Algebra II:
Homological Algebra

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Contents

Homological Algebra

As a motivating example, we review the definition of **singular homology** of a topological space X ([29], Chapter 4; [30], §III.9). Let $n \in \mathbb{N}$ and

$$\Delta^n := \{ (a_0, \dots, a_n) \in \mathbb{R}^{n+1} \mid a_i \geq 0,\ i = 0, \dots, n,\ a_0 + \cdots + a_n = 1 \}$$

the **standard *n*-simplex**. As a subset of $\mathbb{R}^{n+1}$, it inherits a topology from the euclidean topology of $\mathbb{R}^n$. Setting

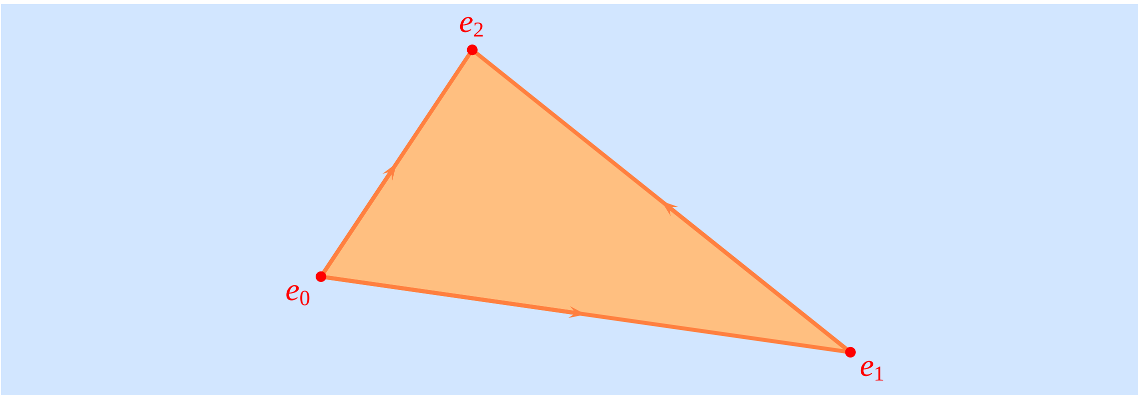
$$e_i := (0, \dots, 0, 1, 0, \dots, 0), \quad 1 \text{ occupying the } i\text{-th entry}, \quad i = 0, \dots, n,$$

we may describe Δ^n as the **convex hull** ([9], Definition I.1.4)

$$[e_0, \dots, e_n]$$

of $e_0, \dots, e_n$. This description involves an **orientation** of Δ^n. The **boundary** of Δ^n may be viewed as a union of oriented simplices

$$\partial\Delta^n = \sum_{i=0}^{n} (-1)^i \cdot [e_0, \dots, e_{i-1}, e_{i+1}, \dots, e_n]. \tag{I.1}$$



Now, a *singular n-simplex* in X is simply a continuous map

$$\sigma\colon \Delta^n \longrightarrow X.$$

The free abelian group $C_n(X,\mathbb{Z})$ generated by all singular n-simplices is the *group of singular n-chains*. The boundary in (I.1) induces a homomorphism

$$\partial_n\colon C_n(X,\mathbb{Z}) \longrightarrow C_{n-1}(X,\mathbb{Z}), \quad n \in \mathbb{N}.$$

One checks that

$$\forall n \in \mathbb{N}: \quad \partial_n \circ \partial_{n+1} = 0. \tag{I.2}$$

We set

$$Z_n(X,\mathbb{Z}) := \mathrm{Ker}(\partial_n) \subset C_n(X,\mathbb{Z}) \quad \text{and} \quad B_n(X,\mathbb{Z}) := \mathrm{Im}(\partial_{n+1}) \subset C_n(X,\mathbb{Z}), \quad n \in \mathbb{N}.$$

Condition (I.2) implies

$$B_n(X,\mathbb{Z}) \subset Z_n(X,\mathbb{Z}),$$

so that we may define the *n-th homology group*

$$H_n(X,\mathbb{Z}) := Z_n(X,\mathbb{Z})/B_n(X,\mathbb{Z})$$

of X, $n \in \mathbb{N}$. The homology groups are very important **topological invariants** of X. They cannot be computed from the above definition. The properties which characterize them and at the same time provide tools for their computation are laid down in the **Eilenberg–Steenrod axioms** ([29], Chapter 4, Section 8).

One of the axioms concerns the **long exact sequence of homology**. The collection $C_n(X,\mathbb{Z})$, $n \in \mathbb{N}$, of free abelian groups together with the boundary maps ∂_n, $n \in \mathbb{N}$, will be denoted by $(C_\bullet(X,\mathbb{Z}),\partial_\bullet)$ and called the *singular chain complex* of X. Suppose $A \subset X$ is a subspace. Then, for every $n \in \mathbb{N}$, $C_n(A,\mathbb{Z})$ is a subgroup of $C_n(X,\mathbb{Z})$, and the inclusions

$$C_n(A,\mathbb{Z}) \subset C_n(X,\mathbb{Z}), \quad n \in \mathbb{N},$$

are compatible with the boundary maps. So, $(C_\bullet(A,\mathbb{Z}),\partial_\bullet)$ is a **subcomplex** of $(C_\bullet(X,\mathbb{Z}), \partial_\bullet)$, and we may form the **quotient complex**

$$(C_\bullet(X,A;\mathbb{Z}),\partial_\bullet) := (C_\bullet(X,\mathbb{Z}),\partial_\bullet)/(C_\bullet(A,\mathbb{Z}),\partial_\bullet)$$

with

$$C_n(X,A;\mathbb{Z}) = C_n(X,\mathbb{Z})/C_n(A;\mathbb{Z}), \quad n \in \mathbb{N}.$$

Associated with the short exact sequence

$$0 \longrightarrow (C_\bullet(A,\mathbb{Z}),\partial_\bullet) \longrightarrow (C_\bullet(X,\mathbb{Z}),\partial_\bullet) \longrightarrow (C_\bullet(X,A;\mathbb{Z}),\partial_\bullet) \longrightarrow 0$$

of complexes, there is a long exact sequence[1]

$$\cdots \longrightarrow H_{n+1}(X,A) \longrightarrow H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X,A) \longrightarrow \cdots$$

$$\vdots$$

$$\cdots \longrightarrow H_1(X,A) \longrightarrow H_0(A) \longrightarrow H_0(X) \longrightarrow H_0(X,A) \longrightarrow 0$$

of homology groups. This is one of the most important tools for actually computing homology groups.

The following questions will be treated in this chapter:

[1] We omit the $\mathbb{Z}$-coefficients here.

- ⋆ What is the general framework for the above constructions?
- ⋆ Do similar constructions exist in other areas of mathematics?

In fact, (co)homology theories play an important role in various mathematical disciplines, such as group theory [7] and algebraic geometry. Of course, we will be mostly interested in the latter case. Note that, in singular homology, all the constructions take place within the realm of abelian groups. It will be important that abelian groups can be replaced by much more general objects, namely by objects of an **abelian category**. Before we can explain this concept, we must develop the basic notions of **category theory**.

I.1 Categories

In many courses in mathematics, you study certain structures and maps between these structures. Examples are vector spaces and linear maps, groups and homomorphisms, topological spaces and continuous maps, differentiable manifolds and differentiable maps, (affine) algebraic varieties and regular maps and so on. The axioms describing these structures are obtained by abstracting from a certain source of examples, and the corresponding maps are those that respect these structures. In a further step of abstraction, we collect these structures and the maps between them into a larger structure, called a category. Now, there are "maps" between categories, called **functors**. Such functors may be used to relate different categories. For example, for a given natural number $n \in \mathbb{N}$, the assignment $X \longmapsto H_n(X, \mathbb{Z})$ which associates with a topological space its n-th homology group is a functor from the category of topological spaces to the category of abelian groups. It relates the complicated structure of a topological space to the much simpler structure of an abelian group.

The theory of categories is developed on the background of set theory according to Neumann, Bernays, and Gödel (NBG) (see [12]; [21], Anhang). In this theory, there exist sets and classes and an element relation "$\in$". Sets are special classes, a set can be an element of a class, but a class which is not a set cannot be an element of a class. In this way, one can form the class of all sets. Russell's antinomy ([23], 1.2.2) is avoided, because it is not allowed to form the class of all classes.

A *category* $\mathscr{C}$ consists of the following data:

- ⋆ a **class** $\mathrm{Ob}(\mathscr{C})$ of objects,
- ⋆ for an ordered pair $A, B \in \mathrm{Ob}(\mathscr{C})$ of objects, a **set** $\mathrm{Mor}_{\mathscr{C}}(A, B)$,
- ⋆ for an ordered triple $A, B, C \in \mathrm{Ob}(\mathscr{C})$ of objects, a **composition map**

$$\begin{aligned} \circ = \circ_{A,B,C} \colon \mathrm{Mor}_{\mathscr{C}}(A, B) \times \mathrm{Mor}_{\mathscr{C}}(B, C) &\longrightarrow \mathrm{Mor}_{\mathscr{C}}(A, C) \\ (\alpha, \beta) &\longmapsto \beta \circ \alpha, \end{aligned}$$

  such that **associativity** holds, i.e., for an ordered quadruple $A, B, C, D \in \mathrm{Ob}(\mathscr{C})$ of objects the diagram

$$\begin{tikzcd} \mathrm{Mor}_{\mathscr{C}}(A, B) \times \mathrm{Mor}_{\mathscr{C}}(B, C) \times \mathrm{Mor}_{\mathscr{C}}(C, D) \arrow[r] \arrow[d] & \mathrm{Mor}_{\mathscr{C}}(A, C) \times \mathrm{Mor}_{\mathscr{C}}(C, D) \arrow[d] \\ \mathrm{Mor}_{\mathscr{C}}(A, B) \times \mathrm{Mor}_{\mathscr{C}}(B, D) \arrow[r] & \mathrm{Mor}_{\mathscr{C}}(A, D) \end{tikzcd}$$

commutes,

- ⋆ for every element $A \in \mathrm{Ob}(\mathscr{C})$, an element $\mathrm{id}_A \in \mathrm{Mor}_{\mathscr{C}}(A, A)$, such that for every object $B \in \mathrm{Ob}(\mathscr{C})$, and every morphism $\alpha \in \mathrm{Mor}_{\mathscr{C}}(A, B)$,

$$\alpha \circ \mathrm{id}_A = \alpha \quad \text{and} \quad \mathrm{id}_B \circ \alpha = \alpha.$$

An element $A \in \mathrm{Ob}(\mathscr{C})$ is an *object* of the category $\mathscr{C}$. For objects $A, B \in \mathrm{Ob}(\mathscr{C})$, an element $\alpha \in \mathrm{Mor}_{\mathscr{C}}(A, B)$ is a *morphism* from A to B. We write this as $\alpha\colon A \longrightarrow B$. For an object $A \in \mathrm{Ob}(\mathscr{C})$, the element $\mathrm{id}_A \in \mathrm{Mor}_{\mathscr{C}}(A, A)$ is the *identity morphism* of A.

A category $\mathscr{C}$ is *small*, if its class $\mathrm{Ob}(\mathscr{C})$ of objects is a set.

I.1.1 Remark. A morphism $\alpha\colon A \longrightarrow B$ need not necessarily be a map from the set A to the set B. An example is provided by quivers (see Example I.1.2, v), below). The reader should keep this in mind. All properties we would like our morphisms to have must be expressed in terms of arrows and diagrams and not in terms of elements of sets. You will find many examples for this, such as the concept of a mono- and an epimorphism (Section I.3), in the sequel.

Let $\mathscr{C}$ be a category, $A, B \in \mathrm{Ob}(\mathscr{C})$ objects of $\mathscr{C}$, and $\alpha\colon A \longrightarrow B$ a morphism from A to B. We say that α is an *isomorphism*, if there is a morphism $\beta\colon B \longrightarrow A$ with

$$\beta \circ \alpha = \mathrm{id}_A \quad \text{and} \quad \alpha \circ \beta = \mathrm{id}_B.$$

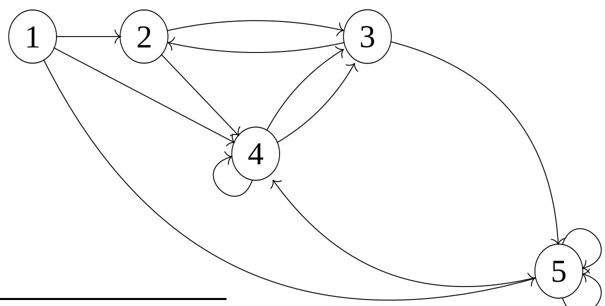
I.1.2 Examples. i) The **category of sets** $\underline{\mathrm{Sets}}$ consists of sets and set theoretic maps, i.e., $\mathrm{Ob}(\underline{\mathrm{Sets}}) = \{\,\mathrm{Sets}\,\}$[2] and, for sets $A, B \in \mathrm{Ob}(\mathscr{C})$, $\mathrm{Mor}_{\underline{\mathrm{Sets}}}(A, B) = \{\,\text{maps } \alpha\colon A \longrightarrow B\,\}$. Isomorphisms are bijective maps.

ii) Let k be a field. The category $\underline{\mathrm{Vect}}_k$ consists of k-vector spaces and linear maps, i.e., $\mathrm{Ob}(\underline{\mathrm{Vect}}_k) = \{\,k\text{-vector spaces}\,\}$ and, for k-vector spaces $A, B \in \mathrm{Ob}(\underline{\mathrm{Vect}}_k)$, $\mathrm{Mor}_{\underline{\mathrm{Vect}}_k}(A, B) = \mathrm{Hom}_k(A, B) = \{\,\alpha\colon A \longrightarrow B \,|\, \alpha \text{ is } k\text{-linear}\,\}$. Isomorphisms are bijective k-linear maps.

iii) More generally, for a commutative ring R, $\underline{\mathrm{Mod}}_R$ is the category of modules and module homomorphisms (see [22], Section III.1). Isomorphisms are bijective homomorphisms ([22], Exercise III.1.3).

iv) The category $\underline{\mathrm{Top}}$ is formed of topological spaces together with continuous maps ([24], Abschnitt 1.4, Aufgabe A.4.2). Isomorphisms in $\underline{\mathrm{Top}}$ are **homeomorphisms**. Recall that not every bijective continuous map is a homeomorphism (see [24], Gegenbeispiel 10.1.5).

v) A *quiver* is a quadruple $Q = (V, A, t, h)$ in which V and A are finite sets and $t, h\colon A \longrightarrow V$ are maps. The elements of V are the *vertices*, the elements of A the *arrows*. For an arrow $a \in A$, $t(a)$ is the *tail* of a, and $h(a)$ the *head*. A quiver may easily be visualized:



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[2] This is the so-called **all class**. It is a genuine class, i.e., a class which is not a set (see [12], Page 5).

We define **paths** in Q:

- ⋆ for every vertex $v \in V$, we have a path e_v of length zero from v to v,
- ⋆ for vertices $v_1, v_2 \in V$ and $n \geq 1$, a *path* of *length* n from v_1 to v_2 is a tuple $p = (a_1, ..., a_n)$ of arrows with $t(a_1) = v_1$, $h(a_i) = t(a_{i+1})$, $i = 1, ..., n-1$, $h(a_n) = v_2$.

There is an obvious composition law "·" for paths, such that $p \cdot e_v = p$ or $e_v \cdot p = p$ for every vertex $v \in V$ and every path p starting or ending at v, respectively.

We form a category $\underline{Q}$ as follows: $\mathrm{Ob}(\underline{Q}) = V$, for $v_1, v_2 \in V$,

$$\mathrm{Mor}_{\underline{Q}}(v_1, v_2) = \{\text{ paths from } v_1 \text{ to } v_2 \}.$$

For every $v \in V$, $\mathrm{id}_V = e_V$.

Let Q be the quiver that consists of a single vertex v and an arrow connecting the arrow to itself. For every $n \in \mathbb{N}$, there is a unique path $p_n \in \mathrm{Mor}_{\underline{Q}}(v, v)$ of length n, such that

$$\forall n_1, n_2 : \quad p_{n_1} \cdot p_{n_2} = p_{n_1+n_2},$$

so that we may say

$$(\mathrm{Mor}_{\underline{Q}}(v, v), \cdot) \cong (\mathbb{N}, +).$$

We leave it to the reader to figure out what the isomorphisms in $\underline{Q}$ are for a given quiver Q.

vi) Given a category $\mathscr{C}$, the *opposite category* $\mathscr{C}^{\mathrm{opp}}$ is obtained by **reversing all arrows** in $\mathscr{C}$, i.e., $\mathrm{Ob}(\mathscr{C}^{\mathrm{opp}}) = \mathrm{Ob}(\mathscr{C})$, and

$$\forall A, B \in \mathrm{Ob}(\mathscr{C}^{\mathrm{opp}}) : \quad \mathrm{Mor}_{\mathscr{C}^{\mathrm{opp}}}(A, B) := \mathrm{Mor}_{\mathscr{C}}(B, A).$$

Let $\mathscr{C}$ be a category. A *subcategory* $\mathscr{D}$ of $\mathscr{C}$ consists of

- ⋆ a subclass $\mathrm{Ob}(\mathscr{D}) \subset \mathrm{Ob}(\mathscr{C})$,
- ⋆ for an ordered pair $A, B \in \mathrm{Ob}(\mathscr{D})$ of objects in $\mathscr{D}$, a subset $\mathrm{Mor}_{\mathscr{D}}(A, B) \subset \mathrm{Mor}_{\mathscr{C}}(A, B)$, such that
  - – for every ordered triple $A, B, C \in \mathrm{Ob}(\mathscr{D})$, every morphism $\alpha \in \mathrm{Mor}_{\mathscr{D}}(A, B)$, and every morphism $\beta \in \mathrm{Mor}_{\mathscr{D}}(B, C)$

$$\beta \circ \alpha \in \mathrm{Mor}_{\mathscr{D}}(A, C).^3$$

  - – for every object $A \in \mathrm{Ob}(\mathscr{D})$ of $\mathscr{D}$, $\mathrm{id}_A \in \mathrm{Mor}_{\mathscr{D}}(A, A)$.

[3] The composition is, of course, formed within the category $\mathscr{C}$.

I.2 Functors

As usual, the theory of categories becomes interesting, because we can study "maps" between different categories. These maps are called **functors** and come in two flavors: covariant and contravariant.

Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A (*covariant*) *functor* from $\mathscr{C}$ to $\mathscr{D}$ consists of

- ⋆ a map $F\colon \mathrm{Ob}(\mathscr{C}) \longrightarrow \mathrm{Ob}(\mathscr{D})$,
- ⋆ for any ordered pair $A, B \in \mathrm{Ob}(\mathscr{C})$, a map $F_{A,B}\colon \mathrm{Mor}_{\mathscr{C}}(A,B) \longrightarrow \mathrm{Mor}_{\mathscr{D}}(F(A), F(B))$, such that
  - $F_{A,A}(\mathrm{id}_A) = \mathrm{id}_{F(A)}$, $A \in \mathrm{Ob}(\mathscr{C})$,
  - $F_{A,C}(\beta \circ \alpha) = F_{B,C}(\beta) \circ F_{A,B}(\alpha)$, $A, B, C \in \mathrm{Ob}(\mathscr{C})$, $\alpha \in \mathrm{Mor}_{\mathscr{C}}(A,B)$, $\beta \in \mathrm{Mor}_{\mathscr{C}}(B,C)$.

We will abusively write F for $F_{A,B}$, $A, B \in \mathrm{Ob}(\mathscr{C})$, and denote the whole functor by $F\colon \mathscr{C} \longrightarrow \mathscr{D}$.

A *contravariant functor* from $\mathscr{C}$ to $\mathscr{D}$ is a covariant functor $F\colon \mathscr{C}^{\mathrm{opp}} \longrightarrow \mathscr{D}$, i.e., a functor which reverses the arrows. We leave it to the reader to define a contravariant functor in terms of $\mathscr{C}$ and $\mathscr{D}$.

A new phenomenon in the setting of categories is that we have "maps between maps": Let $\mathscr{C}$ and $\mathscr{D}$ be categories and $F, G\colon \mathscr{C} \longrightarrow \mathscr{D}$ (covariant) functors. A *natural transformation* $\Phi\colon F \longrightarrow G$ consists of morphisms $\Phi(A)\colon F(A) \longrightarrow G(A)$ in $\mathscr{D}$, $A \in \mathrm{Ob}(\mathscr{C})$, such that the diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(\alpha)"] \arrow[d, "\Phi(A)"'] & F(B) \arrow[d, "\Phi(B)"] \\ G(A) \arrow[r, "G(\alpha)"] & G(B) \end{tikzcd} \tag{I.3}$$

commutes for all $A, B \in \mathrm{Ob}(\mathscr{C})$ and all $\alpha \in \mathrm{Mor}_{\mathscr{C}}(A,B)$. A natural transformation $\Phi\colon F \longrightarrow G$ is an *isomorphism of functors*, if there is a natural transformation $\Psi\colon G \longrightarrow F$, such that $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are the identity transformations (compare Remark I.2.1).

I.2.1 Remark. Given a **small** category $\mathscr{C}$ and a a category $\mathscr{D}$, we can construct the category $\mathrm{Fun}(\mathscr{C}, \mathscr{D})$. Its objects are (covariant) functors from $\mathscr{C}$ to $\mathscr{D}$ and its morphisms are natural transformations of functors. For the technical details, we refer to [21], p. 14.

The "category of small categories" is a 2-category, i.e., we have objects, namely small categories, morphisms, namely functors, and "morphisms between morphism" or **2-morphisms**, namely natural transformation of functors. We refer the reader to [19], Chapter XII, §3, for more information on this delicate business.

I.2.2 Examples. i) For every category $\mathscr{C}$, we have the *identity functor* $\mathrm{id}_{\mathscr{C}}\colon \mathscr{C} \longrightarrow \mathscr{C}$ that is defined in the obvious way.

ii) **Forgetful functors** are very basic functors which just forget a part of the structure that it is given. Here are two examples: Let k be a field. The functor

$$\begin{aligned} F\colon \mathrm{Ob}(\underline{\mathrm{Vect}}_k) &\longrightarrow \mathrm{Ob}(\underline{\mathrm{Sets}}) \\ V &\longmapsto \text{underlying set} \end{aligned}$$

just forgets the additive group structure on V, i.e., the neutral element, the inverse elements, and the addition, as well as scalar multiplication. We will denote the set underlying V again by V. Then, the functor F is given on morphisms, for k-vector spaces V, W, as

$$\begin{array}{rcl} F\colon \mathrm{Mor}_{\underline{\mathrm{Vect}_k}}(V,W) & \longrightarrow & \mathrm{Mor}_{\underline{\mathrm{Sets}}}(V,W) \\ f & \longmapsto & f. \end{array}$$

Let k be an algebraically closed field and $\underline{\mathrm{AffVar}_k}$ the category of affine algebraic varieties over k ([22], Exercise I.9.8 and III.4.2). Then, we have the forgetful functor

$$\begin{array}{rcl} F\colon \mathrm{Ob}(\underline{\mathrm{AffVar}_k}) & \longrightarrow & \mathrm{Ob}(\underline{\mathrm{Top}}) \\ X & \longmapsto & (\text{underlying set, Zariski topology}), \end{array}$$

$$\begin{array}{rcl} F\colon \mathrm{Mor}_{\underline{\mathrm{AffVar}_k}}(X,Y) & \longrightarrow & \mathrm{Mor}_{\underline{\mathrm{Top}}}(X,Y) \\ f & \longmapsto & f, \quad X, Y \in \mathrm{Ob}(\underline{\mathrm{AffVar}_k}). \end{array}$$

iii) We can also add structures. For example, we can equip any set with the discrete or the trivial topology. In this way, we get maps

$$\begin{array}{rcl} F\colon \mathrm{Ob}(\underline{\mathrm{Sets}}) & \longrightarrow & \underline{\mathrm{Top}} \\ A & \longmapsto & (A, \mathscr{P}(A)), \end{array}$$

and

$$\begin{array}{rcl} G\colon \mathrm{Ob}(\underline{\mathrm{Sets}}) & \longrightarrow & \underline{\mathrm{Top}} \\ A & \longmapsto & (A, \{\varnothing, A\}). \end{array}$$

Let $\alpha\colon A \longrightarrow B$ be a set theoretic map. The map α is continuous, if we endow both A and B with the discrete topology or both A and B with the trivial topology. In this way, we extend F and G to functors.

iv) Let k be a field and $Q = (V, A, t, h)$ be a quiver. A *representation* of Q is a functor

$$R\colon \underline{Q} \longrightarrow \underline{\mathrm{Vect}_k}.$$

Note that R is specified by the following data:

- ⋆ a vector space $R_v := R(v)$, $v \in V$,
- ⋆ a k-linear map $\varphi_a := R(a)\colon R_{t(a)} \longrightarrow R_{h(a)}$, $a \in A$.

In principle, we would need to specify $R(p)$ for every path $p = e_v$, $v \in V$, or $p = (a_1, ..., a_n)$ from v_1 to v_2, $v_1, v_2 \in V$. However, the axioms of a functor require

$$R(e_v) = \mathrm{id}_{R_v} \quad \text{and} \quad R(p) = R(a_n) \circ \cdots \circ R(a_1), \quad \text{respectively.}$$

So, we write our representation in the form $R = (R_v, v \in V, \varphi_a, a \in A)$.

Given two representations $R = (R_v, v \in V, \varphi_a, a \in A)$ and $S = (S_v, v \in V, \psi_a, a \in A)$, a *homomorphism* is a natural transformation $\Phi\colon R \longrightarrow S$. It is specified by linear maps $\Phi(v)\colon R_v \longrightarrow S_v$, $v \in V$, such that the diagram

$$\begin{tikzcd} R_{t(a)} \arrow[r, "\varphi_a"] \arrow[d, "\Phi(t(a))"'] & R_{h(a)} \arrow[d, "\Phi(h(a))"] \\ S_{t(a)} \arrow[r, "\psi_a"] & S_{h(a)} \end{tikzcd} \tag{I.4}$$

commutes for every arrow $a \in A$.[4] The notion of a homomorphism induces the notion of an isomorphism. The classification of quiver representations up to isomorphy is an interesting and important problem (see [2], Chapter II). For example, suppose that k is an algebraically closed field and that Q is the quiver with one vertex and one arrow. Then, a representation of Q is a pair $(V, \varphi\colon V \longrightarrow V)$ which consists of a k-vector space V and an endomorphism $\varphi\colon V \longrightarrow V$. The classification of finite dimensional representations is achieved by the Jordan normal form ([28], §54; [22], p. 85f).

v) Let k be a field. Then,

$$\begin{array}{rcl} (\cdot)^\vee\colon \underline{\mathrm{Vect}}_k & \longrightarrow & \underline{\mathrm{Vect}}_k \\ V & \longmapsto & V^\vee := \mathrm{Hom}_k(V, k) \\ f\colon V \longrightarrow W & \longmapsto & \left\{ \begin{array}{rcl} f^\vee\colon W^\vee & \longrightarrow & V^\vee \\ \lambda & \longmapsto & \lambda \circ f \end{array} \right. \end{array}$$

is a contravariant functor.

vi) Let k be an algebraically closed field, $\underline{\mathrm{AffVar}}_k$ the category of affine varieties over k, and $\underline{\mathrm{IntAlg}}_k$ the category of finitely generated k-algebras which are also integral domains. For an affine algebraic variety X, let $\mathscr{O}(X)$ be the k-algebra of regular functions on X ([22], Exercise I.9.8). Then,

$$\begin{array}{rcl} \mathscr{O}\colon \underline{\mathrm{AffVar}}_k & \longrightarrow & \underline{\mathrm{IntAlg}}_k \\ X & \longmapsto & \mathscr{O}(X) \\ \varphi\colon X \longrightarrow Y & \longmapsto & \left\{ \begin{array}{rcl} f^\star\colon \mathscr{O}(Y) & \longrightarrow & \mathscr{O}(X) \\ f & \longmapsto & f \circ \varphi \end{array} \right. \end{array}$$

is a contravariant functor.

Let $\mathscr{C}$ and $\mathscr{D}$ be categories and $F\colon \mathscr{C} \longrightarrow \mathscr{D}$ a functor. Then, F is an *isomorphism*, if there is a functor $G\colon \mathscr{D} \longrightarrow \mathscr{C}$, such that $G \circ F$ is the identity functor on $\mathscr{C}$ and $F \circ G$ the identity functor on $\mathscr{D}$.

