford polystable.

$\iff$ Construction of CSCK metric + generalizations to extremal metrics.

To sum up we have over a curve of genus $\geq 2$, any rank,

Theorem (Della Vedova & Zuddas -2011)

$M = \mathbb{P}(E)$ has a CSCK metric $\iff M$ is A-Chow stable $\iff E$ Mumford polystable

Related to the work of Rollin & Singer for parabolic structures.
Projectivisation of bundles (over a higher dimensional base)

\(B\) base manifold, polarized by \(L\)
\(\pi : E \to B\) bundle on \(B\) of rank \(r\), \(M = \mathbb{P}(E)\) and \(\mathcal{L}_m := \mathcal{O}_M(1) \otimes \pi^* L^m\).

Assume \(\dim B \geq 1\)
Projectivisation of bundles (over a higher dimensional base)

$B$ base manifold, polarized by $L$

$\pi : E \to B$ bundle on $B$ of rank $r$, $M = \mathbb{P}(E)$ and $\mathcal{L}_m := \mathcal{O}_M(1) \otimes \pi^* L^m$.

Assume $\dim B \geq 1$

Theorem (Hong -1999 (weak version))

Assume $\exists$ CSCK metric in $c_1(L)$, $\text{Aut}(B, L)$ discrete, $E$ simple\n
**Hermitian-Einstein** bundle. Then for $m >> 0$, $(M, \mathcal{L}_m)$ carries a CSCK metric.

Generalizations to extremal metrics by Brönnle, Seyyedali.

$\Leftarrow$ Asymptotically, for $m >> 0$, with $\omega_m = c_1(\hat{h}_E) + m\pi^* \omega$,

$$\text{scal}(\omega_m)([v]) = r(r-1) + \frac{1}{m} \left( \text{scal}(\omega) + 2r \Lambda_\omega \text{tr} \left( [F_{h_E}]^0 \frac{v \otimes v^*}{\|v\|^2} \right) \right) + O \left( \frac{1}{m^2} \right)$$
Theorem (Ross-Thomas -2006)

\( E \) Mumford unstable \( \Rightarrow (M, \mathcal{L}_m) \) is K-unstable for \( m \gg 0 \) and thus does not have CSCK metric.

Theorem (Seyyedali -2010)

Assume there is a CSCK in \( c_1(L) \), \( \text{Aut}(B, L) \) discrete, \( E \) Mumford stable. Then \( \mathcal{L}_m \) is Chow stable for \( m \gg 0 \).
Theorem (Ross-Thomas -2006)

\[ \text{E Mumford unstable } \Rightarrow (M, \mathcal{L}_m) \text{ is K-unstable for } m >> 0 \text{ and thus does not have CSCK metric.} \]

Theorem (Seyyedali -2010)

Assume there is a CSCK in \( c_1(L) \), \( \text{Aut}(B, L) \) discrete, E Mumford stable. Then \( \mathcal{L}_m \) is Chow stable for \( m >> 0 \).

\( \Rightarrow \) What can be said under weaker assumptions? For Mumford semistable bundle, everything can happen.
Chow stability & differential geometry (general case)

$\leftrightarrow$ Bergman function.

$\mathcal{L}$ very ample on $X$. Fix $h \in \text{Met}(\mathcal{L})$, $\omega = c_1(h) > 0$ and $(S_i)_{i=1..N}$ orthonormal basis of $H^0(X, \mathcal{L})$ wrt $\int_X h(.,.) \frac{\omega^n}{n!}$. The Bergman function is

$$\rho_h = \sum_i |S_i|_h^2 \in C^\infty(X, \mathbb{R}_+)$$
Chow stability & differential geometry (general case)

→ Bergman function.

\( \mathcal{L} \) very ample on \( X \). Fix \( h \in \text{Met}(\mathcal{L}) \), \( \omega = c_1(h) > 0 \) and \((S_i)_{i=1..N}\) orthonormal basis of \( H^0(X, \mathcal{L}) \) wrt \( \int_X h(.,.) \frac{\omega^n}{n!} \).