It is extremely rare that two categories are isomorphic in the above sense. The reason is that the notion of equality of objects in a category is not very useful, because it cannot be verified. It is much more sensible to use the notion of isomorphic objects. For example, the dualizing functor in Example I.2.2, v), looks pretty harmless. However, the composition of this functor with itself is not the identity: For a non-zero vector space V, we cannot

[4] This example is good for memorizing the concept of a natural transformation. Representations of quivers can be easily defined in the language of linear algebra. The resulting notion of a homomorphism is the one you would have defined yourself for these objects.

say that V and $V^{\vee\vee}$ are equal, although we have a canonical isomorphism $V \longrightarrow V^{\vee\vee}$. Note also that, for k-vector spaces V, W, $(\cdot)^\vee \colon \mathrm{Hom}_k(V,W) \longrightarrow \mathrm{Hom}_k(W^\vee, V^\vee)$ is an isomorphism. We will now introduce the appropriate notion to deal with this situation.

Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A functor $F\colon \mathscr{C} \longrightarrow \mathscr{D}$ is an *equivalence* of categories, if there is a functor $G\colon \mathscr{D} \longrightarrow \mathscr{C}$, such that $G \circ F$ is **isomorphic** to the identity functor on $\mathscr{C}$ and $F \circ G$ to the identity functor on $\mathscr{D}$. A contravariant functor $F\colon \mathscr{C}^{\mathrm{opp}} \longrightarrow \mathscr{D}$ which is an equivalence between $\mathscr{C}^{\mathrm{opp}}$ and $\mathscr{D}$ is an *anti-equivalence* between $\mathscr{C}$ and $\mathscr{D}$.

Let us check what this means. First, we have a natural transformation $\varepsilon\colon\ \mathrm{id}_{\mathscr{C}} \longrightarrow G\circ F$ and an inverse transformation $\eta\colon G \circ F \longrightarrow \mathrm{id}_{\mathscr{C}}$. It follows immediately that

$$\varepsilon_A := \varepsilon(A)\colon A \longrightarrow (G \circ F)(A)$$

is an isomorphism in $\mathscr{C}$. Now, for $A, B \in \mathrm{Ob}(\mathscr{C})$, we obtain the bijection

$$\begin{array}{rcl} C_{A,B}\colon \mathrm{Mor}_{\mathscr{C}}(A,B) & \longrightarrow & \mathrm{Mor}_{\mathscr{C}}((G\circ F)(A), (G \circ F)(B)) \\ \alpha & \longmapsto & \varepsilon_B^{-1} \circ \alpha \circ \varepsilon_A. \end{array}$$

It is readily verified that

$$\mathrm{Mor}_{\mathscr{C}}(A,B) \xrightarrow{F} \mathrm{Mor}_{\mathscr{D}}(F(A), F(B)) \xrightarrow{G} \mathrm{Mor}_{\mathscr{C}}((G\circ F)(A), (G\circ F)(B)) \xrightarrow{C^{-1}}$$

$$\xrightarrow{C^{-1}} \mathrm{Mor}_{\mathscr{C}}(A,B)$$

is the identity. We infer that the map $F\colon \mathrm{Mor}_{\mathscr{C}}(A,B) \longrightarrow \mathrm{Mor}_{\mathscr{D}}(F(A), F(B))$ is injective and the map $G\colon \mathrm{Mor}_{\mathscr{C}}(F(A),F(B)) \longrightarrow \mathrm{Mor}_{\mathscr{C}}((G\circ F)(A), (G \circ F)(B))$ is surjective. The same reasoning for the functor $F \circ G$ shows that the latter map is also injective. Thus,

$$F\colon \mathrm{Mor}_{\mathscr{C}}(A,B) \longrightarrow \mathrm{Mor}_{\mathscr{D}}(F(A), F(B))$$

is a bijection, $A, B \in \mathrm{Ob}(\mathscr{C})$.

Note also that B is isomorphic to $F(G(B))$, $B \in \mathrm{Ob}(\mathscr{D})$. We see that F induces a bijection between the class of isomorphy classes in $\mathrm{Ob}(\mathscr{C})$ and the class of isomorphy classes in $\mathrm{Ob}(\mathscr{D})$.

I.2.3 Example (The skeleton of a category). Here, we need the following **axiom of choice** (see [21], p. 182): If an equivalence relation on a class X is given, then there is a functional class which picks an element in each equivalence class.[5] The axiom of choice also grants the existence of a functional class which assigns to every non-empty set one of its elements ([21], p. 182, Satz).

Let $\mathscr{C}$ be a category. The notion of isomorphy gives an equivalence relation on $\mathrm{Ob}(\mathscr{C})$. By the axiom of choice, there is a subclass $D \subset \mathrm{Ob}(\mathscr{C})$ which contains exactly one element of every isomorphy class in $\mathrm{Ob}(\mathscr{C})$. We form the full subcategory $\mathscr{D}$ of $\mathscr{C}$ with $\mathrm{Ob}(\mathscr{D}) := D$ and $\mathrm{Mor}_{\mathscr{D}}(A,B) := \mathrm{Mor}_{\mathscr{C}}(A,B)$, $A, B \in \mathrm{Ob}(\mathscr{D})$. Let $G\colon \mathscr{D} \longrightarrow \mathscr{C}$ be the inclusion functor.

We have the map $F\colon \mathrm{Ob}(\mathscr{C}) \longrightarrow \mathrm{Ob}(\mathscr{D})$ that assigns to an element $A \in \mathrm{Ob}(\mathscr{C})$ the unique element $F(A) \in \mathrm{Ob}(\mathscr{D})$, such that A is isomorphic to $F(A)$. We would like to

[5]In fact, an equivalence class may be a genuine class, i.e., a class which is not a set. Think of an example.

promote this map to a functor from $\mathscr{C}$ to $\mathscr{D}$. The axiom of choice allows us to pick, for each object $A \in \mathrm{Ob}(\mathscr{C})$, an isomorphism $\varepsilon_A \colon A \longrightarrow F(A)$. Then, for $A, B \in \mathrm{Ob}(\mathscr{C})$, we set

$$\begin{array}{rcl} F\colon \mathrm{Mor}_{\mathscr{C}}(A,B) & \longrightarrow & \mathrm{Mor}_{\mathscr{D}}(F(A),F(B)) \\ \alpha & \longmapsto & \varepsilon_B \circ \alpha \circ \varepsilon_A^{-1}. \end{array}$$

It is clear that F is a functor.

Next, observe

$$\begin{array}{rcl} G\circ F\colon \mathscr{C} & \longrightarrow & \mathscr{C} \\ A & \longmapsto & F(A) \\ \alpha\colon A \longrightarrow B & \longmapsto & F(\alpha)\colon F(A) \longrightarrow F(B) \end{array}$$

and that, for every morphism $\alpha\colon A \longrightarrow B$ in the category $\mathscr{C}$, the diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\varepsilon_A"'] & B \arrow[d, "\varepsilon_B"] \\ F(A) \arrow[r, "F(\alpha)"] & F(B) \end{tikzcd}$$

commutes. From this diagram and the definition of G, we infer that the ε_A, $A \in \mathrm{Ob}(\mathscr{C})$, define a natural transformation $\varepsilon\colon \mathrm{id}_{\mathscr{C}} \longrightarrow G\circ F$. Likewise, the maps $\varepsilon_A^{-1}\colon F(A) \longrightarrow A$ define the inverse $\varepsilon^{-1}\colon G\circ F \longrightarrow \mathrm{id}_{\mathscr{C}}$ of this natural transformation. This shows that $\mathrm{id}_{\mathscr{C}}$ and $G\circ F$ are isomorphic.

Finally,

$$\begin{array}{rcl} F\circ G\colon \mathscr{D} & \longrightarrow & \mathscr{D} \\ A & \longmapsto & F(A) = A \\ \alpha\colon A \longrightarrow B & \longmapsto & F(\alpha)\colon F(A) \longrightarrow F(B). \end{array}$$

Since we have not required that $\varepsilon_A = \mathrm{id}_A$ for $A \in \mathrm{Ob}(\mathscr{D})$, $F(\alpha)$ may be different from α, for a morphism α in the category $\mathscr{D}$. The same argument as before shows that $F\circ G$ is isomorphic to the identity on $\mathscr{D}$.

I.2.4 Examples. i) The functor $(\cdot)^\vee\colon \underline{\mathrm{Vect}}_k \longrightarrow \underline{\mathrm{Vect}}_k$ from Example I.2.2, v), is an anti-equivalence.

ii) The functor $\mathscr{O}\colon \underline{\mathrm{AffAlg}}_k \longrightarrow \underline{\mathrm{IntAlg}}_k$ from Example I.2.2, vi), is an anti-equivalence.

I.2.5 Exercises (The Yoneda[6] lemma). i) Let $\mathscr{C}$ be a category and $A \in \mathrm{Ob}(\mathscr{C})$. Show that

$$\begin{array}{rcl} h_A\colon \mathscr{C} & \longrightarrow & \underline{\mathrm{Sets}} \\ B & \longmapsto & \mathrm{Mor}_{\mathscr{C}}(A,B) \\ f\colon B \longrightarrow C & \longmapsto & (g\colon A \longrightarrow B) \longmapsto (f\circ g\colon A \longrightarrow C) \end{array}$$

is a covariant functor and that

$$\begin{array}{rcl} h^A\colon \mathscr{C} & \longrightarrow & \underline{\mathrm{Sets}} \\ B & \longmapsto & \mathrm{Mor}_{\mathscr{C}}(B,A) \\ f\colon B \longrightarrow C & \longmapsto & (g\colon C \longrightarrow A) \longmapsto (g\circ f\colon B \longrightarrow A) \end{array}$$

[6] Nobuo Yoneda (1930 - 1996), Japanese mathematician and computer scientist.

is a contravariant functor. A functor which is isomorphic to a functor of the form h_A for a suitable object $A \in \mathrm{Ob}(\mathscr{C})$ is called a *representable functors*.

ii) Let $F\colon \mathscr{C} \longrightarrow \underline{\mathrm{Sets}}$ be a (covariant) functor. Let A be an object of $\mathscr{C}$ and $a \in \mathscr{F}(A)$. Show that

$$\begin{aligned} h_a\colon h_A &\longrightarrow F \\ f\colon A \longrightarrow B &\longmapsto F(f)(a) \end{aligned}$$

is a natural transformation of functors.

iii) Let $\Phi\colon h_A \longrightarrow F$ be a natural transformation of functors. Define

$$a_\Phi := \Phi(A)(\mathrm{id}_A) \in F(A).$$

Prove that the assignments $\Phi \longmapsto a_\Phi$ and $a \longmapsto h_a$ are inverse to each other and therefore identify the natural transformations between h_A and F with the elements of $F(A)$.

iv) Rewrite this result for contravariant functors.

v) Conclude that two objects A and B define isomorphic functors h_A and h_B (h^A and h^B) if and only if they are isomorphic.

I.3 Monomorphisms and epimorphisms

In a category, the morphisms need not be maps among the underlying sets. So, in order to define properties of morphisms, we must not apply them to elements. We rather have to describe everything in terms of (universal) properties and diagrams. The following definitions illustrate this fact.

Let $\mathscr{C}$ be a category and $\alpha\colon A \longrightarrow B$ a morphism in the category $\mathscr{C}$. We say that α is a *monomorphism*, if for every object $C \in \mathrm{Ob}(\mathscr{C})$ and every pair $\gamma_1, \gamma_2\colon C \longrightarrow A$, the implication

$$\alpha \circ \gamma_1 = \alpha \circ \gamma_2 \quad \Longrightarrow \quad \gamma_1 = \gamma_2$$

holds true. We call α an *epimorphism*, if, for every object $C \in \mathrm{Ob}(\mathscr{C})$ and every pair $\gamma_1, \gamma_2\colon B \longrightarrow C$, we find

$$\gamma_1 \circ \alpha = \gamma_2 \circ \alpha \quad \Longrightarrow \quad \gamma_1 = \gamma_2.$$

I.3.1 Remark. Let $\alpha\colon A \longrightarrow B$ be a morphism in the category $\mathscr{C}$ and $\alpha^{\mathrm{opp}}\colon B \longrightarrow A$ the corresponding morphism in the opposite category $\mathscr{C}^{\mathrm{opp}}$. Then, α is a monomorphism (epimorphism) in $\mathscr{C}$ if and only if α^{opp} is an epimorphism (monomorphism) in $\mathscr{C}^{\mathrm{opp}}$.

We say that “monomorphism” and “epimorphism” are **dual** notions. In general, we get, for every definition in a category $\mathscr{C}$, a dual definition, by “reversing” all arrows.

I.3.2 Example. Let $\alpha\colon A \longrightarrow B$ be a morphism, such that there exists a morphism $\pi\colon B \longrightarrow A$ with $\pi \circ \alpha = \mathrm{id}_A$. Then, α is a monomorphism. In fact, let $C \in \mathrm{Ob}(\mathscr{C})$ and $\gamma_1, \gamma_2\colon C \longrightarrow A$ morphisms with $\alpha \circ \gamma_1 = \alpha \circ \gamma_2$. Then,

$$\gamma_1 = \mathrm{id}_A \circ \gamma_1 = (\pi \circ \alpha) \circ \gamma_1 = \pi \circ (\alpha \circ \gamma_1) = \pi \circ (\alpha \circ \gamma_2) = \gamma_2.$$

Likewise, α is an epimorphism, if there is a morphism $\iota\colon B \longrightarrow A$ with $\alpha \circ \iota = \mathrm{id}_B$.

I.3.3 Exercise. Prove that a map $\alpha\colon A \longrightarrow B$ in the category $\underline{\mathrm{Sets}}$ is a monomorphism if and only if it is injective and an epimorphism if and only if it is surjective.

I.4 Additive categories

Let $\mathscr{C}$ be a category and $A, B \in \mathrm{Ob}(\mathscr{C})$. A triple $(A \oplus B, \iota_A, \iota_B)$ which consists of an object $A \oplus B$ and morphisms $\iota_A\colon A \longrightarrow A \oplus B$, $\iota_B\colon B \longrightarrow A \oplus B$ is the *direct sum* or *coproduct* of A and B, if it has the following universal property: For every object C and every pair of morphisms $\alpha\colon A \longrightarrow C$ and $\beta\colon B \longrightarrow C$, there is a unique morphism $\alpha \oplus \beta\colon A \oplus B \longrightarrow C$ with

$$\alpha = (\alpha \oplus \beta) \circ \iota_A \quad \text{and} \quad \beta = (\alpha \oplus \beta) \circ \iota_B.$$

We remember this property by the diagram

$$\begin{tikzcd}
 & B \arrow[d, "\iota_B"'] \arrow[ddr, bend left, "\beta"] & \\
A \arrow[r, "\iota_A"] \arrow[drr, bend right, "\alpha"'] & A \oplus B \arrow[dr, dashed, "\exists!\alpha\oplus\beta"] & \\
 & & C.
\end{tikzcd}$$

I.4.1 Remark. The direct sum is, if it exists, unique up to unique isomorphy: Given a triple $(D, i_A\colon A \longrightarrow D, i_B\colon B \longrightarrow D)$, such that, for every object C and every pair of morphisms $\alpha\colon A \longrightarrow C$ and $\beta\colon B \longrightarrow C$, there is a unique morphism $\gamma\colon D \longrightarrow C$ with $\alpha = \gamma \circ i_A$ and $\beta = \gamma \circ i_B$, one has a unique isomorphism $f\colon A \oplus B \longrightarrow D$ with $\iota_A = i_A \circ f$ and $\iota_B = i_B \circ f$. In fact, the morphism f exists by the universal property. Likewise, there is a morphism $g\colon D \longrightarrow A \oplus B$ with $i_A = \iota_A \circ g$ and $i_B = \iota_B \circ g$. For the composition $g \circ f$, we find $\iota_A = \iota_A \circ (g \circ f)$ and $\iota_B = \iota_B \circ (g \circ f)$. Thus, $g \circ f = \mathrm{id}_{A \oplus B}$, by the uniqueness requirement in the universal property. For the same reason $f \circ g = \mathrm{id}_D$.

I.4.2 Exercise (The direct product). Let $\mathscr{C}$ be a category and $A, B \in \mathrm{Ob}(\mathscr{C})$. Define the notion of a **direct product** $A \sqcap B$ that is dual to the notion of a direct sum.

Recall that you have already encountered direct sums and products in the category of modules over a ring R ([22], Section III.1).

Let $\mathscr{C}$ be a category. An object $\star \in \mathrm{Ob}(\mathscr{C})$ is an *initial object*, if, for every object $A \in \mathrm{Ob}(\mathscr{C})$, there is a unique morphism $\star \longrightarrow A$. We say that $\star \in \mathrm{Ob}(\mathscr{C})$ is a *terminal object*, if, for every object $A \in \mathrm{Ob}(\mathscr{C})$, there is a unique morphism $A \longrightarrow \star$. A *null object* is an object $0 \in \mathrm{Ob}(\mathscr{C})$ which is both an initial and a terminal object.

I.4.3 Remark. For two initial objects $\star, \star' \in \mathrm{Ob}(\mathscr{C})$, there is a unique isomorphism $\star \longrightarrow \star'$. This is, because the composition $\star \longrightarrow \star' \longrightarrow \star$ has to be $\mathrm{id}_\star$ and the composition $\star' \longrightarrow \star \longrightarrow \star'$ has to be $\mathrm{id}_{\star'}$. The same goes for terminal objects and null objects.

Let $\mathscr{C}$ be a category which has a null object $0 \in \mathrm{Ob}(\mathscr{C})$. For objects $A, B \in \mathrm{Ob}(\mathscr{C})$, the *null morphism* is the morphism $0\colon A \longrightarrow 0 \longrightarrow B$. Let $\alpha\colon A \longrightarrow B$ a morphism in $\mathscr{C}$. A pair (K, ι), consisting of an object $K \in \mathrm{Ob}(\mathscr{C})$ and a morphism $\iota\colon K \longrightarrow A$, is a *kernel* of α, if

- $\alpha \circ \iota = 0$,[7]

[7] Here, we use that we can say what a zero morphism is.

- ⋆ for every object $C \in \mathrm{Ob}(\mathscr{C})$ and every morphism $\delta\colon C \longrightarrow A$ with $\alpha \circ \delta = 0$, there is a unique morphism $\varkappa\colon C \longrightarrow K$ with $\delta = \iota \circ \varkappa$;

$$\begin{tikzcd} & C \arrow[dl, dashed, "\exists!\varkappa"'] \arrow[d, "\delta"] \arrow[dr, "\alpha\circ\delta=0"] & \\ K \arrow[r, "\iota"] \arrow[rr, bend right, "\alpha\circ\iota=0"'] & A \arrow[r, "\alpha"] & B. \end{tikzcd}$$

As usual, a kernel is, if it exists, determined up to unique isomorphy, i.e., if (K', ι') is another kernel, there is a unique isomorphism $\varkappa\colon K \longrightarrow K'$ with $\iota = \iota' \circ \varkappa$. Thus, we speak of **the** kernel. We will write $\mathrm{Ker}(\alpha)$ for K and often omit ι from the notation.

I.4.4 Remark. Let $\alpha\colon A \longrightarrow B$ be a morphism in $\mathscr{C}$ and $\iota\colon K \longrightarrow A$ the kernel. We claim that ι is a monomorphism. Assume that $\gamma_1, \gamma_2\colon C \longrightarrow K$ are two morphisms with $\iota \circ \gamma_1 = \iota \circ \gamma_2 =: g$. It is easy to see that $\alpha \circ g = 0$. So, there is a **unique** morphism $\overline{g}\colon C \longrightarrow K$ with $g = \iota \circ \overline{g}$. This shows $\gamma_1 = \overline{g} = \gamma_2$.

A *cokernel* of α is a pair (L, p) in which $L \in \mathrm{Ob}(\mathscr{C})$ is an object of $\mathscr{C}$ and $p\colon B \longrightarrow L$ is a morphism, such that

- ⋆ $p \circ \alpha = 0$,
- ⋆ for every object $C \in \mathrm{Ob}(\mathscr{C})$ and every morphism $\delta\colon B \longrightarrow C$ with $\delta \circ \alpha = 0$, there is a unique morphism $\lambda\colon L \longrightarrow C$ with $\delta = \lambda \circ p$;

$$\begin{tikzcd} A \arrow[r, "\alpha"'] \arrow[rr, bend left, "p\circ\alpha=0"] \arrow[dr, "\delta\circ\alpha=0"'] & B \arrow[r, "p"'] \arrow[d, "\delta"] & L \arrow[dl, dashed, "\exists!\lambda"] \\ & C. & \end{tikzcd}$$

A *cokernel* is, if it exists, unique up to unique isomorphy, and will be denoted by $\mathrm{Coker}(\alpha)$.

I.4.5 Remark. i) As in Remark I.4.4, one checks that a cokernel is an epimorphism.

ii) The notions of kernel and cokernel are dual to each other in the sense of Remark I.3.1, i.e., the kernel (cokernel) of a morphism $\alpha\colon A \longrightarrow B$ yields the cokernel (kernel) of the corresponding morphism $\alpha^{\mathrm{opp}}\colon B \longrightarrow A$ in the opposite category $\mathscr{C}^{\mathrm{opp}}$.

An *additive category* $\mathscr{C}$ consists of

- ⋆ a class $\mathrm{Ob}(\mathscr{C})$ of objects,
- ⋆ for an ordered pair $A, B \in \mathrm{Ob}(\mathscr{C})$ of objects, an **abelian group** $\mathrm{Mor}_{\mathscr{C}}(A, B)$,
- ⋆ for an ordered triple $A, B, C \in \mathrm{Ob}(\mathscr{C})$ of objects, a **bilinear** composition map

$$\begin{array}{rcl} \circ\colon \mathrm{Mor}_{\mathscr{C}}(A,B) \times \mathrm{Mor}_{\mathscr{C}}(B,C) & \longrightarrow & \mathrm{Mor}_{\mathscr{C}}(A,C) \\ (\alpha,\beta) & \longmapsto & \beta \circ \alpha, \end{array}$$

- ⋆ for every object $A \in \mathrm{Ob}(\mathscr{C})$, an element id_A, such that

- $\star$ the axioms of a category are verified (see Page 3),
- $\star$ there exists a null object $0 \in \mathrm{Ob}(\mathscr{C})$,
- $\star$ for any two objects $A, B \in \mathrm{Ob}(\mathscr{C})$, their direct sum exists.

I.4.6 Remark. For an alternative definition of an additive category, see [21], Abschnitt 4.1.

I.4.7 Lemma. *Let $\mathscr{C}$ be an additive category.*
i) *An object* $0 \in \mathrm{Ob}(\mathscr{C})$ *is a null object if and only if* $\mathrm{Mor}_{\mathscr{C}}(0,0) = \{0\}$.
ii) *Let $A, B \in \mathrm{Ob}(\mathscr{C})$. The null morphism $A \longrightarrow B$ agrees with the neutral element of* $\mathrm{Mor}_{\mathscr{C}}(A,B)$.

Proof. i) If 0 is a null object, then $\mathrm{Mor}_{\mathscr{C}}(0,0) = \{0\}$, by definition. Now, assume that $0 \in \mathrm{Ob}(\mathscr{C})$ satisfies $\mathrm{Mor}_{\mathscr{C}}(0,0) = \{0\}$. Let $A \in \mathrm{Ob}(\mathscr{C})$. By definition of an additive category, there is a morphism $0\colon 0 \longrightarrow A$. For every morphism $\alpha\colon 0 \longrightarrow A$, we have

$$\alpha = \alpha \circ \mathrm{id}_0 = \alpha \circ 0 = \alpha \circ (0+0) = \alpha \circ 0 + \alpha \circ 0 = \alpha + \alpha.$$

Subtracting α on both sides gives $\alpha = 0$. So, 0 is an initial object in $\mathscr{C}$. In the same vein, we see that 0 is a terminal object, too.

ii) Recall that, for objects $A, B \in \mathrm{Ob}(\mathscr{C})$, the null morphism is the unique morphism $A \longrightarrow 0 \longrightarrow B$. Since $\circ\colon \mathrm{Mor}_{\mathscr{C}}(A,0) \times \mathrm{Mor}_{\mathscr{C}}(0,B) \longrightarrow \mathrm{Mor}_{\mathscr{C}}(A,B)$ is bilinear, this is the zero element of $\mathrm{Mor}_{\mathscr{C}}(A,B)$. □

I.4.8 Remark. Let $\alpha\colon A \longrightarrow B$ be a morphism in $\mathscr{C}$. For an object $C \in \mathrm{Ob}(\mathscr{C})$ and morphisms $\gamma_1, \gamma_2\colon C \longrightarrow A$, we have

$$\alpha \circ \gamma_1 = \alpha \circ \gamma_2 \quad \Longleftrightarrow \quad \alpha \circ (\gamma_1 - \gamma_2) = 0.$$

This shows that α is a monomorphism if and only if, for every object $C \in \mathrm{Ob}(\mathscr{C})$ and every morphism $\gamma\colon C \longrightarrow A$, the condition $\alpha \circ \gamma = 0$ implies $\gamma = 0$. In other words, α is a monomorphism if and only if $0 \longrightarrow A$ is the kernel of α.

Likewise, a morphism $\alpha\colon A \longrightarrow B$ is an epimorphism if and only if $B \longrightarrow 0$ is its cokernel.

Let $\mathscr{C}$ and $\mathscr{D}$ be additive categories. A functor $F\colon \mathscr{C} \longrightarrow \mathscr{D}$ is *additive*, if, for objects $A, B \in \mathrm{Ob}(\mathscr{D})$, the map

$$\mathrm{Mor}_{\mathscr{C}}(A,B) \longrightarrow \mathrm{Mor}_{\mathscr{D}}(F(A), F(B))$$

is a homomorphism of abelian groups.

I.4.9 Exercises (Direct sums and products). i) Let $\mathscr{A}$ be an **additive** category. Show that, for any two objects A and B of $\mathscr{A}$, the direct sum $A \oplus B$ comes with morphisms $\pi_A\colon A \oplus B \longrightarrow A$ and $\pi_B\colon A \oplus B \longrightarrow B$, such that $(A \oplus B, \pi_A, \pi_B)$ is the direct product of A and B.

ii) Let $\mathscr{A}$ be an additive category, $A, B, C \in \mathrm{Ob}(\mathscr{A})$, and $\iota_1\colon A \longrightarrow C$, $\iota_2\colon B \longrightarrow C$, $\pi_1\colon C \longrightarrow A$, and $\pi_2\colon C \longrightarrow B$ morphisms. Suppose

$$\pi_i \circ \iota_j = \begin{cases} 0, & \text{if } i \neq j \\ \mathrm{id}, & \text{if } i = j \end{cases}, \ i, j \in \{1,2\}, \quad \text{and} \quad \iota_1 \circ \pi_1 + \iota_2 \circ \pi_2 = \mathrm{id}_C$$

Show that (C, ι_1, ι_2) is the direct sum of A and B.

I.4.10 Remark. The attentive reader may wonder why there is no requirement regarding direct sums in the definition of additive functors. The reason is that the respective property is automatically satisfied, i.e., for $A, B \in \mathrm{Ob}(\mathscr{C})$, $(F(A \oplus B), F(\iota_A), F(\iota_B))$ is the direct sum of $F(A)$ and $F(B)$. This results easily from the characterization of direct sums in Exercise I.4.9, ii).

Images and coimages

Suppose that $\mathscr{C}$ is an additive category and $\alpha\colon A \longrightarrow B$ is a morphism in $\mathscr{C}$.

- ⋆ If α has a cokernel and the morphism $B \longrightarrow \mathrm{Coker}(\alpha)$ has a kernel, then this kernel is called the *image* of α and denoted by $\mathrm{Im}(\alpha)$.
- ⋆ If α has a kernel and the morphism $\mathrm{Ker}(\alpha) \longrightarrow A$ possesses a cokernel, then this cokernel is referred to as the *coimage* of α and denoted by $\mathrm{Coim}(\alpha)$.

I.4.11 Remark. Let $\alpha\colon A \longrightarrow B$ be a morphism in $\mathscr{D}$ which admits an image and a coimage. Then, the universal properties of kernels and cokernels show that there is an induced morphism

$$\overline{\alpha}\colon \mathrm{Coim}(\alpha) \longrightarrow \mathrm{Im}(\alpha),$$

such that

$$\alpha\colon A \longrightarrow \mathrm{Coim}(\alpha) \xrightarrow{\overline{\alpha}} \mathrm{Im}(\alpha) \longrightarrow B.$$

I.5 Abelian categories

Recall that the **first isomorphism theorem** in group theory ([26], Satz II.10.1) states that, for groups G, H, and a homomorphism $\alpha\colon G \longrightarrow H$, the induced homomorphism

$$\overline{\alpha}\colon G/\mathrm{Ker}(\alpha) \longrightarrow \mathrm{Im}(\alpha)$$

is an isomorphism.

Abelian categories are modelled on the category $\underline{\mathrm{Ab}}$ of abelian groups. In $\underline{\mathrm{Ab}}$, all kernels and cokernels do exist. The image of a homomorphism $\alpha\colon A \longrightarrow B$ coincides with the set theoretic image, and the coimage of α with $A/\mathrm{Ker}(\alpha)$.

An *abelian category* is an **additive category** $\mathscr{A}$, such that

- ⋆ every morphism in $\mathscr{A}$ has a kernel and a cokernel,
- ⋆ for every morphism $\alpha\colon A \longrightarrow B$, the induced morphism $\overline{\alpha}\colon \mathrm{Coim}(\alpha) \longrightarrow \mathrm{Im}(\alpha)$ (see Remark I.4.11) is an isomorphism.

The latter condition says that the first isomorphism theorem holds in $\mathscr{A}$. Of course, we use the other assumptions to be able to formulate it. As a first illustration, we show:

I.5.1 Lemma. *Let $\mathscr{C}$ be an abelian category. Then, a morphism $\alpha\colon A \longrightarrow B$ is an isomorphism if and only if it is both a mono- and an epimorphism.*

Proof. An isomorphism is clearly both a mono- and an epimorphism (compare Remark I.4.8). Now, assume that α is a monomorphism. Then, $0 \longrightarrow A$ is the kernel of α (loc. cit.) and $\mathrm{id}_A\colon A \longrightarrow A$ is the coimage of α. In the same fashion, we see that $\mathrm{id}_B\colon B \longrightarrow B$ is the image of α. Finally, $\overline{\alpha} = \alpha$. The axioms of an abelian category require that α is an isomorphism. □

I.5.2 Examples. i) The category $\underline{\mathrm{Ab}}$ of abelian groups is an abelian category.

ii) Let R be a commutative ring. Then, the category $\underline{\mathrm{Mod}}_R$ is an abelian category. If k is a field, this category is the category of k-vector spaces.

iii) Let R be a non-commutative ring. Then, one may form the category $\underline{\mathrm{Mod}}_R$ of **left** R-modules (see [16], Chapter III.1). It is also an abelian category.

I.5.3 Example (Pairs of vector spaces). Let k be a field. We form the category $\mathscr{C}$ whose objects are pairs (V, U) consisting of a k-vector space U and a sub vector space $V \subset U$. For two such pairs (V_1, U_1) and (V_2, U_2), we set

$$\mathrm{Mor}_{\mathscr{C}}((V_1, U_1), (V_2, U_2)) := \{\, f\colon U_1 \longrightarrow U_2 \mid f \text{ is } k\text{-linear and } f(V_1) \subset V_2 \,\}.$$

It is easy to see that this is an additive category in which kernels and cokernels do exist. In fact, let $f\colon (V_1, U_1) \longrightarrow (V_2, U_2)$ be a morphism. Then, $(V_1 \cap \mathrm{Ker}(f), \mathrm{Ker}(f))$ is the kernel for f, and $(\overline{V}_2, \mathrm{Coker}(f))$, $\overline{V}_2$ the image of V_2 under the projection $U_2 \longrightarrow \mathrm{Coker}(f)$, is the cokernel.

Now, look at the morphism $i\colon (0, k) \longrightarrow (k, k)$ induced by $\mathrm{id}_k\colon k \longrightarrow k$. Its kernel and cokernel are $(0, 0)$, yet it is not an isomorphism. In particular, $\mathscr{C}$ is not an abelian category.

I.5.4 Convention. Let $\mathscr{A}$ be an abelian category. Due to its similarities (see Theorem I.5.9) with the category $\underline{\mathrm{Ab}}$ of abelian groups, we will often refer to morphisms in $\mathscr{A}$ as **homomorphisms** and write $\mathrm{Hom}_{\mathscr{A}}(\cdot, \cdot)$ instead of $\mathrm{Mor}_{\mathscr{A}}(\cdot, \cdot)$.

Diagrams in abelian categories

Let $\mathscr{D}$ be a **small** category, e.g., the category of a quiver as in Example I.1.2, v), and $\mathscr{A}$ an abelian category. A *$\mathscr{D}$-diagram* in $\mathscr{A}$ is a functor $D\colon \mathscr{D} \longrightarrow \mathscr{A}$. According to Remark I.2.1, the $\mathscr{D}$-diagrams form a category $\underline{\mathrm{Diag}}_{\mathscr{D}}(\mathscr{A})$.

I.5.5 Exercise. Show that $\underline{\mathrm{Diag}}_{\mathscr{D}}(\mathscr{A})$ is an abelian category.

We will frequently use some examples coming from quivers. In some cases, we allow the set V of vertices to be infinite.

- ⋆ Let $n \in \mathbb{N}$ be a natural number. We form the quiver A_n with vertices $V := \{\,0, ..., n\,\}$, arrows $A := \{\,a_0, ..., a_{n-1}\,\}$, $t(a_i) = i$, and $h(a_i) = i + 1$, $i = 0, ..., n - 1$:

$$0 \xrightarrow{a_0} 1 \xrightarrow{a_1} 2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-2}} n - 1 \xrightarrow{a_{n-1}} n.$$

  The corresponding category will be denoted by $\mathscr{A}_n$.

- ⋆ The infinite quiver A_∞ has vertices $V = \mathbb{Z}$, arrows $A = \{\,a_k \mid k \in \mathbb{Z}\,\}$, $t(a_k) = k$, and $h(a_k) = k + 1$, $k \in \mathbb{Z}$. The resulting category gets the name $\mathscr{A}_\infty$.

★ The quiver A^3_∞ consists of the vertices $V = \mathbb{Z} \times \{0,1,2\}$, arrows $A = \{a_k, b_k, d_k \,|\, k \in \mathbb{Z}\}$, $t(a_k) = (k,0)$, $t(b_k) = (k,1)$, $t(c_k) = (k,2)$, $h(a_k) = (k,1)$, $h(b_k) = (k,2)$, $h(c_k) = (k+1,0)$, $k \in \mathbb{Z}$:

$$\cdots \xrightarrow{d_{k-1}} (k,0) \xrightarrow{a_k} (k,1) \xrightarrow{b_k} (k,2) \xrightarrow{d_k} (k+1,0) \xrightarrow{a_{k+1}} \cdots.$$

We get the category $\mathscr{A}^3_\infty$.

★ The quiver $A^{3,\star}_\infty$ has the vertices $V = \mathbb{Z} \times \{0,1,2\}$, arrows $A = \{a_k, b_k \,|\, k \in \mathbb{Z}\}$, $t(a_k) = (k,0)$, $t(b_k) = (k,1)$, $h(a_k) = (k,1)$, $h(b_k) = (k,2)$, $k \in \mathbb{Z}$.

I.5.6 Remarks. i) The categories $\mathscr{A}_\infty$ and $\mathscr{A}^3_\infty$ are isomorphic.

ii) Let $\mathscr{D}_1$ and $\mathscr{D}_2$ be small categories and $F\colon \mathscr{D}_1 \longrightarrow \mathscr{D}_2$ a functor. For an abelian category $\mathscr{A}$, it induces a functor $\underline{\mathrm{Diag}}(F)\colon \underline{\mathrm{Diag}}_{\mathscr{D}_2}(\mathscr{A}) \longrightarrow \underline{\mathrm{Diag}}_{\mathscr{D}_1}(\mathscr{A})$. For example, we have an obvious functor $\mathscr{A}^{3,\star}_\infty \longrightarrow \mathscr{A}^3_\infty$.

Let $n \in \mathbb{N}$ be a natural number and

$$A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{n-2}} A_{n-1} \xrightarrow{\alpha_{n-1}} A_n$$

an $\mathscr{A}_n$-diagram. It is an $\mathscr{A}_n$-*sequence*, if

★ $\alpha_k \circ \alpha_{k-1} = 0$,

★ the induced morphism $\mathrm{Im}(\alpha_{k-1}) \longrightarrow \mathrm{Ker}(\alpha_k)$ is an isomorphism, $k = 1, ..., n$.

We will abbreviate this condition as “$\mathrm{Im}(\alpha_{k-1}) = \mathrm{Ker}(\alpha_k)$”, $k = 1, ..., n$.

Likewise, an $\mathscr{A}_\infty$ diagram $D\colon \mathscr{A}_\infty \longrightarrow \mathscr{A}$ is *exact*, if

$$\forall k \in \mathbb{Z}: \quad \mathrm{Im}(D(a_k)) = \mathrm{Ker}(D(a_{k+1})).$$

This notion of exactness yields a notion of exactness for $\mathscr{A}^3_\infty$-diagrams: An $\mathscr{A}^3_\infty$-diagram $D\colon \mathscr{A}^3_\infty \longrightarrow \mathscr{A}$ is *exact*, if

$$\forall k \in \mathbb{Z}: \ \mathrm{Im}(D(a_k)) = \mathrm{Ker}(D(b_k)),\ \mathrm{Im}(D(b_k)) = \mathrm{Ker}(D(d_k)),\ \mathrm{Im}(D(d_k)) = \mathrm{Ker}(D(a_{k+1})).$$

We will refer to an exact $\mathscr{A}^3_\infty$-diagram as a *long exact cohomology sequence*. The full subcategory of $\underline{\mathrm{Diag}}_{\mathscr{A}^3_\infty}(\mathscr{A})$ of long exact cohomology sequences will also be denoted by $\underline{\mathrm{LECS}}(\mathscr{A})$.

An exact $\mathscr{A}_4$-sequence of the form

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

will be called a *short exact sequence*. This means that α is a monomorphism, β is an epimorphism, and $\mathrm{Im}(\alpha) = \mathrm{Ker}(\beta)$. The full subcategory of $\underline{\mathrm{Diag}}_{\mathscr{A}_4}(\mathscr{A})$ formed by the short exact sequences in $\mathscr{A}$ will be written as $\underline{\mathrm{SES}}(\mathscr{A})$.

Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories and $F\colon \mathscr{A} \longrightarrow \mathscr{B}$ a functor. We say that F is *exact*, if, for every short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in $\mathscr{A}$,

$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow 0$$

is a short exact sequence in $\mathscr{B}$. The functor F is *left exact*, if, for every sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C$$

in $\mathscr{A}$, the sequence

$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C)$$

is exact, too. We call F is *right exact*, if, for every sequence

$$A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in $\mathscr{A}$, the sequence

$$F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow 0$$

is exact. We say that F is *half exact*, if it is left or right exact or both.

I.5.7 Exercises (Direct sums in abelian categories). i) Let $\mathscr{A}$ be an **abelian** category and

$$0 \longrightarrow A \xrightarrow{\iota} B \longrightarrow C \longrightarrow 0$$

a short exact sequence in $\mathscr{A}$. Suppose that there exists a morphism $\pi\colon B \longrightarrow A$ with $\pi \circ \iota = \mathrm{id}_A$. Prove that B is isomorphic to the direct sum of A and C.

ii) Let G, H be not necessarily abelian groups, $\iota_G\colon G \longrightarrow G \times H$, $g \longmapsto (g, e)$, $\iota_H\colon H \longrightarrow G \times H$, $h \longmapsto (e, h)$, $p_G\colon G \times H \longrightarrow G$, $(g, h) \longmapsto g$, $p_H\colon G \times H \longrightarrow H$, $(g, h) \longmapsto h$. Show that $(G \times H, p_G, p_H)$ is a direct product in the category $\underline{\text{Groups}}$ of groups. Show by means of an example that $(G \times H, i_G, i_H)$ is, in general, not a direct sum in the category $\underline{\text{Groups}}$.[8] (**Hint**. Let C be a group and $\eta\colon G \longrightarrow C$, $\vartheta\colon H \longrightarrow C$ homomorphisms. Check that there is an induced homomorphism $\varkappa\colon G \times H \longrightarrow C$ with $\varkappa \circ \iota_G = \eta$, $\varkappa \circ \iota_B = \vartheta$ if and only if, for all $g \in G$, $h \in H$, $\eta(g) \cdot \vartheta(h) = \vartheta(h) \cdot \eta(g)$.)

I.5.8 Proposition. *A half exact functor is additive.*

Proof. We first note that F takes direct sums to direct sums. In fact, let A, B be objects in $\mathscr{A}$. The direct sum C of A and B is given by an exact sequence (see Exercise I.5.7, i)

$$0 \longrightarrow A \xrightarrow{\iota_A} C \xrightarrow{\pi_B} B \longrightarrow 0,$$

such that there are maps $\iota_B\colon B \longrightarrow C$ and $\pi_A\colon C \longrightarrow A$ with $\pi_A \circ \iota_A = \mathrm{id}_A$ and $\pi_B \circ \iota_B = \mathrm{id}_B$. Now, applying F gives the exact sequence

$$F(A) \xrightarrow{F(\iota_A)} F(C) \xrightarrow{F(\pi_B)} F(B).$$

Since $F(\pi_A) \circ F(\iota_A) = F(\pi_A \circ \iota_A) = F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ and $F(\pi_B) \circ F(\iota_B) = \mathrm{id}_{F(B)}$, it follows that $F(\iota_A)$ is a monomorphism (Remark I.4.8), $F(\pi_B)$ is an epimorphism, and $F(C)$ is the direct sum of $F(A)$ and $F(B)$, by Exercise I.5.7, i).

[8]The direct sum exists in the category of groups. For groups G, H, it is usually denoted by $G \star H$ and called the *free product* ([1], Exercise 27.11).

Now, let A be an object of $\mathscr{A}$. Since $A \oplus A$ is also the direct product (Exercise I.4.9, ii), $\mathrm{id}_A\colon A \longrightarrow A$ and $\mathrm{id}_A\colon A \longrightarrow A$ give a morphism $\Delta_A\colon A \longrightarrow A \oplus A$. It is readily verified that $F(\Delta_A) = \Delta_{F(A)}$. Next, let B be another object of $\mathscr{A}$ and $f, g\colon A \longrightarrow B$ morphisms. By the universal property of the direct sum, we find an induced map $f \oplus g\colon A \oplus A \longrightarrow B$. We leave it to the reader to verify that

$$f + g = (f \oplus g) \circ \Delta_A$$

and

$$F(f \oplus g) = F(f) \oplus F(g).$$

These facts together yield the claim. For more details, we refer the reader to [21], Abschnitt 4.6.[9] □

The Freyd–Mitchell theorem

The axioms for an abelian category hold certainly true in the category $\underline{\mathrm{Ab}}$ of abelian groups and, more generally, in the category $\underline{\mathrm{Mod}}_R$ of left R-modules, R a not necessarily commutative ring. We state (without proof) that these axioms in a certain sense characterize the category $\underline{\mathrm{Mod}}_R$.

I.5.9 Theorem (Freyd[10], Mitchell[11]). *Let $\mathscr{A}$ be a **small** abelian category. Then, there exist a not necessarily commutative ring and an **exact** functor*

$$F\colon \mathscr{A} \longrightarrow \underline{\mathrm{Mod}}_R$$

with the property that, for all objects $A, B \in \mathrm{Ob}(\mathscr{A})$, the map

$$F\colon \mathrm{Hom}_{\mathscr{A}}(A, B) \longrightarrow \mathrm{Hom}_R(F(A), F(B)) \tag{I.5}$$

is an isomorphism of groups.

Proof. [21], Abschnitt 4.14. □

I.5.10 Remark. A functor which satisfies (I.5) is said to be *fully faithful*. Observe that the definition of a fully faithful functor does not imply that $F\colon \mathrm{Ob}(\mathscr{A}) \longrightarrow \mathrm{Ob}(\underline{\mathrm{Mod}}_R)$ is injective. However, as on Page 9, one sees that a fully faithful functor induces an injection from the set of isomorphy classes in $\mathrm{Ob}(\mathscr{A})$ to the class of isomorphy classes in $\mathrm{Ob}(\underline{\mathrm{Mod}}_R)$.

I.5.11 Exercise (Abelian categories). Let $\mathscr{A}$ be an abelian category, and $\mathscr{C} \subset \mathrm{Ob}(\mathscr{A})$ a **subset** of objects. Show that there exists a **small** full abelian subcategory $\mathscr{B}$ of $\mathscr{A}$, such that $\mathscr{C} \subseteq \mathrm{Ob}(\mathscr{B})$.

Instructions. This reduces to the fact that certain operations on sets lead again to sets. The following two axioms of set theory might be helpful.

- (Axiom of replacement) If $F\colon C \longrightarrow D$ is a map between classes and $A \subset C$ is a set, then $F(A) \subset D$ is also a set.

[9] The notation is slightly different there.
[10] Peter J. Freyd (*1936), U.S. American mathematician.
[11] Barry Mitchell

- (Axiom of union) For a set A, there also exists the set

$$C = \bigcup_{B \in A} B,$$

i.e., C is a set, such that a set D is a member of C if and only if there is a set B such that D is a member of B and B is a member of A.

Theorem I.5.9 and Exercise I.5.11 show that, for a statement in an abelian category which involves, for example, only finitely many objects and morphisms between them, we can pretend to be in a category of the form $\underline{\mathrm{Mod}}_R$. The results provide "local coordinates" for abelian categories.

I.6 Complexes and cohomology

Let $\mathscr{A}$ be an abelian category and $(C^\bullet, \partial^\bullet)$ an $\mathscr{A}_\infty$-diagram in $\mathscr{A}$. Here, $C^\bullet$ stands for a family $(C^k, k \in \mathbb{Z})$ of objects in $\mathscr{A}$ and $\partial^\bullet$ for a family of homomorphisms $(\partial^k \colon C^k \longrightarrow C^{k+1}, k \in \mathbb{Z})$. The diagram $(C^\bullet, \partial^\bullet)$ is a *complex*, if

$$\forall k \in \mathbb{Z} : \quad \partial^{k+1} \circ \partial^k = 0.$$

The complexes of $\mathscr{A}$ form a full abelian subcategory of $\underline{\mathrm{Diag}}_{\mathscr{A}_\infty}(\mathscr{A})$. It will be denoted by $\underline{\mathrm{Compl}}(\mathscr{A})$.

Let $(C^\bullet, \partial^\bullet)$ be a complex in $\mathscr{A}$. We set

$$B^k(C^\bullet, \partial^\bullet) := \mathrm{Im}(\partial^{k-1}) \quad \text{and} \quad Z^k(C^\bullet, \partial^\bullet) := \mathrm{Ker}(\partial^k), \quad k \in \mathbb{Z}.$$

Since $\partial^k \circ \partial^{k-1} = 0$, there is an induced homomorphism $B^k(C^\bullet, \partial^\bullet) \longrightarrow Z^k(C^\bullet, \partial^\bullet)$, $k \in \mathbb{Z}$. The cokernel of this homomorphism is written as $H^k(C^\bullet, \partial^\bullet)$, or, more suggestively,

$$H^k(C^\bullet, \partial^\bullet) := Z^k(C^\bullet, \partial^\bullet)/B^k(C^\bullet, \partial^\bullet), \quad k \in \mathbb{Z}.$$

It is called the *k-th cohomology* of the complex $(C^\bullet, \partial^\bullet)$.

For complexes $(C^\bullet, \partial^\bullet_C)$ and $(D^\bullet, \partial^\bullet_D)$ in $\mathscr{A}$, a homomorphism $\varphi^\bullet \colon (C^\bullet, \partial^\bullet_C) \longrightarrow (D^\bullet, \partial^\bullet_D)$ consists of homomorphisms

$$\varphi^k \colon C^k \longrightarrow D^k,$$

such that the diagram

$$\begin{array}{ccc} C^k & \xrightarrow{\partial^k_C} & C^{k+1} \\ {\scriptstyle \varphi^k}\downarrow & & \downarrow{\scriptstyle \varphi^{k+1}} \\ D^k & \xrightarrow{\partial^k_D} & D^{k+1} \end{array}$$

commutes, i.e.,

$$\varphi^{k+1} \circ \partial^k_C = \partial^k_D \circ \varphi^k, \quad k \in \mathbb{Z}.$$

This implies

★ The composition
$$Z^k(C^\bullet, \partial_C^\bullet) \longrightarrow C^k \xrightarrow{\varphi^k} D^k \xrightarrow{\partial_D^k} D^{k+1}$$
is zero, $k \in \mathbb{Z}$. By the universal property of the kernel (Page 12), there is an induced homomorphism
$$\widetilde{\varphi}^k \colon Z^k(C^\bullet, \partial_C^\bullet) \longrightarrow Z^k(D^\bullet, \partial_D^\bullet), \quad k \in \mathbb{Z}.$$

★ The composition
$$B^k(C^\bullet, \partial_C^\bullet) \longrightarrow Z^k(C^\bullet, \partial_C^\bullet) \xrightarrow{\widetilde{\varphi}^k} Z^k(D^\bullet, \partial_D^\bullet) \longrightarrow H^k(D^\bullet, \partial_D^\bullet)$$
is zero, $k \in \mathbb{Z}$. By the universal property of the cokernel (Page 12), we get an induced homomorphism
$$H^k(\varphi^\bullet) \colon H^k(C^\bullet, \partial_C^\bullet) \longrightarrow H^k(D^\bullet, \partial_D^\bullet), \quad k \in \mathbb{Z}.$$

We call $\varphi^\bullet$ a *quasi-isomorphism* (*qism*, for short), if $H^k(\varphi^\bullet)$ is an isomorphism, $k \in \mathbb{Z}$. The construction shows that we indeed get functors
$$H^k \colon \underline{\mathrm{Compl}}(\mathscr{A}) \longrightarrow \mathscr{A}, \quad k \in \mathbb{Z}.$$
We will view the collection $(H^k, k \in \mathbb{Z})$ as a functor
$$H^\bullet \colon \underline{\mathrm{SES}}(\underline{\mathrm{Compl}}(\mathscr{A})) \longrightarrow \underline{\mathrm{Diag}}_{\mathscr{A}_\infty^{3,\star}}(\mathscr{A}).$$
It associates with a short exact sequence
$$0 \longrightarrow (A^\bullet, \partial_A^\bullet) \xrightarrow{\alpha^\bullet} (B^\bullet, \partial_B^\bullet) \xrightarrow{\beta^\bullet} (C^\bullet, \partial_C^\bullet) \longrightarrow 0$$
in $\underline{\mathrm{Compl}}(\mathscr{A})$ the diagrams
$$H^k(A^\bullet, \partial_A^\bullet) \xrightarrow{H^k(\alpha^\bullet)} H^k(B^\bullet, \partial_B^\bullet) \xrightarrow{H^k(\beta^\bullet)} H^k(C^\bullet, \partial_C^\bullet)$$
in $\mathscr{A}$, $k \in \mathbb{Z}$.

Let $(C^\bullet, \partial_C^\bullet)$ and $(D^\bullet, \partial_D^\bullet)$ be complexes in $\mathscr{A}$ and $\varphi^\bullet \colon (C^\bullet, \partial_C^\bullet) \longrightarrow (D^\bullet, \partial_D^\bullet)$ a homomorphism. We say that $\varphi^\bullet$ is *homotopic to zero*,[12] if there are homomorphisms $h^k \colon C^k \longrightarrow D^{k-1}$ with
$$\varphi^k = h^{k+1} \circ \partial_C^k + \partial_D^{k-1} \circ h^k, \quad k \in \mathbb{Z};$$
$$\begin{tikzcd} C^{k-1} \arrow[r, "\partial_C^{k-1}"] & C^k \arrow[r, "\partial_C^k"] \arrow[dl, "h^k"'] \arrow[d, "\varphi^k"] & C^{k+1} \arrow[dl, "h^{k+1}"'] \\ D^{k-1} \arrow[r, "\partial_D^{k-1}"] & D^k \arrow[r, "\partial_D^k"] & D^{k+1}. \end{tikzcd}$$
Two homomorphisms $\varphi^\bullet, \psi^\bullet \colon (C^\bullet, \partial_C^\bullet) \longrightarrow (D^\bullet, \partial_D^\bullet)$ of complexes are *homotopic*, if their difference $\varphi^\bullet - \psi^\bullet$ is homotopic to zero. A homomorphism $\varphi^\bullet \colon (C^\bullet, \partial_C^\bullet) \longrightarrow (D^\bullet, \partial_D^\bullet)$ is a *homotopy equivalence*, if there is a homomorphism $\psi^\bullet \colon (D^\bullet, \partial_D^\bullet) \longrightarrow (C^\bullet, \partial_C^\bullet)$, such that $\psi^\bullet \circ \varphi^\bullet$ is homotopic to $\mathrm{id}_{(C^\bullet, \partial_C^\bullet)}$ and $\varphi^\bullet \circ \psi^\bullet$ to $\mathrm{id}_{(D^\bullet, \partial_D^\bullet)}$. Two complexes are *homotopic*, if there exists a homotopy equivalence between them.

[12]This terminology has its origins in topology where homotopies between continuous maps lead to homotopies between the induced homomorphisms of complexes of singular (co)chains.

I.6.1 Proposition. *Let $(C^\bullet, \partial^\bullet_C)$ and $(D^\bullet, \partial^\bullet_D)$ be complexes in $\mathscr{A}$ and $\varphi^\bullet, \psi^\bullet \colon (C^\bullet, \partial^\bullet_C) \longrightarrow (D^\bullet, \partial^\bullet_D)$ homomorphisms of complexes. If $\varphi^\bullet$ and $\psi^\bullet$ are homotopic, then*

$$\forall k \in \mathbb{Z} : \quad H^k(\varphi^\bullet) = H^k(\psi^\bullet).$$

Proof. By additivity, it is sufficient to establish the assertion for $\psi^\bullet = 0$. The assumption means that there are homomorphisms $h^k \colon C^k \longrightarrow D^{k-1}$ with

$$\varphi^k = h^{k+1} \circ \partial^k_C + \partial^{k-1}_D \circ h^k, \quad k \in \mathbb{Z}.$$

This implies[13]

$$\varphi^k_{|Z^k(C^\bullet,\partial^\bullet_C)} = (\partial^{k-1}_D \circ h^k)_{|Z^k(C^\bullet,\partial^\bullet_C)},$$

so that

$$\operatorname{Im}(\varphi^k_{|Z^k(C^\bullet,\partial^\bullet_C)}) \subset B^k(D^\bullet, \partial^\bullet_D), \quad k \in \mathbb{Z}.$$

It follows that $H^k(\varphi^\bullet) = 0$ as claimed, $k \in \mathbb{Z}$. □

I.7 The long exact sequence of cohomology

We now present one of the main tools of homological algebra.