The Bergman function is

\[
\rho_h = \sum_i |S_i|^2_h \in C^\infty(X, \mathbb{R}_+) \]

Theorem (Zhang -1996; Luo -1998)

\((X, \mathcal{L})\) Chow polystable \(\iff\) \( \exists h \in \text{Met}(\mathcal{L}) \) balanced, i.e \( \forall x \in X \),

\[
\rho_h(x) = \text{Const} = \frac{h^0(\mathcal{L})}{\text{Vol}_\mathcal{L}(X)}
\]

→ Canonical metrics that can be constructed by iterative process
Fix $h \in \text{Met}(\mathcal{L})$ with curvature $\omega > 0$, and consider the Bergman kernel associated to $\mathcal{L}^k$ then when $k \to +\infty$, 

$$\rho_{hk} = k^n + k^{n-1} \frac{\text{scal}(\omega)}{2} + ...$$

studied by Tian, Bouche, Ruan, Catlin, Zelditch, Lu, Ma & Marinescu etc.
Tian-Yau-Zelditch Asymptotic expansion

Fix $h \in Met(\mathcal{L})$ with curvature $\omega > 0$, and consider the Bergman kernel associated to $\mathcal{L}^k$ then when $k \to +\infty$,

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$\rightarrow$ Higher order terms are polynomials expressions in the covariant derivatives of the curvature
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$\rightarrow$ Higher order terms are polynomials expressions in the covariant derivatives of the curvature

$\rightarrow$ If the metric is real analytic then the asymptotic expansion series converges (Liu-Lu)
Theorem (1)

Set $n = \dim B$. $M = \mathbb{P}(E)$. $\mathcal{L}_m := \mathcal{O}_M(1) \otimes \pi^* L^m$.

Theorem (1)

Assume there is a CSCK metric in $c_1(L)$, $\text{Aut}(B, L)$ discrete, $E$ Gieseker stable with Jordan-Hölder filtration given by subbundles. Then $\mathcal{L}_m$ Chow stable for $m \gg 0$. 
Theorem (1)

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**Theorem (1)**

Assume there is a CSCK metric in \( c_1(L) \), \( \text{Aut}(B, L) \) discrete, \( E \) Gieseker stable with Jordan-Hölder filtration given by subbundles. Then \( \mathcal{L}_m \) Chow stable for \( m \gg 0 \).

**Definition**

\( E \) Gieseker stable (semistable) wrt \( L \) stable if for \( \forall \) proper subsheaf \( \mathcal{F} \),

\[
\frac{h^0(\mathcal{F} \otimes L^k)}{\text{rank}(\mathcal{F})} < \frac{h^0(E \otimes L^k)}{\text{rank}(E)}, \quad k \gg 0 \quad (\leq)
\]

Mumford stable \( \Rightarrow \) Gieseker stable \( \Rightarrow \) Mumford semistable

\Rightarrow \) Gieseker semistable.
Proof of Theorem (1)

(1) Leung’s equation. For $k \gg 0$, $\exists h_k \in \text{Met}(E)$

$$[e^{F_{h_k} + k\omega \text{Id}} \wedge \text{Todd}(B)]^{(n,n)} = \frac{h^0(E \otimes L^k)}{r} \text{Id}_E \frac{\omega^n}{n!}$$
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(2) Bergman kernel for $E$ over $(B, L)$ vs Bergman function for $\mathcal{L}_m$. V vector space. Then $V \sim H^0(\mathbb{P}(V^*), \mathcal{O}_{\mathbb{P}(V^*)}(1))$. Hence $h_V \sim \hat{h}_V \in Met(\mathcal{O}_{\mathbb{P}(V^*)}(1))$

$$\langle \alpha, \beta \rangle_{h_V} = \text{Cst} \int_{\mathbb{P}(V^*)} \langle \hat{\alpha}, \hat{\beta} \rangle_{\hat{h}_V} \omega_{FS}^{\dim V - 1}$$

At $[v] \in \mathbb{P}(E)$, with $B_{h_E \otimes h_L^m}$ Bergman kernel for $E \otimes L^m$, $c_1(h_L) = \omega$,

$$\rho_{\mathcal{L}_m}([v]) = \text{tr} \left( B_{h_E \otimes h_L^m} \Psi_m \frac{v \otimes v^*}{\|v\|^2} \right)$$
Proof of Theorem (1)

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$$[e^{F_{h_k}+k\omega Id} \wedge \text{Todd}(B)]^{(n,n)} = \frac{h^0(E \otimes L^k)^{\omega^n}}{r} \text{Id}_E \frac{\omega^n}{n!}$$