I.7.1 Theorem. *The functor*

$$H^\bullet \colon \underline{\mathrm{SES}}(\underline{\mathrm{Compl}}(\mathscr{A})) \longrightarrow \underline{\mathrm{Diag}}_{\mathscr{A}^{3,\star}_\infty}(\mathscr{A})$$

extends to a functor

$$(H^\bullet, \delta^\bullet) \colon \underline{\mathrm{SES}}(\underline{\mathrm{Compl}}(\mathscr{A})) \longrightarrow \underline{\mathrm{LECS}}(\mathscr{A}).$$

This means:

i) *For every short exact sequence*

$$0 \longrightarrow (A^\bullet, \partial^\bullet_A) \xrightarrow{\alpha^\bullet} (B^\bullet, \partial^\bullet_B) \xrightarrow{\beta^\bullet} (C^\bullet, \partial^\bullet_C) \longrightarrow 0$$

of complexes in $\mathscr{A}$:

a) *The sequences*

$$H^k(A^\bullet, \partial^\bullet_A) \xrightarrow{H^k(\alpha^\bullet)} H^k(B^\bullet, \partial^\bullet_B) \xrightarrow{H^k(\beta^\bullet)} H^k(C^\bullet, \partial^\bullet_C)$$

are exact, $k \in \mathbb{Z}$.

b) *There are* ***connecting homomorphisms***

$$\delta^k \colon H^k(C^\bullet, \partial^\bullet_C) \longrightarrow H^{k+1}(A^\bullet, \partial^\bullet_A),$$

such that the sequences

$$H^k(B^\bullet, \partial^\bullet_B) \xrightarrow{H^k(\beta^\bullet)} H^k(C^\bullet, \partial^\bullet_C) \xrightarrow{\delta^k} H^{k+1}(A^\bullet, \partial^\bullet_A) \xrightarrow{H^{k+1}(\alpha^\bullet)} H^{k+1}(B^\bullet, \partial^\bullet_B)$$

[13] Here, we slightly abuse notation.

are exact, $k \in \mathbb{Z}$.

ii) *Given a morphism*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (A^\bullet, \partial_A^\bullet) & \xrightarrow{\alpha^\bullet} & (B^\bullet, \partial_B^\bullet) & \xrightarrow{\beta^\bullet} & (C^\bullet, \partial_C^\bullet) & \longrightarrow & 0 \\
\| & & {\scriptstyle\varphi^\bullet}\downarrow & & {\scriptstyle\psi^\bullet}\downarrow & & {\scriptstyle\tau^\bullet}\downarrow & & \| \\
0 & \longrightarrow & (A'^\bullet, \partial_{A'}^\bullet) & \xrightarrow{\alpha'^\bullet} & (B'^\bullet, \partial_{B'}^\bullet) & \xrightarrow{\beta'^\bullet} & (C'^\bullet, \partial_{C'}^\bullet) & \longrightarrow & 0
\end{array}$$

between short exact sequences of complexes in $\mathscr{A}$, *the diagram*[14]

$$\begin{array}{ccc}
H^k(C^\bullet, \partial_C^\bullet) & \xrightarrow{\delta^k} & H^{k+1}(A^\bullet, \partial_A^\bullet) \\
{\scriptstyle H^k(\tau^\bullet)}\downarrow & & \downarrow{\scriptstyle H^{k+1}(\varphi^\bullet)} \\
H^k(C'^\bullet, \partial_{C'}^\bullet) & \xrightarrow{\delta^k} & H^{k+1}(A'^\bullet, \partial_{A'}^\bullet)
\end{array} \tag{I.6}$$

commutes, for every $k \in \mathbb{Z}$.

Proof. We will abbreviate $A := (A^\bullet, \partial_A^\bullet)$, $B := (B^\bullet, \partial_B^\bullet)$, $C := (C^\bullet, \partial_C^\bullet)$, $\alpha := \alpha^\bullet$, $\beta := \beta^\bullet$. Let $k \in \mathbb{Z}$ be an integer. The composition

$$A^{k-1} \xrightarrow{\partial_A^{k-1}} A^k \xrightarrow{\partial_A^k} A^{k+1}$$

is zero. Therefore, there is an induced morphism

$$\overline{\partial}_A^k \colon \operatorname{Coker}(\partial_A^{k-1}) \longrightarrow A^{k+1}.$$

Note that $\operatorname{Im}(\overline{\partial}_A^k) = \operatorname{Im}(\partial_A^k)$, and there is an induced monomorphism $\operatorname{Im}(\overline{\partial}_A^k) \longrightarrow \operatorname{Ker}(\partial_A^{k+1})$. By definition, we have the exact sequence

$$0 \longrightarrow H^k(A) \xrightarrow{i_A} \operatorname{Coker}(\partial_A^{k-1}) \xrightarrow{\overline{\partial}_A^k} \operatorname{Ker}(\partial_A^{k+1}) \xrightarrow{p_A} H^{k+1}(A) \longrightarrow 0.$$

There are corresponding sequences for the complexes B and C. We form the commutative

[14]We have refrained from putting the short exact sequence as an index of the connecting homomorphisms. It should be clear to the reader that, in the following diagram, the connecting homomorphisms above and below are associated with the upper and lower short exact sequence in the former diagram, respectively.

diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & H^k(A) \arrow[r, "H^k(\alpha)"] \arrow[d, "i_A"] & H^k(B) \arrow[r, "H^k(\beta)"] \arrow[d, "i_B"] & H^k(C) \arrow[d, "i_C"] & \\
 & \mathrm{Coker}(\partial_A^{k-1}) \arrow[r, "\overline{\alpha}^k"] \arrow[d, "\overline{\partial}^k_A"] & \mathrm{Coker}(\partial_B^{k-1}) \arrow[r, "\overline{\beta}^k"] \arrow[d, "\overline{\partial}^k_B"] & \mathrm{Coker}(\partial_C^{k-1}) \arrow[r] \arrow[d, "\overline{\partial}^k_C"] & 0. \\
0 \arrow[r] & \mathrm{Ker}(\partial_A^{k+1}) \arrow[r, "\alpha^{k+1}"] \arrow[d, "p_A"] & \mathrm{Ker}(\partial_B^{k+1}) \arrow[r, "\beta^{k+1}"] \arrow[d, "p_B"] & \mathrm{Ker}(\partial_C^{k+1}) \arrow[d, "p_C"] & \\
 & H^{k+1}(A) \arrow[r, "H^{k+1}(\alpha)"] \arrow[d] & H^{k+1}(B) \arrow[r, "H^{k+1}(\beta)"] \arrow[d] & H^{k+1}(C) \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{I.7}$$

The columns of this diagram are exact, and it is also clear that the third and the fourth row are exact. We will first show that the second and fifth row are exact, too.

For this, we apply Exercise I.5.11 and the Freyd–Mitchell embedding theorem I.5.9 and assume that Diagram (I.7) is a diagram in the category $\underline{\mathrm{Mod}}_R$, for some ring R. The technique we will employ in the proofs and constructions is called **diagram chasing**.

Exactness at $H^k(B)$. By functoriality, we have

$$H^k(\beta) \circ H^k(\alpha) = H^k(\beta \circ \alpha) = 0, \quad \text{so that} \quad \mathrm{Im}(H^k(\alpha)) \subset \mathrm{Ker}(H^k(\beta)).$$

Let $h \in H^k(B)$ be an element[15] with $H^k(\beta)(h) = 0$. The commutativity of (I.7) implies $\overline{\beta}^k(i_B(h)) = 0$. Using the exactness of the third row, there exists an element $h' \in \mathrm{Coker}(\partial_A^{k-1})$, such that $\overline{\alpha}^k(h') = i_B(h)$. We compute

$$\alpha^{k+1}(\overline{\partial}^k_A(h')) = \overline{\partial}^k_B(\overline{\alpha}^k(h')) = \overline{\partial}^k_B(i_B(h)) = 0.$$

Since α^{k+1} is injective, it follows that $\overline{\partial}^k_A(h') = 0$. By exactness of the first column, there is an element $h'' \in H^k(A)$ with $i_A(h'') = h'$. We find

$$i_B(H^k(\alpha)(h'')) = \overline{\alpha}^k(i_A(h'')) = \overline{\alpha}^k(h') = i_B(h).$$

The injectivity of i_B gives $H^k(\alpha)(h'') = h$. Thus, $\mathrm{Im}(H^k(\alpha)) \supset \mathrm{Ker}(H^k(\beta))$.

Definition of δ^k. Let $h \in H^k(C)$. We choose an element $h' \in \mathrm{Coker}(\partial_B^{k-1})$, such that $\overline{\beta}^k(h') = i_C(h)$. We find

$$\beta^{k+1}(\overline{\partial}^k_B(h')) = \overline{\partial}^k_C(\overline{\beta}^k(h')) = \overline{\partial}^k_C(i_C(h)) = 0.$$

The exactness of the fourth row shows that there is a **unique** element $h'' \in \mathrm{Ker}(\partial_A^{k+1})$ with $\alpha^{k+1}(h'') = \overline{\partial}^k_B(h')$. Now, we set

$$\delta^k(h) = p_A(h''). \tag{I.8}$$

[15] Recall that we assume that we are in the category $\underline{\mathrm{Mod}}_R$, so that we may speak of elements.

In order to see that this is well-defined, pick another element $\widetilde{h}' \in \mathrm{Coker}(\partial_B^{k-1})$ with $\overline{\beta}^k(\widetilde{h}') = i_C(h)$. Then, $\widetilde{h}' - h' \in \mathrm{Ker}(\overline{\beta}^k)$. So, there is an element $\overline{h}' \in \mathrm{Coker}(\partial_A^{k-1})$ with $\overline{\alpha}^k(\overline{h}') = \widetilde{h}' - h'$. Diagram (I.7) shows

$$\widetilde{h}'' = h'' + \overline{\partial}_A^k(\overline{h}'),$$

and we infer $p_A(\widetilde{h}'') = p_A(h'')$.

Exactness at $H^k(C)$. For $h \in H^k(B)$, the element $i_B(h)$ is a lift of $i_C(H^k(\beta))(h)$. Since the second column is exact, $\overline{\partial}^k(B)(i_B(h)) = 0$. This gives $\delta^k \circ H^k(\beta) = 0$.

Next, let $h \in H^k(C)$ be an element with $p_A(h'') = \delta^k(h) = 0$. This implies that there exists an element $h''' \in \mathrm{Coker}(\partial_A^{k-1})$, such that $\overline{\partial}_A^k(h''') = h''$. We may replace h' by $h' - \overline{\alpha}^k(h''')$. This shows that we may assume without loss of generality $\overline{\partial}_B^k(h') = 0$. Under this assumption, there exists an element $\widetilde{h}$ with $i_B(\widetilde{h}) = h'$, and we compute

$$i_C(H^k(\beta)(\widetilde{h})) = \overline{\beta}^k(i_B(\widetilde{h})) = \overline{\beta}^k(h') = i_C(h).$$

The injectivity of i_C finally yields $h = H^k(\beta)(\widetilde{h})$.

Exactness at $H^{k+1}(A)$. For $h \in H^k(C)$, we have

$$H^{k+1}(\alpha)(\delta^k(h)) = p_B(\alpha^{k+1}(h'')) = p_B(\overline{\partial}_B^k(h')) = 0.$$

Now, let $\widetilde{h} \in H^{k+1}(A)$ be an element with $H^{k+1}(\alpha)(\widetilde{h}) = 0$. Fix an element $h'' \in \mathrm{Ker}(\partial_A^{k+1})$ with $p_A(h'') = \widetilde{h}$. The commutativity of (I.7) shows

$$p_B(\alpha^{k+1}(h'')) = H^{k+1}(\alpha)(p_A(h'')) = H^{k+1}(\alpha)(\widetilde{h}) = 0.$$

By the exactness of the second column, there is an element $h' \in \mathrm{Coker}(\partial_B^{k-1})$ with $\overline{\partial}_B^k(h') = \alpha^{k+1}(h'')$. Then,

$$\overline{\partial}_C^k(\overline{\beta}^k(h')) = \beta^{k+1}(\overline{\partial}_B^k(h')) = \beta^{k+1}(\alpha^{k+1}(h'')) = 0,$$

and we find an element $h \in H^k(C)$ with $i_C(h) = \overline{\beta}^k(h')$. The construction gives $\delta^k(h) = \widetilde{h}$.

Functoriality. Diagram (I.7) is a diagram in the category $\mathscr{A}$ in the sense defined on Page 16. It is readily verified that this diagram depends functorially on the short exact sequence. Given a morphism from Diagram (I.7) to the corresponding diagram associated with

$$0 \longrightarrow A' \longrightarrow B' \longrightarrow C' \longrightarrow 0,$$

it is straightforward to check with the definitions that (I.6) commutes. □

I.7.2 Exercise (The snake lemma). Let $\mathscr{A}$ be an abelian category and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ \| & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \| \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

a commutative diagram in $\mathscr{A}$ with exact rows. Show that it induces an exact sequence

$$0 \longrightarrow \mathrm{Ker}(f) \longrightarrow \mathrm{Ker}(g) \longrightarrow \mathrm{Ker}(h) \longrightarrow \mathrm{Coker}(f) \longrightarrow \mathrm{Coker}(g) \longrightarrow \mathrm{Coker}(h) \longrightarrow 0.$$

Theorem I.7.1 gives a general framework for long exact cohomology sequences which includes the long exact sequence in the singular (co)homology of topological spaces. The next question we would like to address is how we get interesting complexes of which we would like to take the cohomology.

I.8 Resolutions

In this section, we will explain how we can associate in certain abelian categories with any object a complex, called **resolution**, in a canonical way. First, we look at two half exact functors which exist on any abelian category.

I.8.1 Lemma. *Let $\mathscr{A}$ be an abelian category and $A \in \mathrm{Ob}(\mathscr{A})$ an object of $\mathscr{A}$.*

i) *The functor* $\mathrm{Hom}_{\mathscr{A}}(A, \cdot)$ *is left exact.*

ii) *The contravariant functor* $\mathrm{Hom}_{\mathscr{A}}(\cdot, A)$ *is left exact.*

Proof. Since

$$\mathrm{Hom}_{\mathscr{A}}(\cdot, A) = \mathrm{Hom}_{\mathscr{A}^{\mathrm{opp}}}(A, \cdot)$$

and $\mathscr{A}^{\mathrm{opp}}$ is also an abelian category (Remark I.4.5, ii), Exercise I.4.9), it suffices to prove Part i).

Given a short exact sequence

$$0 \longrightarrow B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} B_3 \longrightarrow 0$$

in $\mathscr{A}$, we look at the sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathscr{A}}(A, B_1) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(A, B_2) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(A, B_3)$$

of abelian groups.

Exactness at $\mathrm{Hom}_{\mathscr{A}}(A,B_1)$. This means that, for any homomorphism $\alpha\colon A \longrightarrow B_1$, the condition $\beta_1 \circ \alpha = 0$ implies $\alpha = 0$. This results from the fact that β_1 is a monomorphism (Page 11).

Exactness at $\mathrm{Hom}_{\mathscr{A}}(A,B_2)$. The composition

$$\mathrm{Hom}_{\mathscr{A}}(A, B_1) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(A, B_3)$$

is

$$\alpha \longmapsto (\beta_2 \circ \beta_1) \circ \alpha$$

and, thus, zero.

Now, assume that $\alpha \in \mathrm{Hom}_{\mathscr{A}}(A, B_2)$ is a homomorphism with $\beta_2 \circ \alpha = 0$. Then, α factorizes over $\mathrm{Ker}(\beta_2)$. There are induced isomorphisms

$$B_1 \xrightarrow{\widetilde{\beta}_1} \mathrm{Im}(\beta_1) \xrightarrow{\overline{\beta}_2} \mathrm{Ker}(\beta_2).$$

This implies that there is a unique homomorphism $\alpha'\colon A \longrightarrow B_1$ with $\beta_1 \circ \alpha' = \alpha$. □

I.8.2 Exercise (Hom is not right exact). Give an example of an abelian category $\mathscr{A}$ and an object A of $\mathscr{A}$, such that the functor $\mathrm{Hom}_{\mathscr{A}}(A, -)$ is not right exact.

An object $A \in \mathrm{Ob}(\mathscr{A})$ is *projective*, if the functor $\mathrm{Hom}_{\mathscr{A}}(A, \cdot)$ is exact, and *injective*, if the functor $\mathrm{Hom}_{\mathscr{A}}(\cdot, A)$ is exact.

I.8.3 Remark. Projective objects in $\mathscr{A}$ correspond to injective objects in $\mathscr{A}^{\mathrm{opp}}$. So, for general statements and investigations, we may restrict to projective or injective objects. In the sequel, we will mainly speak about injective objects.

The following example shows that projective and injective objects in an abelian category might look quite different.

I.8.4 Example. i) Recall that, for an abelian group A,

$$\begin{array}{rcl} \mathrm{Hom}_{\underline{\mathrm{Ab}}}(\mathbb{Z}, A) & \longrightarrow & A \\ \varphi & \longmapsto & \varphi(1) \end{array}$$

is an isomorphism of abelian groups. Using this fact, it is easy to see that $\mathbb{Z}$ is a projective object in the category $\underline{\mathrm{Ab}}$ of abelian groups. In the same vein, one checks that any free abelian group is projective.

ii) Suppose that A is an **injective** abelian group, and let $n \geq 1$. We apply $\mathrm{Hom}_{\underline{\mathrm{Ab}}}(\cdot, A)$ to the exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{k \to n \cdot k} \mathbb{Z} \longrightarrow \mathbb{Z}/\langle n \rangle \longrightarrow 0.$$

Using the observation from Part i), the homomorphism

$$\mathrm{Hom}_{\underline{\mathrm{Ab}}}(\mathbb{Z}, A) \longrightarrow \mathrm{Hom}_{\underline{\mathrm{Ab}}}(\mathbb{Z}, A)$$

corresponds to

$$\begin{array}{rcl} A & \longrightarrow & A \\ a & \longmapsto & n \cdot a. \end{array}$$

This is surjective if and only if

$$\forall a \in A \exists b \in A : \quad n \cdot b = a.$$

Informally speaking, this means that every element $a \in A$ may be divided by n. A group A satisfying

$$\forall a \in A \forall n \geq 1 \exists b \in A : \quad n \cdot b = a. \tag{I.9}$$

is said to be *divisible*.

Our discussion shows that every injective abelian group is divisible. The converse will be established in Theorem I.12.2. The group $\mathbb{Z}$ is not divisible and, therefore, not injective. The abelian group $(\mathbb{Q}, +)$ is divisible.

An abelian category $\mathscr{A}$ has *enough injectives*, if, for every object $A \in \mathrm{Ob}(\mathscr{A})$, there are an **injective** object $I \in \mathrm{Ob}(\mathscr{A})$ and a monomorphism $i \colon A \longrightarrow I$.

I.8.5 Remark. We can form the category whose objects are triples (A, I, i) which consist of an object $A \in \mathrm{Ob}(\mathscr{A})$, an injective object $I \in \mathrm{Ob}(\mathscr{A})$, and a monomorphism $i \colon A \longrightarrow I$. Let (A, I, i) and (A', I', i') be triples as above. There is no morphism from (A, I, i) to (A', I', i'), if $A \neq A'$. If $A = A'$, a *morphism* from (A, I, i) to (A, I', i') is a morphism $f \colon I \longrightarrow I'$ with $i' = f \circ i$;

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[dr, tail, "i'"'] & I \arrow[d, "f"] \\ & I'. \end{tikzcd}$$

We use the axiom of choice as in Example I.2.3. If $\mathscr{A}$ is a category with enough injectives, we fix in this way, for every object $A \in \mathrm{Ob}(\mathscr{A})$, an injective object $I(A) \in \mathrm{Ob}(\mathscr{A})$ and a monomorphism $i_A \colon A \longrightarrow I(A)$.

Let $\mathscr{A}$ be an abelian category and $A \in \mathrm{Ob}(\mathscr{A})$ an object. A complex $(I^\bullet, \partial_I^\bullet)$ in $\mathscr{A}$ is an *injective resolution* of A, if

* $I^k = 0$, for $k < 0$,
* I_n is an injective object in $\mathscr{A}$, $n \in \mathbb{N}$,
* $\mathrm{Ker}(\partial_I^0)$ is isomorphic to A,
* $\mathrm{Im}(\partial_I^n) = \mathrm{Ker}(\partial_I^{n+1})$, $n \in \mathbb{N}$.[16]

We will frequently write an injective resolution of A in the form

$$0 \longrightarrow A \xrightarrow{\ i\ } I^0 \xrightarrow{\partial_I^0} I^1 \xrightarrow{\partial_I^1} \cdots$$

and even suppress the names of the morphisms.

I.8.6 Lemma. *Suppose that the abelian category $\mathscr{A}$ has* ***enough injectives****. Then, every object of $\mathscr{A}$ has an injective resolution.*

Proof. We define $I^k := 0$, $k < 0$, and $I^0 := I(A)$. The remaining injective objects and derivatives will be defined by recursion:

* $C^0 := \mathrm{Coker}(i_A)$, $I^1 := I(C^0)$, and $\partial_I^0 \colon I^0 \longrightarrow C^0 \xrightarrow{i_{C^0}} I^1$,
* $C^{n+1} := \mathrm{Coker}(\partial_I^n)$, $I^{n+2} := I(C^{n+1})$, and $\partial_I^{n+1} \colon I^{n+1} \longrightarrow C^{n+1} \xrightarrow{i_{C^{n+1}}} I^{n+2}$, $n \in \mathbb{N}$.

It is immediate from the construction that the result is an injective resolution of A. □

I.8.7 Remark. Let $A \in \mathrm{Ob}(\mathscr{A})$ be an object of $\mathscr{A}$ and $(I^\bullet, \partial_I^\bullet)$ an injective resolution of A. We complete this to a complex, also denoted by $(I^\bullet, \partial_I^\bullet)$, by setting

$$I^k := 0 \quad \text{and} \quad \partial_I^k := 0, \quad k < 0.$$

Then,

$$H^k(I^\bullet, \partial_I^\bullet) = \begin{cases} A, & \text{if } k = 0 \\ 0, & \text{if } k \neq 0 \end{cases}.$$

On the other hand, we may form the complex $(A^\bullet, \partial_A^\bullet)$ with

$$A^k = \begin{cases} A, & \text{if } k = 0 \\ 0, & \text{if } k \neq 0 \end{cases}$$

and the obvious differentials ∂_A^k, $k \in \mathbb{Z}$. The monomorphism $A^0 = A \longrightarrow I^0$ from the injective resolution and the zero homomorphisms $A^k = 0 \longrightarrow I^k$, $k \neq 0$, form a homomorphism $(A^\bullet, \partial_A^\bullet) \longrightarrow (I^\bullet, \partial_I^\bullet)$ of complexes which is obviously a quasi-isomorphism (see Page 21).

[16]This is abusive notation, coming from abelian categories of the form $\underline{\mathrm{Mod}}_R$, R a not necessarily commutative ring. We leave it to the reader to define this condition in exact categorical language.

We will denote the injective resolution of A from Lemma I.8.6 by $(I^\bullet(A), \partial^\bullet_{I(A)})$. The injective resolution we have just constructed depends on several choices. We will next demonstrate that, when it comes to cohomology, it is canonical.

I.8.8 Proposition. *Let $A, B \in \mathrm{Ob}(\mathscr{A})$ be objects and*

$$0 \longrightarrow A \longrightarrow I^0 \xrightarrow{\partial_I^0} I^1 \xrightarrow{\partial_I^1} \cdots,$$

$$0 \longrightarrow B \longrightarrow J^0 \xrightarrow{\partial_J^0} J^1 \xrightarrow{\partial_J^1} \cdots$$

injective resolutions of A and B, respectively. For every homomorphism $\alpha \in \mathrm{Hom}_{\mathscr{A}}(A, B)$, there is a homomorphism $\alpha^\bullet \colon (I^\bullet, \partial_I^\bullet) \longrightarrow (J^\bullet, \partial_J^\bullet)$ of complexes, such that[17]

$$\begin{tikzcd}
(A^\bullet, \partial_A^\bullet) \arrow[r] \arrow[d, "\alpha"'] & (I^\bullet, \partial_I^\bullet) \arrow[d, "\alpha^\bullet"] \\
(B^\bullet, \partial_B^\bullet) \arrow[r] & (J^\bullet, \partial_J^\bullet)
\end{tikzcd}$$

is a commutative diagram in the category of complexes in $\mathscr{A}$. The homomorphism $\alpha^\bullet$ is unique up to homotopy.

Proof. We first explain how the homomorphism $\alpha^\bullet$ is constructed. First, we obtain the diagram

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, "\alpha"'] \arrow[dr] & I^0 \arrow[d, dashed, "\alpha^0"] \\
B \arrow[r, tail] & J^0.
\end{tikzcd}$$

Note that α^0 exists, because J^0 is an injective object. The diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\alpha"'] & I^0 \arrow[r, "\partial_I^0"] \arrow[d, "\alpha^0"] & I^1 \\
B \arrow[r] & J^0 \arrow[r, "\partial_J^0"] & J^1
\end{tikzcd}$$

induces

$$\begin{tikzcd}
I^0 \arrow[r] \arrow[d, "\alpha^0"'] & I^0/A \arrow[r, tail] \arrow[d, "\overline{\alpha}^0"'] \arrow[dr] & I^1 \arrow[d, dashed, "\alpha^1"] \\
J^0 \arrow[r] & J^0/B \arrow[r, tail] & J^1.
\end{tikzcd}$$

Here, α^1 exists, because J^1 is an injective object. The construction may clearly be iterated.

Next, we explain how we obtain the homotopy. Here, we may obviously assume that $\alpha = 0$. In this case, we have to explain why $\alpha^\bullet$ is homotopic to zero. The diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "0"'] & I^0 \arrow[r, "\partial_I^0"] \arrow[d, "\alpha^0"] & I^1 \\
B \arrow[r] & J^0 \arrow[r, "\partial_J^0"] & J^1
\end{tikzcd}$$

[17] We use the notation of Remark I.8.7.

yields

$$\begin{tikzcd} I^0/A \arrow[r, tail, "\overline{\partial}^0_I"] \arrow[d] & I^1 \arrow[dl, dashed, "h^1"'] \\ J^0 \arrow[r, "\partial^0_J"] & J^1. \end{tikzcd}$$

The existence of h^1 follows from the injectivity of J^0. Next, the diagram

$$\begin{tikzcd} I^0 \arrow[r, "\partial^0_I"] & I^1 \arrow[r, "\partial^1_I"] \arrow[d, "\alpha^1 - \partial^0_J \circ h^1"] & I^2 \\ J^0 \arrow[r, "\partial^0_J"] & J^1 & \end{tikzcd}$$

leads to

$$\begin{tikzcd} \operatorname{Coker}(\partial^0_I) \arrow[r, tail, "\overline{\partial}^1_I"] \arrow[d, "\overline{(\alpha^1 - \partial^0_J \circ h^1)}"'] & I^2 \arrow[dl, dashed, "h^2"] \\ J^1. & \end{tikzcd}$$

The homomorphism h^2 exists, because J^1 is injective. The construction implies

$$\alpha_1 = \partial^0_J \circ h^1 + h^2 \circ \partial^1_I.$$

Again, we may iterate this construction. □

As usual, all the choices involved in the constructions above may be made uniformly.

I.8.9 Corollary. *Let $A \in \mathrm{Ob}(\mathscr{A})$ be an object and*

$$0 \longrightarrow A \longrightarrow I^0 \xrightarrow{\partial^0_I} I^1 \xrightarrow{\partial^1_I} \cdots,$$

$$0 \longrightarrow A \longrightarrow J^0 \xrightarrow{\partial^0_J} J^1 \xrightarrow{\partial^1_J} \cdots$$

injective resolutions of A. Then, there are homomorphisms $\alpha^\bullet\colon (I^\bullet, \partial^\bullet_I) \longrightarrow (J^\bullet, \partial^\bullet_J)$ and $\beta^\bullet\colon (J^\bullet, \partial^\bullet_J) \longrightarrow (I^\bullet, \partial^\bullet_I)$ of complexes, such that the diagram

$$\begin{tikzcd} I^\bullet \arrow[d, "\alpha^\bullet"'] & A \arrow[l] \arrow[r] \arrow[d, equal] & I^\bullet \\ J^\bullet & A \arrow[l] \arrow[r] & J^\bullet \arrow[u, "\beta^\bullet"'] \end{tikzcd}$$

commutes, and both $\beta^\bullet \circ \alpha^\bullet$ and $\alpha^\bullet \circ \beta^\bullet$ are homotopic to the identity.

I.9 Derived functors

In this section, we will look at abelian categories $\mathscr{A}$ and $\mathscr{B}$ and at an **additive** functor $F\colon \mathscr{A} \longrightarrow \mathscr{B}$.

I.9.1 Lemma. *Let $(C^\bullet, \partial^\bullet_C)$ and $(D^\bullet, \partial^\bullet_D)$ be complexes in $\mathscr{A}$. If they are homotopic in $\mathscr{A}$, then the complexes $F(C^\bullet, \partial^\bullet_C)$ and $F(D^\bullet, \partial^\bullet_D)$ are homotopic in $\mathscr{B}$.*

Proof. This is obvious. □

For the rest of this section, we assume that the abelian category $\mathscr{A}$ has **enough injectives**.

For an object $A \in \mathrm{Ob}(\mathscr{A})$, an injective resolution $(I^\bullet, \partial_I^\bullet)$ of A, and a natural number $n \in \mathbb{N}$, we set

$$R^nF(I^\bullet, \partial_I^\bullet) := H^n(F(I^\bullet, \partial_I^\bullet)).$$

I.9.2 Comments. i) Let $(I^\bullet, \partial_I^\bullet)$ and $(J^\bullet, \partial_J^\bullet)$ be two resolutions of the object A. By Corollary I.8.9, there are maps of complexes $\alpha^\bullet\colon (I^\bullet, \partial_I^\bullet) \longrightarrow (J^\bullet, \partial_J^\bullet)$ and $\beta^\bullet\colon (J^\bullet, \partial_J^\bullet) \longrightarrow (I^\bullet, \partial_I^\bullet)$ which are unique up to homotopy, such that $\beta^\bullet \circ \alpha^\bullet$ and $\alpha^\bullet \circ \beta^\bullet$ are homotopic to the identity.

By Lemma I.9.1, $F(\alpha^\bullet)\colon F(I^\bullet, \partial_I^\bullet) \longrightarrow F(J^\bullet, \partial_J^\bullet)$ and $F(\beta^\bullet)\colon F(J^\bullet, \partial_J^\bullet) \longrightarrow F(I^\bullet, \partial_I^\bullet)$ have similar properties, and we get **canonical** isomorphisms

$$R^nF(I^\bullet, \partial_I^\bullet) \longrightarrow R^nF(J^\bullet, \partial_J^\bullet), \quad n \in \mathbb{N}.$$

ii) In a similar fashion, given objects $A, B \in \mathrm{Ob}(\mathscr{A})$, a homomorphism $\varphi\colon A \longrightarrow B$, and injective resolutions $(I^\bullet, \partial_I^\bullet)$ and $(J^\bullet, \partial_J^\bullet)$ of A and B, respectively, there are canonically induced homomorphisms

$$R^nF(I^\bullet, \partial_I^\bullet) \longrightarrow R^nF(J^\bullet, \partial_J^\bullet), \quad n \in \mathbb{N}.$$

We now set

$$R^nF(A) := R^nF(I^\bullet(A), \partial_{I(A)}^\bullet), \quad n \in \mathbb{N}.$$

The comments show that, for $n \in \mathbb{N}$,

$$R^nF\colon \mathscr{A} \longrightarrow \mathscr{B}$$

is a functor. It is called the *n-th right derived functor of F*.

I.9.3 Lemma. i) *Let $F\colon A \longrightarrow B$ be a* ***left exact*** *functor.*[18] *Then, there is a canonical isomorphism*

$$F \longrightarrow R^0F.$$

ii) *If the object $I \in \mathrm{Ob}(\mathscr{I})$ is injective, then*

$$\forall n \geq 1: \quad R^nF(I) = 0.$$

Proof. i) Since the functor F is left exact, the sequence

$$0 \longrightarrow F(A) \longrightarrow F(I^0(A)) \longrightarrow F(I^1(A)) \longrightarrow \cdots$$

is exact at $F(A)$. This gives i).

ii) We apply Comment I.9.2, i), and use the injective resolution

$$(J^\bullet, \partial_J^\bullet) := (0 \longrightarrow J^0 := I \longrightarrow 0)$$

of I to compute the derived functors. □

[18] Recall from Proposition I.5.8 that F is an additive functor.

I.9.4 Theorem. *Let $\mathscr{A}$ and $\mathscr{B}$ be an abelian category where $\mathscr{A}$ has enough injectives and $F\colon \mathscr{A} \longrightarrow \mathscr{B}$ a left exact functor. For every short exact sequence*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

there exist connecting homomorphisms

$$\delta^n\colon R^nF(C) \longrightarrow RF^{n+1}(A), \quad n \in \mathbb{N},$$

such that the sequence

$$\cdots \xrightarrow{\delta^{n-1}} R^nF(A) \longrightarrow R^nF(B) \longrightarrow R^nF(C) \xrightarrow{\delta^n} R^{n+1}F(A) \longrightarrow \cdots$$

is exact, $n \in \mathbb{N}$. More precisely, the functor from short exact sequences in $\mathscr{A}$ to $\mathscr{A}^3_\infty$-diagrams in $\mathscr{B}$ defined by F extends to a functor $\underline{\mathrm{SES}}(\mathscr{A}) \longrightarrow \underline{\mathrm{LECS}}(\mathscr{B})$.

Proof. We use the injective resolutions $(I^\bullet(A), \partial^\bullet_{I(A)})$ and $(I^\bullet(C), \partial^\bullet_{I(C)})$ for A and C, respectively, but we will work with another injective resolution $(J^\bullet, \partial^\bullet_J)$ for B (keep Comment I.9.2, i), in mind). This injective resolution is constructed as follows: We set

$$J^n := I^n(A) \oplus I^n(C), \quad n \in \mathbb{N}.$$

Using the injectivity of the object $I^0(A)$, we pick a homomorphism $j_1\colon B \longrightarrow I^0(A)$, such that the diagram

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, "i_A"'] & B \arrow[dl, dashed, "j_1"] \\
I^0(A) &
\end{tikzcd}$$

commutes, and set

$$j_2\colon B \longrightarrow C \overset{i_C}{\rightarrowtail} I^0(C).$$

Then,

$$j := j_1 \oplus j_2\colon B \longrightarrow J^0$$

is a monomorphism. Next, we find the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d, "i_A"] & B \arrow[r] \arrow[d, "j"] & C \arrow[r] \arrow[d, "i_C"] & 0 \\
0 \arrow[r] & I^0(A) \arrow[r] \arrow[d] \arrow[dd, bend right, "\partial^0_{I(A)}"'] & J^0 \arrow[r] \arrow[d] & I^0(C) \arrow[r] \arrow[d] \arrow[dd, bend right, "\partial^0_{I(C)}"'] & 0 \\
0 \arrow[r] & \mathrm{Coker}(i_A) \arrow[r] \arrow[d] & \mathrm{Coker}(j) \arrow[r] \arrow[d, dashed, "\overline{\partial}^0_J"] & \mathrm{Coker}(i_C) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^1(A) \arrow[r] & J^1 \arrow[r] & I^1(C) \arrow[r] & 0.
\end{tikzcd}$$

In this diagram, $\overline{\partial}^0_J$ is constructed as j.

Inductively, we construct

$$\partial_J^n \colon J^n \longrightarrow J^{n+1}, \quad n \in \mathbb{N}.$$

In this way, we obtain the exact sequence

$$0 \longrightarrow (I^\bullet(A), \partial^\bullet_{I(A)}) \longrightarrow (J^\bullet, \partial^\bullet_J) \longrightarrow (I^\bullet(C), \partial^\bullet_{I(C)}) \longrightarrow 0$$

of complexes in $\mathscr{A}$. Note that the sequence

$$0 \longrightarrow F(I^\bullet(A), \partial^\bullet_{I(A)}) \longrightarrow F(J^\bullet, \partial^\bullet_J) \longrightarrow F(I^\bullet(C), \partial^\bullet_{I(C)}) \longrightarrow 0 \qquad \text{(I.10)}$$

of complexes in $\mathscr{B}$ is exact, too. This is true, because, for every $n \in \mathbb{N}$, we have a homomorphism $I^n(C) \longrightarrow J^n$ in $\mathscr{A}$, such that the composition

$$I^n(C) \longrightarrow J^n \longrightarrow I^n(C)$$

is the identity.[19] Thus, we have a homomorphism $F(I^n(C)) \longrightarrow F(J^n)$, such that the composition

$$F(I^n(C)) \longrightarrow F(J^n) \longrightarrow F(I^n(C))$$

is the identity, $n \in \mathbb{N}$. This shows that (I.10) is exact. The long exact cohomology sequence (Theorem I.7.1) associated with (I.10) gives the sequence of the theorem.

Finally, we prove the statement about functoriality. For this, we start with a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \arrow[d, equal] \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

in which the rows are exact. We obtain homomorphisms $I^\bullet(A) \longrightarrow I^\bullet(A')$ and $I^\bullet(C) \longrightarrow I^\bullet(C')$ of complexes. For every $n \in \mathbb{N}$, we get

$$J^n = I^n(A) \oplus I^n(C) \longrightarrow I^n(A') \oplus I^n(C') = J'^n.$$

It is readily verified that these homomorphisms commute with the differentials of the complexes $(J^\bullet, \partial^\bullet_J)$ and $(J'^\bullet, \partial^\bullet_{J'})$. This means that

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, equal] & (I^\bullet(A), \partial^\bullet_{I(A)}) \arrow[r] \arrow[d] & (J^\bullet, \partial^\bullet_J) \arrow[r] \arrow[d] & (I^\bullet(C), \partial^\bullet_{I(C)}) \arrow[r] \arrow[d] & 0 \arrow[d, equal] \\ 0 \arrow[r] & (I^\bullet(A'), \partial^\bullet_{I(A')}) \arrow[r] & (J'^\bullet, \partial^\bullet_{J'}) \arrow[r] & (I^\bullet(C'), \partial^\bullet_{I(C')}) \arrow[r] & 0 \end{tikzcd}$$

is a commutative diagram of complexes in $\mathscr{A}$. Applying the functor F, we find a similar diagram of complexes in $\mathscr{B}$. So, our statement becomes an application of the functoriality statement in Theorem I.7.1. □

We will denote the functor from $\underline{\mathrm{SES}}(\mathscr{A})$ to $\underline{\mathrm{LECS}}(\mathscr{B})$ in Theorem I.9.4 by $\{ R^\bullet F, \delta^\bullet \}$. A functor $\underline{\mathrm{SES}}(\mathscr{A}) \longrightarrow \underline{\mathrm{LECS}}(\mathscr{B})$ will be referred to as a *δ-functor*. The δ-functor $\{ R^\bullet F, \delta^\bullet \}$ is characterized by the following property:

[19] It is important to note that, in general, the homomorphisms $I^n(C) \longrightarrow J^n$, $n \in \mathbb{N}$, do not combine to a homomorphism $I^\bullet(C) \longrightarrow J^\bullet$ of complexes.

I.9.5 Proposition. *Let* $\{G^\bullet, \delta_G^\bullet\}\colon \underline{\mathrm{SES}}(\mathscr{A}) \longrightarrow \underline{\mathrm{LECS}}(\mathscr{B})$ *be a* δ*-functor and*

$$\Phi\colon F \longrightarrow G^0$$

an isomorphism. Assume that

$$G^n(I) = 0$$

holds true for every injective object $I \in \mathrm{Ob}(\mathscr{A})$ *and every* $n > 0$*. Then, there is an induced isomorphism*

$$\Phi^\bullet\colon \{R^\bullet F, \delta^\bullet\} \longrightarrow \{G^\bullet, \delta_G^\bullet\}$$

of δ*-functors.*

Proof. Let $A \in \mathrm{Ob}(\mathscr{A})$ be an object and $I^0 := I^0(A)$. We form the exact sequence

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow K^1 := I^0/A \longrightarrow 0.$$

This sequence gives rise to the commutative diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & R^0F(A) & \longrightarrow & R^0F(I^0) & \longrightarrow & R^0F(K^1) & \longrightarrow & R^1F(A) & \longrightarrow & 0 \\
\| & & \uparrow & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & F(A) & \longrightarrow & F(I^0) & \longrightarrow & F(K^1) & \longrightarrow & R^1F(A) & \longrightarrow & 0 \\
\| & & {\scriptstyle\Phi(A)}\downarrow & & {\scriptstyle\Phi(I^0)}\downarrow & & {\scriptstyle\Phi(K^1)}\downarrow & & & & \\
0 & \longrightarrow & G^0(A) & \longrightarrow & G^0(I^0) & \longrightarrow & G^0(K^1) & \longrightarrow & G^1(A) & \longrightarrow & 0
\end{array}$$

with exact rows. (Recall that I^0 is injective.) Since $\Phi(\cdot)$ is an isomorphism, we can define an isomorphism

$$\Phi^1(A)\colon R^1F(A) \longrightarrow G^1(A)$$

by this diagram. The fact that Φ is a natural transformation of functors implies that Φ^1 is also a natural transformation of functors.

We continue by recursion. Assume that, for $n \geq 2$, $\Phi^0 := \Phi$, Φ^1,..., and Φ^{n-1} have already been defined. Using the diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & R^{n-1}(K^1) & \xrightarrow{\delta^{n-1}} & R^nF(A) & \longrightarrow & 0 \\
\| & & {\scriptstyle\Phi^{n-1}(K^1)}\downarrow{\scriptstyle\cong} & & & & \| \\
0 & \longrightarrow & G^{n-1}(K^1) & \xrightarrow{\delta_G^{n-1}} & G^n(A) & \longrightarrow & 0,
\end{array}$$

we define the isomorphism

$$\Phi^n(A)\colon R^nF(A) \longrightarrow G^n(A),$$

for every object $A \in \mathrm{Ob}(\mathscr{A})$. In this way, we get a natural transformation

$$\Phi^n\colon R^nF \longrightarrow G^n$$

of functors. It remains to show that this isomorphism commutes with coboundary maps. Let

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be an exact sequence in $\mathscr{A}$. We complete it to the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \arrow[d, equal] \\
0 \arrow[r] & A \arrow[r] & I^0 \arrow[r] & K^1 \arrow[r] & 0.
\end{tikzcd}$$

Now, study the diagram

$$\begin{tikzcd}
R^nF(C) \arrow[rrr, "\delta^n"] \arrow[dr] \arrow[ddd, "\Phi^n(C)"'] & & & R^{n+1}F(A) \arrow[dl, equal] \arrow[ddd, "\Phi^{n+1}(A)"] \\
& R^nF(K^1) \arrow[r, "\delta^n"] \arrow[d, "\Phi^n(K^1)"'] & R^{n+1}F(A) \arrow[d, "\Phi^{n+1}(A)"] & \\
& G^n(K^1) \arrow[r, "\delta^n_G"] & G^{n+1}(A) \arrow[dr, equal] & \\
G^n(C) \arrow[rrr, "\delta^n_G"] \arrow[ur] & & & G^{n+1}(A).
\end{tikzcd}$$

The right hand square trivially commutes. The upper square commutes, because $\{ R^\bullet F, \delta^\bullet \}$ is a δ-functor. Likewise, the bottom square commutes, because $\{ G^\bullet, \delta^\bullet_G \}$ is a δ-functor. The left hand square is commutative, because Φ^n is a natural transformation of functors. Finally, the inner square commutes by construction of Φ^n and Φ^{n+1}. Therefore, also the outer square commutes. $\square$

I.9.6 Exercise (Degree shift). Let $F\colon \mathscr{A} \longrightarrow \mathscr{B}$ be a left exact functor between abelian categories, and

$$0 \longrightarrow A \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_r \longrightarrow B \longrightarrow 0$$

an exact sequence in $\mathscr{A}$. Suppose that I_1, ..., I_r are injective objects. Show that

$$R^nF(B) \cong R^{n+r}F(A), \quad n \geq 1.$$

I.10 Acyclic resolutions

The formalism of injective resolutions is not suited for explicitly computing derived functors. For this reason, we will develop some other tools. Let $\mathscr{A}$, $\mathscr{B}$ be abelian categories, $F\colon \mathscr{A} \longrightarrow \mathscr{B}$ a left exact functor, and suppose that $\mathscr{A}$ has enough injectives. An *acyclic resolution* of an object $A \in \mathrm{Ob}(\mathscr{A})$ is a complex $(J^\bullet, \partial^\bullet_J)$, such that

- ⋆ $A \cong \mathrm{Ker}(\partial^0_J)$ and $0 \longrightarrow A \longrightarrow J^0 \longrightarrow J^1 \longrightarrow \cdots$ is an exact complex,
- ⋆ $R^nF(J^k) = 0$, $k \in \mathbb{N}$, $n \geq 1$.

I.10.1 Proposition. *If $(J^\bullet, \partial^\bullet_J)$ is an acyclic resolution of A, then the n-th derived functor $R^nF(A)$ of F evaluated at A is isomorphic to the n-th cohomology of the complex $F(J^\bullet, \partial^\bullet_J)$.*

Proof. i) The fact that F is left exact implies

$$F(A) \cong H^0(F(J^\bullet, \partial_J^\bullet)).$$

ii) We prove the statement on the cohomology by induction on n. In the proof, we use the exact short sequence

$$0 \longrightarrow A \longrightarrow J^0 \longrightarrow B := J^0/A \longrightarrow 0. \tag{I.11}$$

Note that the complex $(J'^\bullet, \partial_{J'}^\bullet)$ with $J'^n := J^{n+1}$ and $\partial_{J'}^n := \partial_J^{n+1}$, $n \in \mathbb{N}$, is an acyclic resolution of B.

***n* = 1.** We have the exact sequences

$$F(J^0) \longrightarrow F(B) \longrightarrow R^1F(A) \longrightarrow 0 = R^1F(J^0)$$

and

$$0 \longrightarrow F(B) \longrightarrow F(J^1) \longrightarrow F(J^2).$$

The second sequence shows $F(B) = Z^1(F(J^\bullet, \partial_J^\bullet))$, and, with the first one, we find

$$R^1F(A) \cong H^1(F(J^\bullet, \partial_J^\bullet)).$$

***n* ⟶ *n* + 1.** The long exact cohomology sequence associated with (I.11) and the fact that J^0 is acyclic yield an isomorphism

$$R^nF(B) \xrightarrow{\cong} R^{n+1}F(A).$$

By induction hypothesis,

$$R^nF(B) \cong H^n(F(J'^\bullet, \partial_{J'}^\bullet)) = H^{n+1}(F(J^\bullet, \partial_J^\bullet)).$$

This finishes the argument. □

I.11 The Ext functors

Let $\mathscr{A}$ be an abelian category which has enough injectives. For every object $A \in \mathrm{Ob}(\mathscr{A})$ the functor

$$\mathrm{Hom}_{\mathscr{A}}(A, \cdot)\colon \mathscr{A} \longrightarrow \underline{\mathrm{Ab}}$$

is left exact (Lemma I.8.1, i). We can look at its right derived functors

$$\mathrm{Ext}^n_{\mathscr{A}}(A, \cdot) := R^n\mathrm{Hom}_{\mathscr{A}}(A, \cdot), \quad n \in \mathbb{N}.$$

I.11.1 Proposition. i) *For every object $B \in \mathrm{Ob}(\mathscr{A})$, we have*

$$\mathrm{Ext}^0_{\mathscr{A}}(A, B) \cong \mathrm{Hom}_{\mathscr{A}}(A, B).$$

ii) *For an injective object $B \in \mathrm{Ob}(\mathscr{A})$, it follows that*

$$\mathrm{Ext}^n_{\mathscr{A}}(A, B) = 0, \quad n \geq 1.$$

iii) *A short exact sequence*

$$0 \longrightarrow B_1 \longrightarrow B_2 \longrightarrow B_3 \longrightarrow 0$$

in $\mathscr{A}$ gives rise to a long exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_{\mathscr{A}}(A, B_1) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(A, B_2) \longrightarrow& \\ \longrightarrow \mathrm{Hom}_{\mathscr{A}}(A, B_3) \longrightarrow \mathrm{Ext}^1_{\mathscr{A}}(A, B_1) \longrightarrow \cdots& \\ \cdots& \\ \cdots \longrightarrow \mathrm{Ext}^n_{\mathscr{A}}(A, B_1) \longrightarrow \mathrm{Ext}^n_{\mathscr{A}}(A, B_2) \longrightarrow& \\ \longrightarrow \mathrm{Ext}^n_{\mathscr{A}}(A, B_3) \longrightarrow \mathrm{Ext}^{n+1}_{\mathscr{A}}(A, B_1) \longrightarrow \cdots& \end{aligned}$$

in the category of abelian groups.

iv) *If A is **projective**, then, for any object $B \in \mathrm{Ob}(\mathscr{A})$ and any $n \geq 1$,*

$$\mathrm{Ext}^n_{\mathscr{A}}(A, B) = 0.$$

v) *For a short exact sequence*

$$0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow 0$$

in $\mathscr{A}$ and an object $B \in \mathrm{Ob}(\mathscr{A})$, one finds a long exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_{\mathscr{A}}(A_3, B) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(A_2, B) \longrightarrow& \\ \longrightarrow \mathrm{Hom}_{\mathscr{A}}(A_1, B) \longrightarrow \mathrm{Ext}^1_{\mathscr{A}}(A_3, B) \longrightarrow \cdots& \\ \cdots& \\ \cdots \longrightarrow \mathrm{Ext}^n_{\mathscr{A}}(A_3, B) \longrightarrow \mathrm{Ext}^n_{\mathscr{A}}(A_2, B) \longrightarrow& \\ \longrightarrow \mathrm{Ext}^n_{\mathscr{A}}(A_1, B) \longrightarrow \mathrm{Ext}^{n+1}_{\mathscr{A}}(A_3, B) \longrightarrow \cdots& \end{aligned}$$

in the category of abelian groups.

Proof. The assertions i) - iii) are specializations of the general results (Lemma I.9.3, i), and Theorem I.9.4) to the functor $\mathrm{Hom}_{\mathscr{A}}(A, \cdot)$.

iv) If A is projective, then the functor $\mathrm{Hom}_{\mathscr{A}}(A, \cdot)$ is exact. So, if $B \in \mathrm{Ob}(\mathscr{A})$ is an object, then the complex

$$0 \longrightarrow \mathrm{Hom}_{\mathscr{A}}(A, B) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(A, I^0(B)) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(A, I^1(B)) \longrightarrow \cdots$$

is exact. Since, by definition, the Ext's are the cohomology objects of this complex, the claim follows.

v) Let $(I^\bullet(B), \partial^\bullet_{I(B)})$ be the usual injective resolution of B. By definition, the contravariant functor $\mathrm{Hom}_{\mathscr{A}}(\cdot, I)\colon \mathscr{A} \longrightarrow \underline{\mathrm{Ab}}$ is exact, if $I \in \mathrm{Ob}(\mathscr{A})$ is an injective object. Therefore,

$$0 \longrightarrow \mathrm{Hom}_{\mathscr{A}}(A_3, I^\bullet(B)) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(A_2, I^\bullet(B)) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(A_1, I^\bullet(B)) \longrightarrow 0$$

is an exact sequence of complexes of abelian groups, and the displayed sequence is the long exact cohomology sequence (Theorem I.7.1) associated with it. □

I.11.2 Remarks. i) It is important to note that Part v) of the proposition holds even, if the category $\mathscr{A}$ has **not** enough projectives.

ii) Assume that $\mathscr{A}$ has enough projectives. This means that the opposite category $\mathscr{A}^{\mathrm{opp}}$ has enough injectives. By definition,

$$\mathrm{Hom}_{\mathscr{A}}(\cdot, B) = \mathrm{Hom}_{\mathscr{A}^{\mathrm{opp}}}(B, \cdot).$$

In this case, we set

$$\mathrm{Ext}^n_{\mathscr{A}}(A, B) := \mathrm{Ext}^n_{\mathscr{A}^{\mathrm{opp}}}(B, A) := \mathrm{Ext}^n_{\mathscr{A}^{\mathrm{opp}}}(B, \cdot)(A), \quad n \in \mathbb{N}.$$

Now, if the category $\mathscr{A}$ has both enough injectives and projectives, then Part iv) and v) of the proposition and the universality of δ-functors (Proposition I.9.5) imply that both possible definitions of Ext-objects yield canonically isomorphic results.

I.11.3 Exercises (Double complexes and Ext). Let $\mathscr{A}$ be an abelian category. A *double complex in* $\mathscr{A}$ consists of

- objects K^{ij}, $i, j \in \mathbb{Z}$
- horizontal differentials $d'_{ij}\colon K^{ij} \longrightarrow K^{i+1j}$, $i, j \in \mathbb{Z}$,
- vertical differentials $d''_{ij}\colon K^{ij} \longrightarrow K^{ij+1}$, $i, j \in \mathbb{Z}$,

such that

- $d'_{i+1j} \circ d'_{ij} = 0$, $i, j \in \mathbb{Z}$,
- $d''_{ij+1} \circ d''_{ij} = 0$, $i, j \in \mathbb{Z}$,
- $d''_{i+1j} \circ d'_{ij} = d'_{ij+1} \circ d''_{ij}$, $i, j \in \mathbb{Z}$.

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
\cdots \arrow[r, "d''_{i+1j-2}"] & K^{i+1j-1} \arrow[u, "d'_{i+1j-1}"] \arrow[r, "d''_{i+1j-1}"] & K^{i+1j} \arrow[u, "d'_{i+1j}"] \arrow[r, "d''_{i+1j}"] & K^{i+1j+1} \arrow[u, "d'_{i+1j+1}"] \arrow[r, "d''_{i+1j+1}"] & \cdots \\
\cdots \arrow[r, "d''_{ij-2}"] & K^{ij-1} \arrow[u, "d'_{ij-1}"] \arrow[r, "d''_{ij-1}"] & K^{ij} \arrow[u, "d'_{ij}"] \arrow[r, "d''_{ij}"] & K^{ij+1} \arrow[u, "d'_{ij+1}"] \arrow[r, "d''_{ij+1}"] & \cdots \\
\cdots \arrow[r, "d''_{i-1j-2}"] & K^{i-1j-1} \arrow[u, "d'_{i-1j-1}"] \arrow[r, "d''_{i-1j-1}"] & K^{i-1j} \arrow[u, "d'_{i-1j}"] \arrow[r, "d''_{i-1j}"] & K^{i-1j+1} \arrow[u, "d'_{i-1j+1}"] \arrow[r, "d''_{i-1j+1}"] & \cdots \\
 & \vdots \arrow[u, "d'_{i-2j-1}"] & \vdots \arrow[u, "d'_{i-2j}"] & \vdots \arrow[u, "d'_{i-2j+1}"] & .
\end{tikzcd}$$

i) Assume

- $K_{ij} = 0$ for $i < -1$ or $j < -1$,
- the complex $(K^{\bullet j}, d'_{\bullet j})$ is exact for all $j \geq 0$,
- the complex $(K^{i\bullet}, d''_{i\bullet})$ is exact for all $i \geq 0$.

Prove that
$$H^n(K^{\bullet -1}, d'_{\bullet -1}) \cong H^n(K^{-1\bullet}, d''_{-1\bullet}), \quad n \ge 0.$$

ii) Use this result to show that
$$\mathrm{Ext}^n_{\underline{\mathrm{Mod}}_R}(M,N) \cong \mathrm{Ext}^n_{\underline{\mathrm{Mod}}_R^{\mathrm{op}}}(N,M)$$
for all n and all R-modules M and N. (Use an injective resolution of N and a projective resolution of M in order to produce a double complex as in i).)

I.12 Injective and projective objects in categories of modules

Let R be a ring (commutative with identity element). In this section, we will prove that the category $\underline{\mathrm{Mod}}_R$ of R-modules has both enough injectives and projectives.

Injective objects

I.12.1 Proposition (Baer)[20]]. *An R-module I is an injective object in the category $\underline{\mathrm{Mod}}_R$ if and only if, for every ideal $\mathfrak{a} \subset R$ and every homomorphism $f\colon \mathfrak{a} \longrightarrow I$ of R-modules, there is a homomorphism $F\colon R \longrightarrow I$ with $F_{|\mathfrak{a}} = f$.*

Proof. If I is an injective R-module, then it has the stated property, by definition. For the converse direction, assume that I has the stated property. Let $\varphi\colon M \longrightarrow N$ be an injective homomorphism of R-modules, i.e., a monomorphism in $\underline{\mathrm{Mod}}_R$, and $f\colon M \longrightarrow I$ a homomorphism. We have to extend this homomorphism to N. For simplicity, we assume that M is a submodule of N, and φ is the inclusion. We look at all pairs (P, f_P) which consist of an R-submodule $M \subset P \subset N$ and a homomorphism $f_P\colon P \longrightarrow I$ with $f_{P|M} = f$. The set Σ of all such pairs is non-empty, because it contains (M, f). We introduce a partial ordering on Σ: $(P_1, f_{P_1}) \le (P_2, f_{P_2})$, if $P_1 \subset P_2$ and $f_{P_2|P_1} = f_{P_1}$. If $(P_t, f_{P_t})_{t\in T}$ is a chain in Σ, we set
$$P := \bigcup_{t\in T} P_t$$
and
$$\begin{aligned} f_P\colon P &\longrightarrow I \\ p &\longmapsto f_{P_t}(p), \quad \text{if } p \in P_t,\ t \in T. \end{aligned}$$

By definition of "$\le$", this is well-defined. Zorn's lemma ([22], I.4.7) states that Σ contains a maximal element (Q, f_Q). We have to prove that $Q = N$. If $Q \subsetneq N$, pick $x \in N \setminus Q$. Then,
$$\mathfrak{a} := \{\, r \in R \mid r \cdot x \in Q \,\}$$
is an ideal of R, and
$$\begin{aligned} \psi\colon \mathfrak{a} &\longrightarrow I \\ r &\longmapsto f_Q(r \cdot x) \end{aligned}$$

[20] Reinhold Baer (1902 - 1979), German mathematician.

is a homomorphism of R-modules. By the property which we assume, there is a homomorphism $\Psi\colon R \longrightarrow I$ with $\Psi_{|\mathfrak{a}} = \psi$.