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$$\rho_{\mathcal{L}_m}([v]) = \text{tr} \left( B_{h_E \otimes h_L^m} \Psi_m \frac{v \otimes v^*}{\|v\|^2} \right)$$

(3) $\Psi_m$ encodes the difference of volume forms between $\omega^n$ and $(c_1(h_E) + m\pi^* \omega)^{n+r-1}$.
Proof of Theorem (1)

(1) Leung’s equation. For \( k >> 0 \), \( \exists h_k \in Met(E) \)

\[
\left[ e^{F_{h_k} + k\omega Id} \wedge Todd(B) \right]^{(n,n)} = \frac{h^0(E \otimes L^k)}{r} \Id_E \frac{\omega^n}{n!}
\]

(2) Bergman kernel for \( E \) over \((B, L)\) vs Bergman function for \( \mathcal{L}_m \). 

V vector space. Then \( V \sim H^0(\mathbb{P}(V^*), \mathcal{O}_\mathbb{P}(V^*)(1)) \). 

Hence \( h_V \sim \hat{h}_V \in Met(\mathcal{O}_\mathbb{P}(V^*)(1)) \)

\[
\langle \alpha, \beta \rangle_{h_V} = \text{Cst} \int_{\mathbb{P}(V^*)} \langle \hat{\alpha}, \hat{\beta} \rangle_{\hat{h}_V} \omega^{\text{dim} V - 1}_{\text{FS}}
\]

At \([v] \in \mathbb{P}(E)\), with \( B_{h_E \otimes h_L^m} \) Bergman kernel for \( E \otimes L^m \), \( c_1(h_L) = \omega \),

\[
\rho_{\mathcal{L}_m}([v]) = \text{tr} \left( B_{h_E \otimes h_L^m} \Psi_m \frac{v \otimes v^*}{\|v\|^2} \right)
\]

(3) \( \Psi_m \) encodes the difference of volume forms between \( \omega^n \) and \( (c_1(h_E) + m\pi^*\omega)^{n+r-1} \).

(4) \( \bar{\mathcal{B}} := B_{h_E \otimes h_L^m} \Psi_m \), distorted Bergman kernel.
Proof of Theorem (1)

Then $\tilde{B}$ has an asymptotic expansion for $k >> 0$,

$$\tilde{B} = k^n Id_E + k^{n-1} \left( \left[ \Lambda \omega F_{h_E} \right]^0 + \frac{r + 1}{2r} \text{scal}(\omega) Id_E \right) + O(k^{n-2})$$
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$$

(5) Deform recursively the metrics $h_E$ on $E$ and $h_L$ on $L$ to obtain $\tilde{B}$ almost constant. Linearization at $(\omega + \sqrt{-1} \partial \bar{\partial} \phi, h_E(1 + \Phi))$:

$$
\frac{r+1}{2r} \text{Lic}(\phi) Id_E + \left[ \sqrt{-1} \Lambda_{\omega} \partial \bar{\partial} \Phi + \Lambda_{\omega}^2 F_{h_E} \wedge \partial \bar{\partial} \phi - \Delta_{\omega} \phi \Lambda_{\omega} F_{h} \right]^0
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$\rightarrow$ Invertible if $\text{Aut}(B, L)$ discrete + $\omega$ CSCK + $E$ simple.
Proof of Theorem (1)

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$\leftrightarrow$ Invertible if $\text{Aut}(B, L)$ discrete $+$ $\omega$ CSCK $+$ $E$ simple.
The solution is not selfadjoint:

$$\left( \Lambda_\omega \partial\bar{\partial}\Phi^{*_{h_E}} \right) = \left( \sqrt{-1}\Lambda_\omega \partial\bar{\partial}\Phi - [\Lambda_\omega F_h, \Phi] \right)^{*_{h_E}}$$

$\leftrightarrow$ From almost constant Bergman function $\rho_{L_m}$, obtain a balanced metric by using moment map framework (Donaldson’s method).
Theorem (2)

Consider a non split extension $0 \to F \to E \to G \to 0$.
$\to$ a family of bundles with general fibre $E$ and central fibre $F \oplus G$ at 0.
$\to$ provides a test configuration $\mathcal{M} : \mathbb{P}(\mathcal{E}) \to \mathbb{C}$.
Theorem (2)

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$\mathcal{E}$ family of bundles with general fibre $E$ and central fibre $F \oplus G$ at 0.
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Theorem (2)

Assume $E$ of rank 2 and $\dim B = 2$. The Futaki-Donaldson invariant of the test configuration $(\mathcal{M}, \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \otimes \pi^* L^m)$ is

$$F_1 = \text{Cst}(\mu(E) - \mu(F)) m^3 \left\{ \left( \frac{c_1(E)}{2} - c_1(F) \right) c_1(B) + 4 \left( \frac{ch_2(E)}{2} - ch_2(F) \right) \right. $$
$$+ \left( \mu(E) - \mu(F) \right)(\star) \left. \right\} m^2 + O(m)$$

Lower order terms are known.
Proof of Theorem (2)

Proof.
\[ d_k = \chi(S^k E^* \otimes L^{km}) = a_0 k^3 + a_1 k^2 + \ldots \]
Proof of Theorem (2)

Proof.

→ Equivariant Riemann-Roch theorem.

\[
d_k = \chi(S^k E^* \otimes L^{km}) = a_0 k^3 + a_1 k^2 + ...
\]

\[
H^0(\mathbb{P}(F \oplus G), \mathcal{L}_m^k) = \bigoplus_{i=0}^{k} H^0(B, F^i \otimes G^{k-i} \otimes L^{km})
\]

so the weight of the \( \mathbb{C}^* \) action is

\[
w(k) = \sum_i -ih^0(F^i \otimes G^{k-i} \otimes L^{km}) = b_0 k^4 + b_1 k^3 + ...
\]

Finally

\[
F_1 = b_0 a_1 - b_1 a_0.
\]
Applications

Consider $E$ Mumford semistable with $F \subset E$ such that $\mu(F) = \mu(E)$. If \[
\left( \frac{c_1(E)}{2} - c_1(F) \right) c_1(B) + 4 \left( \frac{ch_2(E)}{2} - ch_2(F) \right) < 0 \]
then $(\mathbb{P}(E), \mathcal{L}_m)$ is K-unstable for $m >> 0$.

Applying Theorem 1 + Theorem 2 with a well chosen Gieseker stable bundle $E$, we obtain the following corollary:

There exist examples of ruled manifolds $(\mathbb{P}(E), \mathcal{L}_m)$ such that $\mathcal{L}_m$ is Chow stable and asymptotically Chow unstable.

Define $L_{q,m} = O_{\mathbb{P}(E)}(q) \otimes \pi^* \mathcal{L}_m$. There exist Mumford stable bundles $E \to B$ over CSCK surfaces $(B, \mathcal{L})$ such that $L_{q,m}$ is K-stable and admits a CSCK metric for $m >> 0$ and is K-unstable for $q >> 0$. 
Consider $E$ Mumford semistable with $F \subset E$ such that $\mu(F) = \mu(E)$. If \( \left( \frac{c_1(E)}{2} - c_1(F) \right) c_1(B) + 4 \left( \frac{ch_2(E)}{2} - ch_2(F) \right) < 0 \) then \((\mathbb{P}(E), \mathcal{L}_m)\) is K-unstable for \(m >> 0\).

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**Corollary**

*There exist examples of ruled manifolds \((\mathbb{P}(E), \mathcal{L}_m)\) such that \(\mathcal{L}_m\) is Chow stable and asymptotically Chow unstable.*
Applications

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There exist examples of ruled manifolds \((\mathbb{P}(E), \mathcal{L}_m)\) such that $\mathcal{L}_m$ is Chow stable and asymptotically Chow unstable.

Corollary

Define $\mathcal{L}_{q,m} = \mathcal{O}_{\mathbb{P}(E)}(q) \otimes \pi^* \mathcal{L}_m$. There exist Mumford stable bundles $E \to B$ over CSCK surfaces $(B, L)$ such that $\mathcal{L}_{q,m}$ is $K$-stable and admits a CSCK metric for $m >> 0$ and is $K$-unstable for $q >> 0$. 
Conjecture

There exist a rank 2 vector bundle $E \to B$ and a polarized surface $B$ such that $(\mathbb{P}(E), \mathcal{L}_m)$ is A-Chow stable, $\text{Aut}(\mathbb{P}(E), \mathcal{L}_m)$ discrete and not K-stable.
Thank you!
Dziękuję!