For $r \in \mathfrak{a}$ and $q := r \cdot x$, we have

$$\psi(r) = f_Q(q),$$

so that

$$\begin{array}{rcl} \widetilde{f}\colon Q + R\cdot x & \longrightarrow & I \\ q + r\cdot x & \longmapsto & f_Q(q) + \psi(r) \end{array}$$

is a well-defined homomorphism. It satisfies $\widetilde{f}_{|Q} = f_Q$, i.e., $(Q, f_Q) \preceq (Q + R\cdot x, \widetilde{f})$. Since $Q \subsetneq Q + R\cdot x$, this contradicts the maximality of (Q, f_Q). □

Let us apply this criterion to the ring $\mathbb{Z}$. Then, $\underline{\mathrm{Mod}}_{\mathbb{Z}} = \underline{\mathrm{Ab}}$ is the category of abelian groups. We know already that an injective abelian group has to be divisible. Now, assume that A is a divisible abelian group. The ideals of $\mathbb{Z}$ are of the form $\langle n\rangle$, $n \in \mathbb{N}$. For $n = 0$, the zero homomorphism $\langle 0\rangle \longrightarrow A$ extends to a homomorphism $\mathbb{Z} \longrightarrow A$. For $n \geq 1$, a homomorphism $\varphi\colon \langle n\rangle \longrightarrow A$, and $a := \varphi(n)$, we have

$$\forall k \in \mathbb{Z}: \quad \varphi(k\cdot n) = k\cdot a.$$

Since the group A is divisible, there exists an element $b \in A$ with $n\cdot b = a$. Set

$$\begin{array}{rcl} \Psi\colon \mathbb{Z} & \longrightarrow & A \\ k & \longmapsto & k\cdot b. \end{array}$$

This homomorphism extends ψ. So, in view of Baer's criterion I.12.1, we have proved:

I.12.2 Theorem. *An abelian group A is injective if and only if it is divisible.*

I.12.3 Example. The group $\mathbb{Q}/\mathbb{Z}$ is divisible and, therefore, injective.

I.12.4 Proposition. *The category* $\underline{\mathrm{Ab}}$ *of abelian groups has enough injectives.*

Proof. Let A be an abelian group. Set

$$I(A) := \prod_{f \in \mathrm{Hom}_{\underline{\mathrm{Ab}}}(A, \mathbb{Q}/\mathbb{Z})} \mathbb{Q}/Z.$$

This is an injective abelian group, and

$$\begin{array}{rcl} i_A\colon A & \longrightarrow & I(A) \\ a & \longmapsto & \big(f(a), f \in \mathrm{Hom}_{\underline{\mathrm{Ab}}}(A, \mathbb{Q}/\mathbb{Z})\big) \end{array}$$

is a homomorphism. We have to verify that i_A is injective. Let $a \in A \setminus \{0\}$ and $\langle a\rangle \subset A$ the subgroup generated by a ([26], Definition II.4.11). Denote by $\mathrm{ord}(a)$ the **order** of a, i.e., $\mathrm{ord}(a) := \#\langle a\rangle$, if $\langle a\rangle$ is finite, and $\mathrm{ord}(a) = \infty$, else. Then,

$$\begin{array}{rcl} g_a\colon \langle a\rangle & \longrightarrow & \mathbb{Q}/\mathbb{Z} \\ a & \longmapsto & \begin{cases} \left[\dfrac{1}{n}\right], & \text{if } \mathrm{ord}(a) = n \in \mathbb{N} \\[2ex] \left[\dfrac{1}{2}\right], & \text{if } \mathrm{ord}(a) = \infty \end{cases} \end{array}$$

is a homomorphism with $g_a(a) \neq 0$. Since $\mathbb{Q}/\mathbb{Z}$ is injective, g_a extends to a homomorphism $f_a\colon A \longrightarrow \mathbb{Q}/\mathbb{Z}$ with $f_a(a) = g_a(a) \neq 0$. □

Now, let R be a ring, A an R-module, and B an abelian group. Define

$$\begin{array}{rcl} \cdot\colon R \times \mathrm{Hom}_{\underline{\mathrm{Ab}}}(A,B) & \longrightarrow & \mathrm{Hom}_{\underline{\mathrm{Ab}}}(A,B) \\ (r,f) & \longmapsto & (r \cdot f\colon a \longmapsto f(r\cdot a)). \end{array}$$

This defines the structure of an R-module on $\mathrm{Hom}_{\underline{\mathrm{Ab}}}(A,B)$. In fact,

- $1 \cdot f = f$, $f \in \mathrm{Hom}_{\underline{\mathrm{Ab}}}(A,B)$,
- $(r\cdot(f_1+f_2))(a) = (f_1+f_2)(r\cdot a) = f_1(r\cdot a) + f_2(r\cdot a) = (r\cdot f_1)(a) + (r\cdot f_2)(a)$, $r \in R$, $f_1, f_2 \in \mathrm{Hom}_{\underline{\mathrm{Ab}}}(A,B)$, $a \in A$,
- $((r_1+r_2)\cdot f)(a) = f((r_1+r_2)\cdot a) = f(r_1\cdot a + r_2\cdot a) = f(r_1\cdot a) + f(r_2\cdot a) = (r_1\cdot f)(a) + (r_2\cdot f)(a)$, $r_1, r_2 \in R$, $f \in \mathrm{Hom}_{\underline{\mathrm{Ab}}}(A,B)$, $a \in A$.

Let A be an R-module and B an abelian group. Then,

$$\begin{array}{rcl} \Phi\colon \mathrm{Hom}_{\underline{\mathrm{Mod}}_R}(A, \mathrm{Hom}_{\underline{\mathrm{Ab}}}(R,B)) & \longrightarrow & \mathrm{Hom}_{\underline{\mathrm{Ab}}}(A,B) \\ f & \longmapsto & (a \longmapsto f(a)(1)) \end{array}$$

is an isomorphism of R-modules. Indeed,

⋆ Φ is injective. For a non-zero homomorphism $f\colon A \longrightarrow \mathrm{Hom}_{\underline{\mathrm{Ab}}}(R,B)$, there is an element $a \in A$ with $f(a) \neq 0$. Let $r \in R$ with $f(a)(r) \neq 0$. Since $f(a)(r) = (r\cdot f(a))(1) = f(r\cdot a)(1)$, we see $\Phi(f) \neq 0$.

⋆ Φ is surjective. As above, one sees that, for a homomorphism $\varphi\colon A \longrightarrow B$ of abelian groups,

$$\begin{array}{rcl} f\colon A & \longrightarrow & \mathrm{Hom}_{\underline{\mathrm{Ab}}}(R,B) \\ a & \longmapsto & (r \longmapsto (r\cdot\varphi)(a)) \end{array}$$

is a homomorphism of R-modules with $\Phi(f) = \varphi$.

⋆ $\Phi(f_1+f_2)(a) = (f_1+f_2)(a)(1) = (f_1(a)+f_2(a))(1) = f_1(a)(1) + f_2(a)(1) = \Phi(f_1)(a) + \Phi(f_2)(a)$, $f_1, f_2\colon A \longrightarrow \mathrm{Hom}_{\underline{\mathrm{Ab}}}(R,B)$, $a \in A$.

⋆ $\Phi(r\cdot f)(a) = (r\cdot f)(a)(1) = f(r\cdot a)(1) = (r\cdot\Phi(f))(a)$, $r \in R$, $f\colon A \longrightarrow \mathrm{Hom}_{\underline{\mathrm{Ab}}}(R,B)$, $a \in A$.

This construction can be promoted to an isomorphism

$$\mathrm{Hom}_{\underline{\mathrm{Mod}}_R}(\cdot, \mathrm{Hom}_{\underline{\mathrm{Ab}}}(R,B)) \longrightarrow \mathrm{Hom}_{\underline{\mathrm{Ab}}}(\cdot, B)$$

of contravariant functors from $\underline{\mathrm{Mod}}_R$ to $\underline{\mathrm{Mod}}_R$.

I.12.5 Proposition. *For every injective abelian group I, the R-module* $\mathrm{Hom}_{\underline{\mathrm{Ab}}}(R,I)$ *is an injective object in* $\underline{\mathrm{Mod}}_R$.

I.12.6 Example. The R-module $\mathrm{Hom}_{\underline{\mathrm{Ab}}}(R,\mathbb{Q}/\mathbb{Z})$ is injective.

I.12.7 Corollary. *The category* $\underline{\mathrm{Mod}}_R$ *has enough injectives.*

Projective objects

Let S be a set. Recall from [22], Page 76, that the **free *R*-module** generated by S is

$$F(S) := \bigoplus_{s\in S} R := \{\, f\colon S \longrightarrow R \mid f(s) = 0 \text{ for all but finitely many } s \in S\,\}. \tag{I.12}$$

There is the injection

$$\begin{array}{rcl} S & \longrightarrow & F(S) \\ s & \longmapsto & \left(e_s\colon t \longmapsto \begin{cases} 1, & \text{if } s = t \\ 0, & \text{if } s \neq t \end{cases}\right) \end{array}$$

of sets. Then, for any R-module B, the map

$$\begin{array}{rcl} \mathrm{Hom}_{\underline{\mathrm{Mod}}_R}(F(S), B) & \longrightarrow & \mathrm{Mor}_{\underline{\mathrm{Sets}}}(S, B) \\ g & \longmapsto & (s \longmapsto g(e_s)) \end{array}$$

is an isomorphism of R-modules.

I.12.8 Example. For $S = \{1\}$, we find the isomorphism

$$\begin{array}{rcl} \mathrm{Hom}_{\underline{\mathrm{Mod}}_R}(R, B) & \longrightarrow & B \\ f & \longmapsto & f(1). \end{array}$$

We infer:

I.12.9 Proposition. *For every set S, the free R-module $F(S)$ is projective.*

I.12.10 Corollary. *The category $\underline{\mathrm{Mod}}_R$ has enough projectives.*

Proof. Let A be an R-module. The identity $\mathrm{id}_A\colon A \longrightarrow A$, viewed as a bijection of sets, gives rise to a surjection $F(A) \longrightarrow A$ of R-modules. □

I.12.11 Theorem. *An R-module A is projective if and only if it is isomorphic to a direct summand of a free module, i.e., there are a set S and an R-module B with $A \oplus B \cong F(S)$.*

Proof. Let A be a direct summand of a free module, $\varphi\colon M \longrightarrow N$ a surjective homomorphism of R-modules, and $f\colon A \longrightarrow N$ a homomorphism. There are a set S and homomorphisms $\iota\colon A \longrightarrow F(S)$ and $\pi\colon F(S) \longrightarrow A$ with $\pi \circ \iota = \mathrm{id}_A$. Since $F(S)$ is projective (Proposition I.12.9), there is a homomorphism $F\colon F(S) \longrightarrow M$ with

$$\varphi \circ F = f \circ \pi;$$

$$\begin{tikzcd} F(S) \arrow[r, "\pi"] \arrow[d, dashed, "F"'] & A \arrow[d, "f"] \\ M \arrow[r, "\varphi"] & N. \end{tikzcd}$$

Then,

$$\varphi \circ (F \circ \iota) = f \circ (\pi \circ \iota) = f,$$

i.e., $F \circ \iota\colon A \longrightarrow M$ lifts f.

Let A be a projective R-module. There is a surjection $\pi\colon F(A) \longrightarrow A$. Since A is projective, the identity $\mathrm{id}_A\colon A \longrightarrow A$ lifts to a homomorphism $\iota\colon A \longrightarrow F(A)$ with $\pi \circ \iota = \mathrm{id}_A$. One readily checks that

$$F(A) \cong A \oplus \mathrm{Ker}(\pi).$$

This settles the claim. □

I.12.12 Exercises (Injectives and projectives). Let R be a noetherian ring and $\underline{\mathrm{fgMod}}_R$ the abelian category of **finitely generated** R-modules.

i) Does $\underline{\mathrm{fgMod}}_R$ have enough projectives?

ii) Does $\underline{\mathrm{fgMod}}_R$ have enough injectives? (Consider the special cases that R is a field and $R = \mathbb{Z}$.)

I.12.13 Exercises (Projective quiver representations). Determine the projective representations of the quiver $\bullet \longrightarrow \bullet$ in the category $\underline{\mathrm{Vect}}_k$, k a field.

I.13 The tensor product and Tor

Let R be a commutative ring and A, B, and C R-modules. A map

$$\varphi\colon A \times B \longrightarrow C$$

is said to be *R-bilinear*, if

- ★ $\forall a_1, a_2 \in A$, $b \in B$: $\varphi(a_1 + a_2, b) = \varphi(a_1, b) + \varphi(a_2, b)$,
- ★ $\forall a \in A$, $b_1, b_2 \in B$: $\varphi(a, b_1 + b_2) = \varphi(a, b_1) + \varphi(a, b_2)$,
- ★ $\forall r \in R$, $a \in A$, $b \in B$: $\varphi(r \cdot a, b) = r \cdot \varphi(a, b) = \varphi(a, r \cdot b)$.

I.13.1 Theorem. *Let R be a ring and A, B R-modules. There exist an R-module T and an R-bilinear map*

$$\Phi\colon A \times B \longrightarrow T,$$

such that, for any R-module C and any R-bilinear map

$$\varphi\colon A \times B \longrightarrow C,$$

*there is a **unique** homomorphism $\psi\colon T \longrightarrow C$ of R-modules with*

$$\varphi = \psi \circ \Phi.$$

Proof. Let $S := A \times B$, $F(S)$ the free R-module generated by S (I.12), and $Q \subset F(S)$ the submodule generated by the following elements

- ★ $(a_1 + a_2, b) - (a_1, b) - (a_2, b)$, $a_1, a_2 \in A$, $b \in B$,
- ★ $(a, b_1 + b_2) - (a, b_1) - (a, b_2)$, $a \in A$, $b_1, b_2 \in B$,
- ★ $(r \cdot a, b) - r \cdot (a, b)$, $r \in R$, $a \in A$, $b \in B$,
- ★ $(a, r \cdot b) - r \cdot (a, b)$, $r \in R$, $a \in A$, $b \in B$.

We set
$$T := F(S)/Q.$$
The class of an element (a,b) in T will be denoted by $a \otimes b$, $(a,b) \in A \times B$. We set
$$\begin{aligned} \Phi\colon A \times B &\longrightarrow T \\ (a,b) &\longmapsto a \otimes b. \end{aligned}$$
It is easy to see that Φ is R-bilinear. Now, let C be an R-module and $\varphi\colon A \times B \longrightarrow C$ an R-bilinear map. The assignment $(a,b) \longmapsto \varphi(a,b)$ extends to a homomorphism
$$\widetilde{\psi}\colon F(S) \longrightarrow C.$$
Since φ is R-bilinear, $\widetilde{\psi}$ vanishes on the generators of Q and, therefore, on Q. This means that $\widetilde{\psi}$ factorizes over a homomorphism $\psi\colon T \longrightarrow C$. By construction, $\varphi = \psi \circ \Phi$. The homomorphism $\widetilde{\psi}$ is uniquely determined and so is ψ. □

As usual, the R-module T is unique up to canonical isomorphy. It is called the *tensor product* of A and B over R.

Notation. $A \otimes_R B := T$.

I.13.2 Remark. It is important to note that not every element of $A \otimes_R B$ is of the form $a \otimes b$ with $a \in A$, $b \in B$. The reader should check this, for example, for $\mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2$. However, $A \otimes_R B$ is **generated** as an R-module by the elements $a \otimes b$, $a \in A$, $b \in B$.

Let A, B_1, B_2 R-modules and $\alpha\colon B_1 \longrightarrow B_2$ a homomorphism. It induces the homomorphism
$$\begin{aligned} \mathrm{id}_A \otimes \alpha\colon A \otimes_R B_1 &\longrightarrow A \otimes_R B_2 \\ a \otimes b &\longmapsto a \otimes \alpha(b). \end{aligned}$$
Attention. We have defined $\mathrm{id}_A \otimes \alpha$ on a set of generators. So, we will have to verify that $\mathrm{id}_A \otimes \alpha$ is well-defined. To do so, we simply observe that
$$\begin{aligned} A \times B_1 &\longrightarrow A \otimes_R B_2 \\ (a,b) &\longmapsto a \otimes \alpha(b) \end{aligned}$$
is an R-bilinear map. By Theorem I.13.1, it induces a unique homomorphism $A \otimes_R B_1 \longrightarrow A \otimes_R B_2$ that maps $a \otimes b$ to $a \otimes \alpha(b)$, $(a,b) \in A \times B_1$. This is $\mathrm{id}_A \otimes \alpha$. The same remark will apply to many other homomorphisms which we will introduce in the sequel.

The above constructions provide us with a functor
$$A \otimes_R \cdot\colon \underline{\mathrm{Mod}}_R \longrightarrow \underline{\mathrm{Mod}}_R.$$
I.13.3 Remarks. i) The universal property defines the tensor product only up to isomorphy. In order to get a map $\mathrm{Ob}(\underline{\mathrm{Mod}}_R) \longrightarrow \mathrm{Ob}(\underline{\mathrm{Mod}}_R)$, we have to fix one R-module $A \otimes_R B$, for every R-module B. We may take, for example, the module $F(A \times B)/Q$ from the proof of Theorem I.13.1.

ii) Let $\varphi\colon R \longrightarrow S$ be a homomorphism of rings and M an R-module. Then, S acts on $M \otimes_R S$ by multiplication on the second factor. In this way, $M \otimes_R S$ becomes an S-module. We say that it is obtained from M by *extension of the scalars* via φ.

I.13.4 Proposition. *The functor $A \otimes_R \cdot$ is right exact.*

Proof. We look at an exact sequence

$$B_1 \xrightarrow{\alpha} B_2 \xrightarrow{\beta} B_3 \longrightarrow 0$$

of R-modules. Since β is surjective, the image of $\mathrm{id}_A \otimes \beta$ contains all elements of the form $a \otimes b$, $a \in A$, $b \in B_3$. These elements generate $A \otimes_R B_3$, so that $\mathrm{id}_A \otimes \beta$ is surjective. It is also clear that

$$(\mathrm{id}_A \otimes \beta) \circ (\mathrm{id}_A \otimes \alpha) = (\mathrm{id}_A \otimes (\beta \circ \alpha)) = 0.$$

For this reason, we obtain the commutative diagram

$$\begin{tikzcd} A \otimes_R B_1 \arrow[r, "\mathrm{id}_A \otimes \alpha"] \arrow[d, equal] & A \otimes_R B_2 \arrow[r] \arrow[d, equal] & L \arrow[r] \arrow[d, "\sigma"] & 0 \arrow[d, equal] \\ A \otimes_R B_1 \arrow[r, "\mathrm{id}_A \otimes \alpha"] & A \otimes_R B_2 \arrow[r, "\mathrm{id}_A \otimes \beta"] & A \otimes_R B_3 \arrow[r] & 0 \end{tikzcd}, \quad L := \mathrm{Coker}(\mathrm{id}_A \otimes \alpha),$$

in which the top row is exact. The homomorphism σ is obviously surjective. In order to see that it is an isomorphism, we define the bilinear map

$$\begin{array}{rcl} \varphi\colon A \times B_3 & \longrightarrow & L \\ (a, b) & \longmapsto & [a \otimes \widetilde{b}]. \end{array}$$

Here, $\widetilde{b} \in B_2$ is an element with $\beta(\widetilde{b}) = b$. Note that φ is well-defined. In fact, for another element $b' \in B_2$ with $\beta(b') = b$, we have $\widetilde{b} - b' \in \alpha(B_1)$. This shows

$$[a \otimes \widetilde{b}] - [a \otimes b'] = [a \otimes (\widetilde{b} - b')] = 0.$$

The R-bilinear map φ defines a homomorphism $\psi\colon A \otimes_R B_3 \longrightarrow L$. Note

$$\forall a \in A,\ b \in B_2: \quad \psi\Big(\sigma([a \otimes b])\Big) = \psi[a \otimes \beta(b)] = [a \otimes b].$$

Thus, $\psi \circ \sigma = \mathrm{id}_L$, and σ is injective, too. □

I.13.5 Proposition. i) *Let A, B be R-modules. Then,*

$$A \otimes_R B \cong B \otimes_R A.$$

ii) *For every R-module A, we have*

$$R \otimes_R A \cong A.$$

iii) *Let $(A_i)_{i \in I}$ be a family of R-modules and B an R-module. Then,*

$$\Big(\bigoplus_{i \in I} A_i\Big) \otimes_R B \cong \bigoplus_{i \in I} (A_i \otimes_R B).$$

Proof. i) The required isomorphism is

$$\begin{array}{rcl} A\otimes_R B & \longrightarrow & B\otimes_R A \\ a\otimes b & \longmapsto & b\otimes a. \end{array}$$

ii) Scalar multiplication

$$\begin{array}{rcl} R\times A & \longrightarrow & A \\ (r,a) & \longmapsto & r\cdot a \end{array}$$

is R-bilinear. So, it gives $R\otimes_R A \longrightarrow A$, $r\otimes a \longmapsto r\cdot a$. The inverse to this map is supplied by

$$\begin{array}{rcl} A & \longrightarrow & R\otimes_R A \\ a & \longmapsto & 1\otimes a. \end{array}$$

iii) The map

$$\begin{array}{rcl} \Big(\bigoplus_{i\in I} A_i\Big)\times_R B & \longrightarrow & \bigoplus_{i\in I}(A_i\otimes_R B) \\ ((a_i, i\in I), b) & \longmapsto & (a_i\otimes b, i\in I) \end{array}$$

is R-bilinear and induces a homomorphism

$$\varphi\colon \Big(\bigoplus_{i\in I} A_i\Big)\otimes_R B \longrightarrow \bigoplus_{i\in I}(A_i\otimes_R B).$$

Recall ([22], Exercise III.1.7) that the direct sum comes with inclusion homomorphisms

$$\iota_j\colon A_j \longrightarrow \bigoplus_{i\in I} A_i, \quad j\in I,$$

and these induce homomorphisms

$$\psi_j := \iota_j\otimes \mathrm{id}_B\colon A_j\otimes_R B \longrightarrow \Big(\bigoplus_{i\in I} A_i\Big)\otimes_R B, \quad j\in I.$$

We form

$$\psi := \bigoplus_{i\in I}\psi_i\colon \bigoplus_{i\in I}(A_i\otimes_R B) \longrightarrow \Big(\bigoplus_{i\in I} A_i\Big)\otimes_R B,$$

and this homomorphism is inverse to φ. □

Let R be a ring and A an R-module. Applying the general theory from Section I.9, define the functors

$$\mathrm{Tor}_n^R(A,\cdot)\colon \underline{\mathrm{Mod}}_R \longrightarrow \underline{\mathrm{Mod}}_R, \quad n\in\mathbb{N},$$

as the left derived functors of $A\otimes_R \cdot$. More concretely, given an R-module B and a projective resolution

$$\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow B \longrightarrow 0$$

of B, the R-module $\mathrm{Tor}_n^R(A,B)$ is defined as the n-th homology module of the complex

$$\cdots \longrightarrow A\otimes_R P_n \longrightarrow \cdots \longrightarrow A\otimes_R P_1 \longrightarrow A\otimes_R P_0 \longrightarrow 0.$$

Lemma I.9.3, i), and Theorem I.9.4 specialize to

I.13.6 Proposition. i) *For R-modules A, B, we find*

$$\mathrm{Tor}_0^R(A,B) \cong A \otimes_R B.$$

ii) *A short exact sequence*

$$0 \longrightarrow B_1 \longrightarrow B_2 \longrightarrow B_3 \longrightarrow 0$$

of R-modules gives rise to a long exact sequence

$$\cdots \longrightarrow \mathrm{Tor}_{n+1}^R(A,B_3) \longrightarrow \mathrm{Tor}_n^R(A,B_1) \longrightarrow \mathrm{Tor}_n^R(A,B_2) \longrightarrow$$
$$\longrightarrow \mathrm{Tor}_n^R(A,B_3) \longrightarrow \mathrm{Tor}_{n-1}^R(A,B_1) \longrightarrow \cdots$$
$$\cdots$$
$$\cdots \longrightarrow \mathrm{Tor}_1^R(A,B_3) \longrightarrow A \otimes_R B_1 \longrightarrow A \otimes_R B_2 \longrightarrow A \otimes_R B_3 \longrightarrow 0$$

of Tor-*modules.*

I.13.7 Remarks. Let A and B be R-modules. The basic symmetry $A \otimes_R B \cong B \otimes_R A$ from Proposition I.13.5, i), is reflected by the left derived functors.

i) Let

$$\cdots \longrightarrow M_n \longrightarrow \cdots \longrightarrow M_1 \longrightarrow M_0 \longrightarrow A \longrightarrow 0$$

be a projective resolution of A. Then, $\mathrm{Tor}_n^R(A,B)$ is isomorphic to the n-th homology module of the complex

$$\cdots \longrightarrow M_n \otimes_R B \longrightarrow \cdots \longrightarrow M_1 \otimes_R B \longrightarrow M_0 \otimes_R B \longrightarrow 0.$$

This is an easy consequence of the universality of δ-functors (Proposition I.9.5).

ii) For every $n \in \mathbb{N}$, there is an isomorphism

$$\mathrm{Tor}_n^R(A,B) \cong \mathrm{Tor}_n^R(B,A).$$

With the help of the tensor functor, we can characterize a new class of R-modules: An R-module A is *flat*, if $A \otimes_R \cdot$ is an exact functor.

I.13.8 Examples. i) Projective modules are flat. Indeed, note that projective modules are acyclic for every right exact functor (see Lemma I.9.3, ii) and apply symmetry (Proposition I.13.5, i), and Remark I.13.7, ii). Alternatively, you may apply Part ii) and iii).

ii) Free modules are flat. This follows from Proposition I.13.5, ii) and iii), or the fact that free modules are projective (Proposition I.12.9).

iii) Direct summands of flat modules are flat, by Proposition I.13.5, iii), and the resulting fact that the Tor-functors commute with direct sums (compare Remark I.13.7).

I.13.9 Proposition. *For an R-module A, the following conditions are equivalent:*

i) *A if flat.*

ii) *For all R-modules B and all positive natural numbers n* > 0, $\mathrm{Tor}_n^R(A,B) = 0$.

iii) *For all R-modules B,* $\mathrm{Tor}_1^R(A,B) = 0$.

Proof. The implication "ii)$\Longrightarrow$iii)" is trivial. The conclusion "iii)$\Longrightarrow$i)" results from the long exact Tor-sequence (Proposition I.13.6, ii). Finally, we address "i)$\Longrightarrow$ii)". Let B be an R-module and

$$\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow B \longrightarrow 0$$

a projective resolution of B. The flatness of A implies that the complex

$$\cdots \longrightarrow A \otimes_R P_n \longrightarrow \cdots \longrightarrow A \otimes_R P_0 \longrightarrow A \otimes_R B \longrightarrow 0$$

is also exact, and this gives the claim. □

I.13.10 Exercises (Examples of tensor products). i) Let n be a natural number. Compute the tensor products

$$\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Z}/\langle n\rangle) \quad \text{and} \quad (\mathbb{Q}/\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/\langle n\rangle).$$

ii) Let m and n be natural numbers. Determine the tensor product

$$(\mathbb{Z}/\langle m\rangle) \otimes_{\mathbb{Z}} (\mathbb{Z}/\langle n\rangle).$$

iii) Is the tensor product of the exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{n \longmapsto 2n} \mathbb{Z} \longrightarrow \mathbb{Z}/\langle 2\rangle \longrightarrow 0$$

with $\mathbb{Z}/\langle 2\rangle$ exact?

I.13.11 Exercise (The tensor product of algebras). Let R be a commutative ring and A and B R-algebras. Show that there is an R-algebra structure on $A \otimes_R B$ with

$$(a \otimes b) \cdot (a' \otimes b') = (aa') \otimes (bb'), \quad \forall a, a' \in A,\ b, b' \in B.$$

I.13.12 Exercises (Tor). Let R be a commutative ring.

i) Show that[21]

$$\mathrm{Tor}_n^R(M, -) \cong L_n(- \otimes_R M), \quad n \in \mathbb{N},\ M \text{ an } R\text{-module}.$$

(This means that Tor may be computed by taking projective resolutions in the first variable.)

ii) Verify that

$$\mathrm{Tor}_n^R(M, N) \cong \mathrm{Tor}_n^R(N, M)$$

holds for all $n \in \mathbb{N}$ and all R-modules M and N.

[21] Here, L_n stands for the n-th **left derived functor** of a right exact functor defined in analogy to n-th right derived functor of a left exact functor (Section I.9), $n \in \mathbb{N}$.

Relations between Hom and $\otimes$

Let R be a ring and A an R-module. We say that A has a *finite presentation*, if there are natural numbers m, n and an exact sequence

$$R^{\oplus n} \longrightarrow R^{\oplus m} \longrightarrow A \longrightarrow 0.$$

I.13.13 Remark. By definition, the module A is finitely presented if and only if it can be written as a quotient of a free module in such a way that the kernel is also finitely generated. Thus, if the ring R is noetherian, a module over R has a finite presentation if and only if it is finitely generated ([22], Proposition III.1.30).

I.13.14 Proposition. *Let R be a ring, S a **flat** R-algebra, and B an R-module. There is a natural transformation*

$$h\colon \mathrm{Hom}_R(\cdot, B) \otimes_R S \longrightarrow \mathrm{Hom}_S(\cdot \otimes_R S, B \otimes_R S)$$

*of contravariant functors from $\underline{\mathrm{Mod}}_R$ to $\underline{\mathrm{Mod}}_S$, such that $h(A)$ is an isomorphism for every R-module A which has a **finite presentation**.*

Proof. We abbreviate $F := \mathrm{Hom}_R(\cdot, B) \otimes_R S$ and $G := \mathrm{Hom}_S(\cdot \otimes_R S, B \otimes_R S)$. For an R-module A, we define

$$\begin{aligned} h(A)\colon F(A) &\longrightarrow G(A) \\ f \otimes b &\longmapsto b \cdot (f \otimes \mathrm{id}_S). \end{aligned} \tag{I.13}$$

We leave it to the reader to verify that this is a homomorphism of S-modules and gives a natural transformation as A varies over all R-modules.

Next, observe

- ⋆ $h(A)$ is an isomorphism for a **free** R-module of finite rank,
- ⋆ both F and G are left exact contravariant functors.

Fix a finite presentation

$$R^{\oplus n} \longrightarrow R^{\oplus m} \longrightarrow A \longrightarrow 0$$

of A. We obtain the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, equal] & F(A) \arrow[r] \arrow[d, "h(A)"'] & F(R^{\oplus m}) \arrow[r] \arrow[d, "h(R^{\oplus m})"'] & F(R^{\oplus n}) \arrow[d, "h(R^{\oplus n})"'] \\ 0 \arrow[r] & G(A) \arrow[r] & G(R^{\oplus m}) \arrow[r] & G(R^{\oplus n}) \end{tikzcd}$$

with exact rows. Since $h(R^{\oplus m})$ and $h(R^{\oplus n})$ are isomorphisms, it is easy to infer that $h(A)$ is an isomorphism, too. □

I.13.15 Exercise. Let R be a ring, S a flat R-algebra, and A an R-module. Verify that (I.13) also gives rise to a natural transformation

$$h'\colon \mathrm{Hom}_R(A, \cdot) \otimes_R S \longrightarrow \mathrm{Hom}_S(A \otimes_R S, \cdot \otimes_R S).$$

The localization functor

Let R be a ring and $S \subset R$ a multiplicatively closed subset. We define a functor

$$(\cdot)_S : \underline{\mathrm{Mod}}_R \longrightarrow \underline{\mathrm{Mod}}_{R_S}.$$

It assigns to an R-module A the R_S-module A_S. Next, let $f\colon A \longrightarrow B$ be a homomorphism of R-modules. Set

$$\begin{array}{rcl} f_S : A_S & \longrightarrow & B_S \\ \dfrac{x}{s} & \longmapsto & \dfrac{f(x)}{s}. \end{array}$$

We first need to verify that this is well-defined. Assume that $x, y \in R$ and $s, t \in S$ are such that

$$\frac{x}{s} = \frac{y}{t}.$$

By definition ([22], Page 53), there is an element $u \in S$ with

$$u \cdot (t \cdot x - s \cdot y) = 0.$$

We apply f and find

$$0 = f(u \cdot t \cdot x - u \cdot s \cdot y) = u \cdot t \cdot f(x) - u \cdot s \cdot f(y).$$

This shows

$$\frac{f(x)}{s} = \frac{f(y)}{t}.$$

Now, let us check that f_S is a homomorphism. For $x, y \in A$ and $s, t \in S$, we compute

$$f_S\left(\frac{x}{s} + \frac{y}{t}\right) = f_S\left(\frac{t \cdot x + s \cdot y}{s \cdot t}\right) = \frac{t \cdot f(x) + s \cdot f(y)}{s \cdot t} = \frac{f(x)}{s} + \frac{f(y)}{t} = f_S\left(\frac{x}{s}\right) + f_S\left(\frac{y}{t}\right).$$

Finally, let $x \in A$, $r \in R$, and $s, t \in S$. Then,

$$f_S\left(\frac{r}{s} \cdot \frac{x}{t}\right) = f_S\left(\frac{r \cdot x}{s \cdot t}\right) = \frac{f(r \cdot x)}{s \cdot t} = \frac{r \cdot f(x)}{s \cdot t} = \frac{r}{s} \cdot \frac{f(x)}{t} = \frac{r}{s} \cdot f_S\left(\frac{x}{t}\right).$$

I.13.16 Proposition. *The localization functor* $(\cdot)_S$ *is isomorphic to the functor* $\cdot \otimes_R R_S$.

Proof. The map

$$\begin{array}{rcl} A \times R_S & \longrightarrow & A_S \\ \left(x, \dfrac{r}{s}\right) & \longmapsto & \dfrac{r \cdot x}{s} \end{array}$$

is R-bilinear, so it induces a (surjective) homomorphism

$$\alpha\colon A \otimes_R R_S \longrightarrow A_S$$

of R-modules. It is easy to check that it is also a homomorphism of R_S-modules.

Next, we introduce

$$\begin{aligned} \beta\colon A_S &\longrightarrow A \otimes_R R_S \\ \frac{x}{s} &\longmapsto x \otimes \frac{1}{s}. \end{aligned}$$

Again, we begin by verifying that β is well-defined. Let $x, y \in A$ and $s, t, u \in S$ with

$$u \cdot (t \cdot x - s \cdot y) = 0.$$

Using this identity, we compute

$$x \otimes \frac{1}{s} = x \otimes \frac{u \cdot t}{u \cdot t \cdot s} = (u \cdot t \cdot x) \otimes \frac{1}{u \cdot t \cdot s} = (u \cdot s \cdot y) \otimes \frac{1}{u \cdot t \cdot s} = y \otimes \frac{u \cdot s}{u \cdot t \cdot s} = y \otimes \frac{1}{t}.$$

We will also check that β is a homomorphism. For $x, y \in A$ and $s, t \in S$, we have

$$\begin{aligned} \beta\left(\frac{x}{s} + \frac{y}{t}\right) &= (t \cdot x + s \cdot y) \otimes \frac{1}{s \cdot t} = (t \cdot x) \otimes \frac{1}{s \cdot t} + (s \cdot y) \otimes \frac{1}{s \cdot t} \\ &= x \otimes \frac{t}{s \cdot t} + y \otimes \frac{s}{s \cdot t} = x \otimes \frac{1}{s} + y \otimes \frac{1}{t} = \beta\left(\frac{x}{s}\right) + \beta\left(\frac{y}{t}\right). \end{aligned}$$

Next, for $x \in A$, $r \in R$, and $s, t \in S$, we compute

$$\beta\left(\frac{r}{s} \cdot \frac{x}{t}\right) = x \otimes \frac{r}{s \cdot t} = \frac{r}{s} \cdot \left(x \otimes \frac{1}{t}\right) = \frac{r}{s} \cdot \beta\left(\frac{x}{t}\right).$$

For the second equality, we used the R_S-module structure of $A \otimes_R R_S$ (see Remark I.13.3, ii).

It is straightforward to check that the homomorphisms α and β are inverse to each other.

To conclude, we must explain that the above constructions give natural transformations of functors. This amounts to the fact that, for R-modules A, B, and a homomorphism $f\colon A \longrightarrow B$, the diagrams

$$\begin{array}{ccc} A \otimes_R R_S & \longrightarrow & B \otimes_R R_S \\ {\scriptstyle \alpha_A}\big\downarrow & & \big\downarrow{\scriptstyle \alpha_B} \\ A_S & \longrightarrow & B_S \end{array}, \qquad \begin{array}{ccc} A_S & \longrightarrow & B_S \\ {\scriptstyle \beta_A}\big\downarrow & & \big\downarrow{\scriptstyle \beta_B} \\ A \otimes_R R_S & \longrightarrow & B \otimes_R R_S \end{array}$$

commute. Again, this is an easy task. □

I.13.17 Proposition. *The localization functor* $(\cdot)_S : \underline{\mathrm{Mod}}_R \longrightarrow \underline{\mathrm{Mod}}_{R_S}$ *is exact.*

Proof. Let

$$0 \longrightarrow A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C \longrightarrow 0$$

be a short exact sequence of R-modules. By Proposition I.13.16 and I.13.4, the sequence

$$A_S \xrightarrow{\ f_S\ } B_S \xrightarrow{\ g_S\ } C_S \longrightarrow 0$$

is exact. It remains to verify that the homomorphism $f_S : A_S \longrightarrow B_S$ is injective. We pick $x \in A$ and $s \in R$ with

$$f_S\left(\frac{x}{s}\right) = \frac{f(x)}{s} = 0.$$

This means that there is an element $u \in S$ with

$$0 = u \cdot f(x) = f(u \cdot x).$$

The injectivity of f gives $u \cdot x = 0$, so that $x/s = 0$ in A_S. □

Via the localization homomorphism $R \longrightarrow R_S$, the ring R_S becomes an R-algebra. The last proposition says that it is a flat R-algebra.

I.13.18 Corollary. *Let R be a ring, $S \subset R$ a multiplicatively closed subset, and B and R-module. There is a natural transformation*

$$h\colon \mathrm{Hom}_R(\cdot, B)_S \longrightarrow \mathrm{Hom}_{R_S}((\cdot)_S, B_S)$$

of contravariant functors from $\underline{\mathrm{Mod}}_R$ to $\underline{\mathrm{Mod}}_{R_S}$, such that $h(A)$ is an isomorphism for every R-module A which has a ***finite presentation****.*

Proof. By Proposition I.13.16, the localization functor $(\cdot)_S$ is isomorphic to the tensor functor $\cdot \otimes_R R_S$. Since R_S is a flat R-algebra, we may apply Proposition I.13.14. □

I.13.19 Exercises (Some properties of Hom and ⊗). Let R be a commutative ring.

i) Let A, B, and C be R-modules. Verify that there is a canonical isomorphism

$$\mathrm{Hom}_R(A \otimes_R B, C) \cong \mathrm{Hom}_R\big(A, \mathrm{Hom}_R(B, C)\big).$$

ii) Let A be an R-module and B an abelian group. Show that there is a natural isomorphism

$$\mathrm{Hom}_R\big(A, \mathrm{Hom}_{\mathbb{Z}}(R, B)\big) \cong \mathrm{Hom}_{\mathbb{Z}}(A, B)$$

of R-modules.

iii) Let A, B, and C be R-modules. Prove that

$$(A \otimes_R B) \otimes_R C \cong A \otimes_R (B \otimes_R C)$$

(The tensor product is associative.)

I.14 Global dimension

Let R be a commutative ring and A an R-module. We know that A has both injective and projective resolutions. A priori, they might be infinite. Here, we will investigate some important situations where we do have resolutions of finite length.

An exact sequence

$$0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

in which $P_0, ..., P_n$ are projective R-modules is a *projective resolution of length n*. In the same way, we define an *injective resolution of length n*. The *projective dimension* of A is the minimal length of a projective resolution of A. It can be infinite. Likewise, the *injective dimension* of A is the minimal length of an injective resolution of A. The *global dimension* of R is the supremum of the **projective** dimensions of all R-modules.

I.14.1 Remark. One could also define a global dimension of R in terms of the injective dimensions of its modules. Proposition I.14.3 shows, however, that one would get the same result in this way.

I.14.2 Proposition. *Let A be an R-module and $n \in \mathbb{N}$ a natural number. The following conditions are equivalent:*

i) *For every R-module B, we have* $\mathrm{Ext}^{n+1}_R(A, B) = 0$.

ii) *For every exact sequence*

$$0 \longrightarrow C \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0 \qquad \text{(I.14)}$$

in which P_0, ..., P_{n-1} are projective, C is projective, too.

iii) *The module A has a projective resolution of length at most n.*

Proof. "i)$\Longrightarrow$ii)". We apply Exercise I.9.6 to the exact sequence (I.14) and find, for every R-module B,

$$\mathrm{Ext}^1_R(C, B) \cong \mathrm{Ext}^{n+1}_R(A, B) = 0.$$

The long exact Ext-sequence (Proposition I.11.1, iii) shows that $\mathrm{Hom}_R(C, \cdot)$ is an exact functor, i.e., that C is projective.

"ii)$\Longrightarrow$iii)". Let $P_\bullet$ be a projective resolution of A and define $C := \mathrm{Ker}(P_{n-1} \longrightarrow P_{n-2})$. According to ii), the resulting exact sequence

$$0 \longrightarrow C \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

is a projective resolution of A.

"iii)$\Longrightarrow$i)". This follows from the fact that we may compute $\mathrm{Ext}^{n+1}_R(A, B)$ from a projective resolution of A (see Remark I.11.2, i). □

I.14.3 Proposition. *Let R be a commutative ring and $n \in \mathbb{N}$ a natural number. The following statements are equivalent:*

i) *Every R-module has projective dimension at most n.*

ii) *Every* ***finitely generated*** *R-module has projective dimension at most n.*

iii) *Every R-module has injective dimension at most n.*

iv) *For all R-modules A, B, one has* $\mathrm{Ext}^{n+1}_R(A, B) = 0$.

Proof. The implication "i)$\Longrightarrow$ii)" is trivial, "iii)$\Longrightarrow$iv)" results from the fact that $\mathrm{Ext}^{n+1}_R(A, B)$ can be computed from any injective resolution of B, and "iv)$\Longrightarrow$i)" is contained in Proposition I.14.2.

We finally turn to the implication "ii)$\Longrightarrow$iii)". Let B be an R-module and $I^\bullet$ an injective resolution of B. We form the exact sequence

$$0 \longrightarrow B \longrightarrow I^0 \longrightarrow \cdots \longrightarrow I^{n-1} \longrightarrow J \longrightarrow 0$$

with

$$J := \mathrm{Coker}(I^{n-2} \longrightarrow I^{n-1}).$$

By Exercise I.9.6,

$$\mathrm{Ext}^1_R(A, J) = \mathrm{Ext}^{n+1}_R(A, B), \qquad \text{(I.15)}$$

for every R-module A. If A is finitely generated, then, by the assumption and the fact that $\mathrm{Ext}_R^{n+1}(A,B)$ may be computed from any projective resolution of A (Remark I.11.2, i), and Comment I.9.2, i), we have

$$\mathrm{Ext}_R^{n+1}(A,B) = 0. \tag{I.16}$$

We will show that J is an injective R-module. Using an argument similar to the one in the proof of Proposition I.12.1, we only have to verify that, for R-modules K, Q, and a monomorphism $K \longrightarrow Q$, such that $A := Q/K$ is **finitely generated**, any homomorphism $K \longrightarrow J$ extends to Q. The short exact sequence

$$0 \longrightarrow K \longrightarrow Q \longrightarrow A \longrightarrow 0$$

yields the exact sequence

$$\mathrm{Hom}_R(Q,J) \longrightarrow \mathrm{Hom}_R(K,J) \longrightarrow \mathrm{Ext}_R^1(A,J).$$

The last term vanishes by (I.15) and (I.16). □

I.14.4 Proposition. *Let R be a noetherian local ring, $\mathfrak{m}$ its maximal ideal, and $k := R/\mathfrak{m}$ its residue field. A **finitely generated** R-module A with*

$$\mathrm{Tor}_1^R(A,k) = 0$$

is free.

Proof. By assumption, there are a natural number t and a surjection $\pi\colon R^{\oplus t} \longrightarrow A$. We apply the functor $\cdot \otimes_R k$ and get the surjection

$$k^{\oplus t} \cong R^{\oplus t} \otimes_R k \longrightarrow A \otimes_R k \cong A/(\mathfrak{m}\cdot A).$$

We choose elements $m_1,...,m_s$ whose classes $\overline{m}_1,...,\overline{m}_s$ in $A/(\mathfrak{m}\cdot A)$ form a basis for that k-vector space. Nakayama's lemma ([22], III.1.31) shows that

$$\begin{aligned} \widetilde{\pi}\colon R^{\oplus s} &\longrightarrow A \\ (r_1,...,r_s) &\longmapsto r_1\cdot m_1 + \cdots + r_s \cdot m_s \end{aligned}$$

is surjective. With $K := \mathrm{Ker}(\widetilde{\pi})$, we form the short exact sequence

$$0 \longrightarrow K \longrightarrow R^{\oplus s} \xrightarrow{\widetilde{\pi}} A \longrightarrow 0. \tag{I.17}$$

We would like to show that $K = 0$. The ring R is noetherian, by assumption. It follows that K is finitely generated ([22], Proposition III.1.30). It is enough to check $K \otimes_R k = 0$, again by Nakayama's lemma. For this, we look at the exact sequence

$$\mathrm{Tor}_1^R(A,k) \longrightarrow K \otimes_R k \longrightarrow k^{\oplus s} \xrightarrow{\widetilde{\pi}\otimes \mathrm{id}_k} A \otimes_R k \longrightarrow 0,$$

obtained from applying $\cdot \otimes_R k$ to (I.17). By construction, $\widetilde{\pi}\otimes\mathrm{id}_k$ is an isomorphism. Since $\mathrm{Tor}_1^R(A,k) = 0$, by assumption, we are done. □

I.14.5 Corollary. *Let R be a noetherian local ring and A a **finitely generated** R-module. Then, A is projective if and only if it is free.*

I.14.6 Remark. i) We can prove the corollary without using Proposition I.14.4. A free module is projective. Assume that A is projective. We use the notation of the proof of Proposition I.14.4 and set $F := R^{\oplus s}$. We infer that

$$K \subset \mathfrak{m} \cdot F.$$

Since A is projective, there is a homomorphism $\iota\colon A \longrightarrow F$ with $\widetilde{\pi} \circ \iota = \mathrm{id}_A$. Thus, there is an isomorphism $F \cong K \oplus A$ under which $\widetilde{\pi}$ identifies with the projection onto A. We conclude

$$K = \mathfrak{m} \cdot K.$$

The Nakayama lemma ([22], III.1.31) shows $K = 0$. □

ii) Corollary I.14.5 is valid without the assumption that A be finitely generated (see [20], Theorem 2.5).

I.14.7 Proposition. *Let R be a noetherian local ring, $\mathfrak{m}$ its maximal ideal, $k := R/\mathfrak{m}$ its residue field, and $n \in \mathbb{N}$ a natural number.*

i) *A finitely generated R-module A has projective dimension at most n if and only if* $\mathrm{Tor}^R_{n+1}(A, k) = 0$.

ii) *The global dimension of R is at most n if and only if* $\mathrm{Tor}^R_{n+1}(k, k) = 0$.

Proof. i) The implication "$\Longrightarrow$" results from the fact that $\mathrm{Tor}^R_{n+1}(A, k)$ may be computed with the help of a projective resolution of A. For the converse implication "$\Longleftarrow$", we observe that A has a projective resolution $P_\bullet$ in which P_l is a free module of finite rank, $l \in \mathbb{N}$, because A is finitely generated and R is noetherian. Set $C := \mathrm{Ker}(P_{n-1} \longrightarrow P_{n-2})$ in order to form the exact sequence

$$0 \longrightarrow C \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0. \quad \text{(I.18)}$$

Exercise I.9.6 implies

$$\mathrm{Tor}^R_1(C, k) \cong \mathrm{Tor}^R_{n+1}(A, k) = 0.$$

Since R is assumed to be noetherian, C is a finitely generated R-module. By Proposition I.14.4, C is free and, in particular, projective. Therefore, (I.18) is a projective resolution of A of length at most n.

ii) Note that k is a finitely generated R-module. If the global dimension of R is at most n, then k has a projective resolution of length at most n, and $\mathrm{Tor}^R_{n+1}(k, k) = 0$ by Part i). If, conversely, $\mathrm{Tor}^R_{n+1}(k, k) = 0$, then k has a projective resolution of length at most n. For any R-module A, we thus find $\mathrm{Tor}^R_{n+1}(A, k) = 0$. This is because the module $\mathrm{Tor}^R_{n+1}(A, k)$ can be computed from a projective resolution of k. Part i) implies that every **finitely generated** R-module has projective dimension at most n. According to Proposition I.14.3, this yields the claimed estimate on the global dimension of R. □

I.15 Serre's theorem and Hilbert's syzygy theorem

In this section, we will prove that **regular** noetherian local rings can be characterized by purely cohomological conditions.

Let R be a noetherian local ring with maximal ideal $\mathfrak{m}$. We say that a tuple $(r_1, ..., r_s)$ of elements of R is a *regular set of generators* for $\mathfrak{m}$, if

★ $\mathfrak{m} = \langle r_1, ..., r_s \rangle$,

★ the image $\overline{r}_j$ of r_j in $R/\langle r_1, ..., r_{j-1} \rangle$ is not a zero divisor, $j = 1, ..., s$.

I.15.1 Example. Let k be a field, $s \in \mathbb{N}$ a natural number, $\mathfrak{p} := \langle x_1, ..., x_s \rangle \subset k[x_1, ..., x_s]$, and $R := k[x_1, ..., x_s]_{\mathfrak{p}}$. Then, R is a noetherian local ring with maximal ideal

$$\mathfrak{m} = \mathfrak{p}^{\mathrm{e}} = \langle x_1, ..., x_s \rangle.$$

It is easy to check that $x_1, ..., x_s$ is a regular set of generators for $\mathfrak{m}$. Of course, a similar discussion applies to $\mathfrak{p} = \langle x_1 - a_1, ..., x_s - a_s \rangle$, $(a_1, ..., a_s) \in \mathbb{A}^s_k$.

I.15.2 Lemma. *Let R be a regular local ring, $\mathfrak{m} \subset R$ its maximal ideal, and $k := R/\mathfrak{m}$ its residue field. If $(r_1, ..., r_s)$ is a tuple of elements in $\mathfrak{m}$, such that their classes $\overline{r}_1, ..., \overline{r}_s \in \mathfrak{m}/\mathfrak{m}^2$ form a k-basis for that vector space, then $(r_1, ..., r_s)$ is a regular set of generators for $\mathfrak{m}$.*

Proof. The Nakayama lemma (see [22], III.1.31) implies

$$\langle r_1, ..., r_s \rangle = \mathfrak{m}.$$

Recall from [22], Proof of Proposition IV.8.3, that, for an element $x \in \mathfrak{m} \setminus \mathfrak{m}^2$, the ring $R/\langle x \rangle$ is a regular local ring of dimension $\dim(R) - 1 = s - 1$. This gives that

$$R/\langle r_1, ..., r_j \rangle$$

is a regular local ring of dimension $s - j$, $j = 1, ..., s$.

Finally, we remind the reader that a regular local ring is an integral domain ([22], Proposition IV.8.3). In particular, $\overline{r}_j \neq 0$ is not a zero divisor in $R/\langle r_1, ..., r_{j-1} \rangle$, $j = 1, ..., s$. □

I.15.3 Theorem (Serre[22]). *Let R be a regular local ring of dimension s. Then, R has global dimension s.*

Proof. We will prove

★ $\mathrm{Tor}^R_{s+1}(k, k) = 0$,

★ $\mathrm{Tor}^R_s(k, k) \neq 0$.

According to Proposition I.14.7, this will settle the result.

Lemma I.15.2 grants the existence of a regular set of generators for the maximal ideal $\mathfrak{m}$. Let us fix such a set $r_1, ..., r_s$ of generators. Furthermore, we define

$$R_0 := R, \quad R_j := R/\langle r_1, ..., r_j \rangle, \quad j = 1, ..., s.$$

Note that

$$R_s = R/\mathfrak{m} = k.$$

[22] Jean-Pierre Serre (born 1926), French mathematician.

By an inductive argument, we will prove

$$\mathrm{Tor}_n^R(R_j, k) = 0, \quad n > j, \quad \mathrm{Tor}_j^R(R_j, k) \neq 0, \quad j = 0, ..., s.$$

***j* = 0.** Since $R_0 = R$ is a projective R-module, we have

$$\forall n > 0: \quad \mathrm{Tor}_n^R(R_0, k) = 0.$$

In addition,

$$\mathrm{Tor}_0^R(R_0, k) \cong R_0 \otimes_R k \cong k.$$

(We used the vanishing of higher Tors for projective modules (Example I.13.8, i), Proposition I.13.6, i), and I.13.5, ii).)

***j* ⟶ *j* + 1.** Since the image $\overline{r}_{j+1}$ of r_{j+1} in R_j is not a zero divisor, the sequence

$$0 \longrightarrow R_j \xrightarrow{r_{j+1}\cdot} R_j \longrightarrow R_{j+1} \longrightarrow 0$$

is exact. The corresponding long exact Tor-sequence I.13.6, ii), gives

$$\forall n > j+1: \quad \mathrm{Tor}_n^R(R_{j+1}, k) = 0,$$

and the exact sequence

$$0 \longrightarrow \mathrm{Tor}_{j+1}^R(R_{j+1}, k) \longrightarrow \mathrm{Tor}_j^R(R_j, k) \xrightarrow{r_{j+1}\cdot} \mathrm{Tor}_j^R(R_j, k). \tag{I.19}$$

Claim. *The multiplication by r_{j+1} on $\mathrm{Tor}_j^R(R_j, k)$ is the zero map.*

The module $\mathrm{Tor}_j^R(R_j, k)$ is computed from a projective resolution $(P_\bullet, \partial_\bullet)$ of k. All the maps in the resolution are homomorphisms of R-modules, so that the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & P_1 \arrow[r, "\partial_1"] \arrow[d, "r_{j+1}\cdot"] & P_0 \arrow[r, "\partial_0"] \arrow[d, "r_{j+1}\cdot"] & k \arrow[r] \arrow[d, "r_{j+1}\cdot"] & 0 \arrow[d, equal] \\
\cdots \arrow[r] & P_1 \arrow[r, "\partial_1"] & P_0 \arrow[r, "\partial_0"] & k \arrow[r] & 0
\end{tikzcd}$$

commutes. Since multiplication by r_{j+1} on k is the zero map, we infer our claim from Proposition I.8.8. ✓

Sequence (I.19) and the claim show that $\mathrm{Tor}_{j+1}^R(R_{j+1}, k)$ is isomorphic to $\mathrm{Tor}_j^R(R_j, k)$ and, therefore, non-zero, by the induction hypothesis. □

Given a finitely generated R-module A, a *chain of syzygies* for A is a resolution of A of the form

$$\cdots \longrightarrow R^{\oplus n_s} \xrightarrow{\delta_s} R^{\oplus n_{s-1}} \xrightarrow{\delta_{s-1}} \cdots \longrightarrow R^{\oplus n_1} \xrightarrow{\delta_1} R^{\oplus n_0} \longrightarrow A \longrightarrow 0$$

with natural numbers n_l, $l \in \mathbb{N}$. We say that it *terminates at level s*, if $\mathrm{Ker}(\delta_{s-1})$ is free.

I.15.4 Hilbert's syzygy theorem (local version). *Let R be a regular noetherian local ring of dimension s and A a finitely generated R-module. Then, any chain of syzygies of A terminates at level s.*

Proof. By Theorem I.15.3, the global dimension of R is s. Proposition I.14.2, ii), shows that $\mathrm{Ker}(\delta_{s-1})$ is projective for every chain of syzygies for A. Corollary I.14.5 finally shows that it is free. □

We will also give a variant of this theorem. Let $s > 0$ be a positive natural number. A *finite free resolution* (*FFR*) of A of *length s* is an exact sequence of the form

$$0 \longrightarrow R^{\oplus n_s} \longrightarrow \cdots \longrightarrow R^{\oplus n_1} \longrightarrow R^{\oplus n_0} \longrightarrow A \longrightarrow 0.$$

Here, we assume $n_i > 0$, $i = 0, ..., s$.

I.15.5 Remark. A module which has a finite free resolution has also a finite presentation.

As a consequence of Theorem I.15.4 and the fact that a finitely generated module over a noetherian ring has a resolution by free modules of finite rank we note:

I.15.6 Theorem. *Let R be a regular noetherian local ring. Then, every finitely generated R-module has a finite free resolution of length at most* $\dim(R)$.

In the following, we will establish the converse to Theorem I.15.3. We will need a few preparations.

I.15.7 Lemma. *Let R be a noetherian local ring with maximal ideal* $\mathfrak{m}$. *If every element of* $\mathfrak{m} \setminus \mathfrak{m}^2$ *is a zero divisor,* $\mathfrak{m}$ *is also a minimal prime ideal of R.*

Proof. The assertion is only non-trivial, if $\mathfrak{m} \neq 0$. Let $\mathfrak{p}_1$, ..., $\mathfrak{p}_k$ be the minimal prime ideals of R. By [22], Theorem II.4.28, the assumption means

$$\mathfrak{m} \subset \mathfrak{m}^2 \cup \bigcup_{i=1}^{k} \mathfrak{p}_i.$$

Since $\mathfrak{m} \not\subset \mathfrak{m}^2$, by Nakayama's lemma ([22], III.1.31), there must be an index $i_0 \in \{1, ..., k\}$ of R with

$$\mathfrak{m} \subset \mathfrak{p}_{i_0},$$

by [22], Lemma IV.8.4. We must actually have equality. □

I.15.8 Lemma. *Let R be a noetherian local ring with maximal ideal* $\mathfrak{m}$ *and residue field* $k := R/\mathfrak{m}$. *For an element* $r \in \mathfrak{m} \setminus \mathfrak{m}^2$, *the R-module* $\mathfrak{m}/\langle r\rangle$ *is a direct summand of the R-module* $\mathfrak{m}/(r \cdot \mathfrak{m})$.

Proof. We may pick $r_2, ..., r_s \in \mathfrak{m}$, such that their classes $\overline{r}, \overline{r}_2, ..., \overline{r}_s$ in $\mathfrak{m}/\mathfrak{m}^2$ form a k-basis for that k-vector space. Set $I := \langle r_2, ..., r_s \rangle$. We note that $\langle r\rangle \cap I \subset r \cdot \mathfrak{m}$. In fact, let $a_1, ..., a_s \in R$ be elements with $a_1 \cdot r = a_2 \cdot r_2 + \cdots + a_s \cdot r_s$, i.e., $a_1 \cdot r - a_2 \cdot r_2 - \cdots - a_s \cdot r_s = 0$. This equality implies that $a_1, ..., a_s \in \mathfrak{m}$ and gives our claim. So, the inclusion $I \subset \mathfrak{m}$ induces a homomorphism

$$f\colon I/(\langle r\rangle \cap I) \longrightarrow \mathfrak{m}/(r \cdot \mathfrak{m}).$$

It is straightforward to check that the homomorphism

$$\mathfrak{m}/\langle r\rangle = (\langle r\rangle + I)/\langle r\rangle \cong I/(\langle r\rangle \cap I) \xrightarrow{f} \mathfrak{m}/(r \cdot \mathfrak{m}) \longrightarrow \mathfrak{m}/\langle r\rangle$$

is the identity, and this implies the contention of the lemma. □

Let R be a ring, A an R-module. An element $r \in R$ is *regular* for A, if the multiplication map $A \longrightarrow A$, $x \longmapsto r \cdot x$, is injective.

I.15.9 Lemma. *Let R be a noetherian local ring, A a finitely generated R-module, and $r \in R$ an element which is regular for A. If A has finite projective dimension, then $A/(r \cdot A)$ has finite projective dimension as $(R/\langle r \rangle)$-module.*

Proof. We carry out an induction over the projective dimension of A. If the projective dimension is zero, A is projective and, therefore, free of finite rank, by Corollary I.14.5. Then, $A/(r \cdot A)$ is a free $(R/\langle r \rangle)$-module and, therefore, projective.

Let A be a finitely generated projective R-module and $h \geq 1$ its projective dimension. We find a natural number $s \geq 1$ and a surjection $\pi\colon R^{\oplus s} \longrightarrow A$. With $K := \operatorname{Ker}(\pi)$, we get the exact sequence

$$0 \longrightarrow K \longrightarrow R^{\oplus s} \xrightarrow{\pi} A \longrightarrow 0.$$

Using Exercise I.9.6 and Proposition I.14.7, i), we deduce that the projective dimension of K is $h - 1$. We tensorize this sequence by $R/\langle r \rangle$ and get

$$K/(r \cdot K) \longrightarrow (R/\langle r \rangle)^{\oplus s} \longrightarrow A/(r \cdot A) \longrightarrow 0.$$

We compute the kernel of the map $K \longrightarrow (R/\langle r \rangle)^{\oplus s}$. It is $K \cap \langle r \rangle^{\oplus s}$. Let $x \in K$ and $y \in R^{\oplus s}$ with $x = r \cdot y$. We see that

$$r \cdot \pi(y) = \pi(r \cdot y) = \pi(x) = 0.$$

Since r is regular for A, we have $\pi(y) = 0$, i.e., $y \in K$. So, $K \cap \langle r \rangle^{\oplus s} = r \cdot K$. We infer that $K/(r \cdot K) \longrightarrow (R/\langle r \rangle)^{\oplus s}$ is injective. Thus, we have the exact sequence

$$0 \longrightarrow K/(r \cdot K) \longrightarrow (R/\langle r \rangle)^{\oplus s} \longrightarrow A/(r \cdot A) \longrightarrow 0.$$

The induction hypothesis says that the projective dimension of $K/(r \cdot K)$ is at most $h - 1$. As before, the above sequence shows that the projective dimension of $A/(r \cdot A)$ is at most h. □

I.15.10 Theoren (Serre). *Assume that R is a noetherian local ring of finite global dimension. Then, R is a regular local ring.*

Proof. We perform an induction on the embedding dimension ([22], Page 131). If we have $\operatorname{edim}(R) = 0$, then $\mathfrak{m} = 0$ (see [22], Proposition IV.6.2), R is a field, and there is nothing to prove.

Case 1. Every element of $\mathfrak{m} \setminus \mathfrak{m}^2$ is a zero divisor. By Lemma I.15.7, $\mathfrak{m}$ is also a minimal prime ideal. In other words, $\mathfrak{m}$ is the radical of R. Since $\mathfrak{m}$ is finitely generated, there is a natural number $k \geq 1$ with $\mathfrak{m}^k = 0$. We choose $k_0 \geq 1$ minimal with $\mathfrak{m}^{k_0} = 0$. If $k_0 = 1$, R is a field, and we are done. Otherwise, pick $r \in \mathfrak{m}^{k_0 - 1} \setminus \{0\}$. Then, $\operatorname{Ann}(r) = \mathfrak{m}$, and we obtain the exact sequence

$$0 \longrightarrow k \xrightarrow{[a] \longmapsto a \cdot r} R \longrightarrow A := R/\langle r \rangle \longrightarrow 0.$$

As in the proof of Lemma I.15.9, this shows that the projective dimension of A is greater than the projective dimension of k. But this is impossible, because the projective dimension of k equals the global dimension of R (Proposition I.14.7, ii).

Case 2. The exact sequence

$$0 \longrightarrow \mathfrak{m} \longrightarrow R \longrightarrow k \longrightarrow 0,$$

Exercise I.9.6 and Proposition I.14.7, i), imply that the projective dimension of $\mathfrak{m}$ equals the projective dimension of k minus one. In particular, the projective dimension of $\mathfrak{m}$ is finite if and only if the projective dimension of k is finite. Moreover, by Proposition I.14.7, the projective dimension of k is finite if and only if the global dimension of R is finite.

Let $r \in \mathfrak{m} \setminus \mathfrak{m}^2$ be an element which is not a zero divisor. Then,

$$\operatorname{edim}(R/\langle r\rangle) = \operatorname{edim}(R) - 1. \tag{I.20}$$

The maximal ideal of $R/\langle r\rangle$ is $\mathfrak{m}/\langle r\rangle$. According to Lemma I.15.9, $\mathfrak{m}/(r \cdot \mathfrak{m})$ has finite projective dimension. Lemma I.15.8 shows that $\mathfrak{m}/\langle r\rangle$ is isomorphic to a direct summand of $\mathfrak{m}/(r \cdot \mathfrak{m})$. We infer that $\mathfrak{m}/\langle r\rangle$ has finite projective dimension. Indeed, if $\operatorname{Tor}_n^R(\mathfrak{m}/(r \cdot \mathfrak{m}), k)$ vanishes, then also $\operatorname{Tor}_n^R(\mathfrak{m}/\langle r\rangle, k)$, $n \in \mathbb{N}$, because the Tor-functors commute with direct sums (compare Example I.13.8, iii). By the previous remark, $R/\langle r\rangle$ has finite global dimension. By the induction hypothesis, $R/\langle r\rangle$ is a regular noetherian local ring. By [22], Lemma IV.8.2,

$$\dim(R/\langle r\rangle) = \dim(R) - 1.$$

Together with (I.20), we see that R is regular, too. □

Local properties of modules

Local properties in commutative algebra are properties which can be tested in all localizations at prime or even maximal ideals of a ring. This terminology comes from geometry: The prime or maximal ideals of a ring R are the points and closed points of the topological space $\operatorname{Spec}(R)$ ([22], Exercise I.9.11).

I.15.11 Proposition. *Let R be a ring, A an R-module, and $x \in A$. If, for every maximal ideal $\mathfrak{m} \subset R$, we have*

$$\frac{x}{1} = 0 \quad in \quad A_{\mathfrak{m}},$$

then $x = 0$.

Proof. Let $\operatorname{Ann}(x)$ be the annihilator ideal of x ([22], Page 37). Our assertion is that, under the stated assumption, $\operatorname{Ann}(x) = R$. If $\operatorname{Ann}(x) \neq R$, there is a maximal ideal $\mathfrak{m}_0 \subset R$ with $\operatorname{Ann}(x) \subset \mathfrak{m}_0$ ([22], Corollary I.4.8, i). The condition $x/1 = 0$ in $A_{\mathfrak{m}_0}$ means that there is an element $s \in R \setminus \mathfrak{m}_0$ with $s \cdot x = 0$ in A. So, $s \in \operatorname{Ann}(x) \setminus \mathfrak{m}_0$, a contradiction. □

I.15.12 Theorem. *Let R be a ring and A an R-module which has a finite presentation. Then, A is projective if and only if the localization $A_{\mathfrak{m}}$ is free, for every maximal ideal $\mathfrak{m} \subset R$.*

Proof. Suppose that A is projective. Since A is, by assumption, finitely generated, the proof of Theorem I.12.11 shows that there are a natural number $s > 0$ and a module B with

$$A \oplus B \cong R^{\oplus s}.$$

For every maximal ideal $\mathfrak{m} \subset R$, we find

$$A_\mathfrak{m} \oplus B_\mathfrak{m} \cong R_\mathfrak{m}^{\oplus s}.$$

Thus, by Theorem I.12.11 again, $A_\mathfrak{m}$ is projective. Corollary I.14.5 shows that $A_\mathfrak{m}$ is free of finite rank.

Let

$$B_2 \longrightarrow B_1 \longrightarrow 0$$

be an exact sequence of R-modules. We get the exact sequence

$$\mathrm{Hom}_R(A, B_2) \longrightarrow \mathrm{Hom}_R(A, B_1) \longrightarrow C \longrightarrow 0.$$

Here, C is just the cokernel of the map $\mathrm{Hom}_R(A, B_2) \longrightarrow \mathrm{Hom}_R(A, B_1)$. Localization is an exact functor (Proposition I.13.17). Hence, the sequence

$$\mathrm{Hom}_R(A, B_2)_\mathfrak{m} \longrightarrow \mathrm{Hom}_R(A, B_1)_\mathfrak{m} \longrightarrow C_\mathfrak{m} \longrightarrow 0$$

is also exact, for every maximal ideal $\mathfrak{m} \subset R$. By Corollary I.13.18 and Exercise I.13.15, this is the sequence that is obtained from the sequence

$$B_{2,\mathfrak{m}} \longrightarrow B_{1,\mathfrak{m}} \longrightarrow 0$$

by applying the functor $\mathrm{Hom}_{R_\mathfrak{m}}(A_\mathfrak{m}, \cdot)$. The assumption gives $C_\mathfrak{m} = 0$. Using Proposition I.15.11, we see that $C = 0$. □

I.16 The Auslander–Buchsbaum theorem

The aim of this section is to prove the following result:

I.16.1 Theorem (Auslander[23]–Buchsbaum[24]). *Let R be a regular noetherian local ring. Then, R is factorial.*

A central ingredient is the following criterion for factoriality.

I.16.2 Proposition. *A noetherian ring R is factorial if and only if every prime ideal $\mathfrak{p} \subset R$ of **height one** is principal.*

Proof. Assume, first, that R is factorial, and let $\mathfrak{p} \subset R$ be a prime ideal of height one. Pick a non-zero element $a \in \mathfrak{p}$. It is not a unit. Since R is factorial, we can find $s \geq 1$ and prime elements $p_1, ..., p_s \in R$ with

$$a = p_1 \cdot \dots \cdot p_s.$$

Since $\mathfrak{p}$ is a prime ideal, there exists an index $i_0 \in \{1, ..., s\}$ with $p_{i_0} \in \mathfrak{p}$. We get

$$\langle 0 \rangle \subsetneq \langle p_{i_0} \rangle \subset \mathfrak{p}.$$

Now, $\langle p_{i_0} \rangle$ is a prime ideal and $\mathfrak{p}$ has height one. So, the inclusion $\langle p_{i_0} \rangle \subset \mathfrak{p}$ must be an equality.

[23]Maurice Auslander (1926 - 1994), US American mathematician.
[24]David Alvin Buchsbaum (born 1929), US American mathematician.

For the converse direction, we first note that, in a noetherian ring R, every element $r \in R \setminus (\{0\} \cup R^\star)$ can be written as a product of irreducible elements (Exercise). By [26], Theorem I.6.1, it suffices to prove that every irreducible element is a prime element. Let $r \in R$ be an irreducible element and $\mathfrak{p}$ a minimal prime ideal containing r. Krull's principal ideal theorem ([26], Theorem IV.5.6) implies that $\mathrm{ht}(\mathfrak{p}) = 1$, so that $\mathfrak{p}$ is a principal ideal, by assumption. Let $p \in R$ be a prime element with $\mathfrak{p} = \langle p \rangle$. There exists an element $a \in R$ with $r = a \cdot p$. Since r is irreducible and p is not a unit, a must be a unit. So, r is a prime element. □

In the next step, we will prove that we may check factoriality after an appropriate localization.

I.16.3 Proposition. *Let R be a noetherian* ***integral*** *domain, $\Gamma \subset R$ a subset which consists of* ***prime elements*** *in R, and*

$$S = \{1\} \cup \{\, p_1 \cdot \cdots \cdot p_n \,|\, n \geq 1,\ p_1, ..., p_n \in \Gamma \,\}$$

the multiplicatively closed subset generated by Γ. If R_S is factorial, then so is R.

Proof. We apply Proposition I.16.2. So, let $\mathfrak{p} \subset R$ be a prime ideal of height one. We distinguish two cases.

Case 1. We have $\mathfrak{p} \cap S \neq \emptyset$. This is equivalent to $\mathfrak{p} \cap \Gamma \neq \emptyset$. Pick $p \in \mathfrak{p} \cap \Gamma$. Then,

- ⋆ $\langle 0 \rangle \subsetneq \langle p \rangle \subset \mathfrak{p}$,
- ⋆ $\langle p \rangle$ is a prime ideal.

As before, this and $\mathrm{ht}(\mathfrak{p}) = 1$ imply $\langle p \rangle = \mathfrak{p}$.

Case 2. We have $\mathfrak{p} \cap S = \emptyset$. Then, by [22], Corollary II.3.7, i), $\mathfrak{p}^{\mathrm{e}} \subset R_S$ is a prime ideal of height one. By assumption and Proposition I.16.2, there is an element $r \in R$ with $\mathfrak{p}^{\mathrm{e}} = \langle r \rangle$. Since any non-empty subset of ideals in a noetherian ring has maximal elements ([22], Theorem II.1.1, iii), we may pick an element $r_0 \in R$ with $\mathfrak{p}^{\mathrm{e}} = \langle r_0 \rangle$, such that[25] $\langle r_0 \rangle \subset R$ and $\langle r_0 \rangle$ is a maximal element of

$$\Sigma := \{\, \langle r \rangle \subset R \,|\, r \in R,\ \mathfrak{p}^{\mathrm{e}} = \langle r \rangle \,\}.$$

If $p \in \Gamma$, then $p \nmid r_0$. Otherwise, there would be an element $r_0' \in R$ with $r_0 = p \cdot r_0'$. The element r_0' is not a unit, because $\mathfrak{p} \cap \Gamma = \emptyset$. This implies $\langle r_0 \rangle \subsetneq \langle r_0' \rangle$ and $\mathfrak{p}^{\mathrm{e}} = \langle r_0 \rangle = \langle r_0' \rangle$ and contradicts the maximality of $\langle r_0 \rangle$ in Σ.

For an element $a \in \mathfrak{p}$, there exist $b \in R$ and $s \in S$ with

$$\frac{a}{1} = \frac{b \cdot r_0}{s} \quad \text{in} \quad R_S,$$

i.e.,

$$a \cdot s = b \cdot r_0 \quad \text{in} \quad R,$$

because R is an integral domain. If $s = 1$, there is nothing else to do. Otherwise, there are a positive natural number $n \geq 1$ and elements $p_1, ..., p_n \in \Gamma$ with $p_1 \cdot \cdots \cdot p_n = s$. We find

[25] In the following, the reader has to pay attention whether a principal ideal is taken in R or R_S.

$p_1|(b\cdot r_0)$. As we observed before, $p_1 \nmid r_0$, so that $p_1|b$, because p_1 is a prime element. Let $b' \in R$ be an element with $b = b'\cdot p_1$. It follows that

$$a\cdot(p_2\cdot\dots\cdot p_n)\cdot p_1 = (b'\cdot r_0)\cdot p_1.$$

Since R is an integral domain, this yields

$$a\cdot p_2\cdot\dots\cdot p_n = b'\cdot r_0.$$

We iterate this argument and find an element b'' with $a = b''\cdot r_0$. Altogether, we have shown that $\mathfrak{p}$ is the principal ideal generated by r_0. □

Next, we present a criterion that allows to verify that an ideal is a principal ideal. It generalizes [22], Lemma III.1.10.

I.16.4 Lemma. *Let R be an* ***integral domain*** *and $I \subset R$ an ideal. If there are natural numbers $m, n \in \mathbb{N}$ with*

$$I \oplus R^{\oplus n} \cong R^{\oplus(m+n)},$$

then I is a principal ideal.

Proof. We may clearly assume $I \neq \langle 0\rangle$. Let $K := Q(R)$ be the quotient field of R. The functor $\cdot\otimes_R K$ corresponds to localization at the multiplicatively closed subset $S = R\setminus\{0\}$ (Proposition I.13.16). By Proposition I.13.17, it is exact. The short exact sequence

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

leads to

$$0 \longrightarrow I\otimes_R K \xrightarrow{\cong} K \cong R\otimes_R K \longrightarrow 0 = R/I\otimes_R K \longrightarrow 0.$$

To see that $R/I\otimes_R K = 0$, note that $R/I\otimes_R K$ is generated as K-vector space by $[1]\otimes 1$. Pick $r \in I\setminus\{0\}$,

$$[1]\otimes 1 = [1]\otimes(r\cdot r^{-1}) = [r]\otimes r^{-1} = 0\otimes r^{-1} = 0.$$

This discussion shows $m = 1$.

Denote by $e_0, e_1, ..., e_n$ the standard basis of R^{n+1}, fix an isomorphism $\varphi\colon R^{\oplus(n+1)} \longrightarrow I\oplus R^{\oplus n}$, and define $\iota\colon I\oplus R^{\oplus n} \longrightarrow R^{\oplus(n+1)}$ as the inclusion. Finally, set

$$\Phi\colon R^{\oplus(n+1)} \xrightarrow{\varphi} I\oplus R^{\oplus n} \xrightarrow{\iota} R^{\oplus(n+1)}.$$

With respect to the fixed basis of $R^{\oplus(n+1)}$, Φ is determined by an $((n+1)\times(n+1))$-matrix

$$M = (m_{ij})_{i,j=0,\dots,n} \in \mathrm{Mat}(n+1, R).$$

Note that

$$d := \mathrm{Det}(M) \neq 0,$$

because $\Phi\otimes_R \mathrm{id}_K$ is an isomorphism, by the discussion from the beginning.

Let M_j be the matrix that is obtained from M by deleting row zero and colum j and

$$d_j := (-1)^j\cdot\mathrm{Det}(M_j),\quad j = 0, ..., n.$$

We observe
$$\sum_{j=0}^{n} m_{ij} \cdot d_j = \begin{cases} d, & \text{for } i = 0 \\ 0, & \text{for } i = 1, ..., n \end{cases}.$$
For the element $f_0 := \sum_{i=0}^{n} d_i \cdot e_i$, we find
$$M \cdot f_0 = \Phi(f_0) = d \cdot e_0.$$
There are elements $f_1, ..., f_n$ with $\Phi(f_i) = e_i$, $i = 1, ..., n$, and a homomorphism
$$\Psi\colon R^{\oplus(n+1)} \longrightarrow R^{\oplus(n+1)}$$
with
$$\Psi(e_i) = f_i, \quad i = 0, ..., n.$$
Let $N \in \mathrm{Mat}(n+1, R)$ be the corresponding matrix with respect to the basis $e_0, ..., e_n$ of $R^{\oplus(n+1)}$. By construction,
$$M \cdot N = \begin{pmatrix} d & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix},$$
so that
$$d \cdot \mathrm{Det}(N) = \mathrm{Det}(M) \cdot \mathrm{Det}(N) = d, \quad \text{i.e.,} \quad \mathrm{Det}(N) = 1.$$
This shows that Ψ is an isomorphism and the image of $\Phi \circ \Psi$ is the same as the image of Φ. Let $\Pi\colon R^{\oplus(n+1)} \longrightarrow R$ be the projection onto the zeroth factor. Then,
$$I = (\Pi \circ \iota)(I \oplus R^{\oplus n}) = (\Pi \circ \Phi)(R^{\oplus(n+1)}) = (\Pi \circ \Phi \circ \Psi)(R^{\oplus(n+1)}) = \langle d \rangle.$$
This proves our assertion. □

Let R be a ring and A a finitely generated R-module. We call A *stably free*, if there exist natural numbers $m, n \in \mathbb{N}$ with
$$A \oplus R^{\oplus m} \cong R^{\oplus(m+n)}.$$
The above lemma states that an ideal in an integral domain is principal if and only if it is stably free as module.

I.16.5 Proposition. *Let R be a ring and P an R-module which has a* ***finite free resolution****. Then, P is projective if and only if P is stably free.*

Proof. A module which is stably free is projective (Theorem I.12.11). Now, assume that P is projective. We prove the following statement by induction on s: If
$$0 \longrightarrow R^{\oplus n_s} \xrightarrow{\pi_s} \cdots \xrightarrow{\pi_2} R^{\oplus n_1} \xrightarrow{\pi_1} R^{\oplus n_0} \xrightarrow{\pi_0} P \longrightarrow 0$$
is an exact sequence in which P is projective, then P is stably free.

If $s = 0$, then P is free, and we are done. For the induction step, we note
$$R^{\oplus n_0} \cong \mathrm{Ker}(\pi_0) \oplus P,$$

because P is projective. By Theorem I.12.11, $\mathrm{Ker}(\pi_0)$ is projective. There is an exact sequence

$$0 \longrightarrow R^{\oplus n_s} \xrightarrow{\pi_s} \cdots \xrightarrow{\pi_2} R^{\oplus n_1} \xrightarrow{\pi_1} \mathrm{Ker}(\pi_0) \longrightarrow 0.$$

By the induction hypothesis, $\mathrm{Ker}(\pi_0)$ is stably free, i.e., there are natural numbers $m, n \in \mathbb{N}$ with

$$\mathrm{Ker}(\pi_0) \oplus R^{\oplus m} \cong R^{\oplus n}.$$

Now, we have

$$P \oplus R^{\oplus n} \cong P \oplus \mathrm{Ker}(\pi_0) \oplus R^{\oplus m} \cong R^{\oplus n_0} \oplus R^{\oplus m} \cong R^{\oplus (m+n_0)},$$

so that P is stably free. □

Proof of Theorem I.16.1. As usual, $\mathfrak{m}$ will be the maximal ideal of R. Recall that a regular noetherian ring is an integral domain. We carry out an induction on $n := \dim(R)$. If $n = 0$, then R is a field and there is nothing to show. For the induction step "$(n-1) \longrightarrow n$", we pick an element $x \in \mathfrak{m} \setminus \mathfrak{m}^2$. (Since $\mathfrak{m} \neq 0$, $\mathfrak{m} \neq \mathfrak{m}^2$, by Nakayama's lemma ([22], III.1.31).) Then, $\langle x \rangle$ is a prime ideal ([22], Proof of Proposition IV.8.3). We apply Proposition I.16.3 to $\Gamma = \{x\}$. The multiplicatively closed subset generated by Γ is $S = \{\, x^k \mid k \in \mathbb{N} \,\}$.

We point out that R_x need not be a local ring, so that we cannot directly apply the induction hypothesis.

We will show that R_x is a factorial ring. For this, we invoke Proposition I.16.2, i.e., we show that every prime ideal $\mathfrak{p}$ of height one in R_x is principal. The contraction $\mathfrak{p}^{\mathrm{c}} \subset R$ is a finitely generated R-module. By Theorem I.15.6, it has a finite free resolution. Since localization is an exact functor (Proposition I.13.17),

$$\mathfrak{p} = \mathfrak{p}^{\mathrm{ce}} = \mathfrak{p}^{\mathrm{c}}_S$$

has a finite free resolution as R_x-module. By Lemma I.16.4 and Proposition I.16.5, it is now sufficient to show that $\mathfrak{p}^{\mathrm{c}}_S$ is projective as R_x-module. This can be checked locally, by Theorem I.15.12.

Let $\mathfrak{n} \subset R_x$ be a maximal ideal and $\mathfrak{n}^{\mathrm{c}} \subset R$ its contraction. Then,

$$(R_x)_{\mathfrak{n}} = R_{\mathfrak{n}^{\mathrm{c}}}.$$

Claim. *The ring $R_{\mathfrak{n}^{\mathrm{c}}}$ is regular.*

To see this, we will apply Serre's criterion I.15.10. It suffices to verify that

$$K = R_{\mathfrak{n}^{\mathrm{c}}}/(\mathfrak{n}^{\mathrm{c}} \cdot R_{\mathfrak{n}^{\mathrm{c}}}) = (R/\mathfrak{n}^{\mathrm{c}})_{\mathfrak{n}^{\mathrm{c}}}$$

has finite global dimension as $R_{\mathfrak{n}^{\mathrm{c}}}$-module. This results from the fact that $R/\mathfrak{n}^{\mathrm{c}}$ has finite projective dimension as R-module and localization is an exact functor. ✓

Next, we have

$$\dim(R_{\mathfrak{n}^{\mathrm{c}}}) = \mathrm{ht}(\mathfrak{n}^{\mathrm{c}}) < \mathrm{ht}(\mathfrak{m}) = \dim(R).$$

The inequality follows, because $x \in \mathfrak{m} \setminus \mathfrak{n}^{\mathrm{c}}$. Now, we can apply the induction hypothesis to $R_{\mathfrak{n}^{\mathrm{c}}}$. There are two cases.

Case 1. If $\mathfrak{p} \subset \mathfrak{n}$, then the extension of $\mathfrak{p}$ to $R_{\mathfrak{n}^c}$ is a prime ideal of height one and, therefore, a principal ideal (Proposition I.16.2).

Case 2. If $\mathfrak{p} \not\subset \mathfrak{n}$, then the extension of $\mathfrak{p}$ to $R_{\mathfrak{n}^c}$ equals $R_{\mathfrak{n}^c}$.

In both cases, $\mathfrak{p}_{\mathfrak{n}^c}$ is a free $R_{\mathfrak{n}^c}$-module of rank one. As indicated before, Theorem I.15.12 shows that $\mathfrak{p}$ is a projective R-module, and this finishes the argument. □

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