# Periods map: combinatorial approach

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# Motivation

Solutions of several rational approximation problems in C-norm

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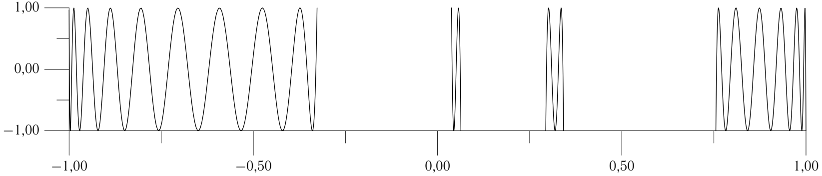


Figure: Degree 29 Chebyshev polynomial on four segments

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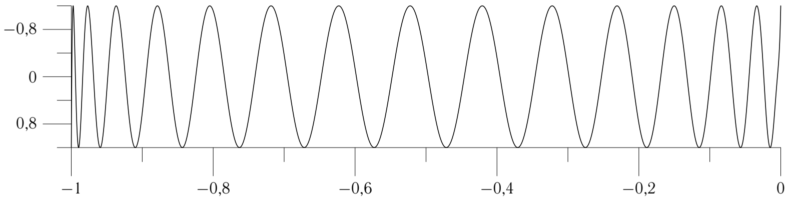


Figure: Degree $n = 31$ Optimal Stability Polynomial for RK method of accuracy degree $p = 3$

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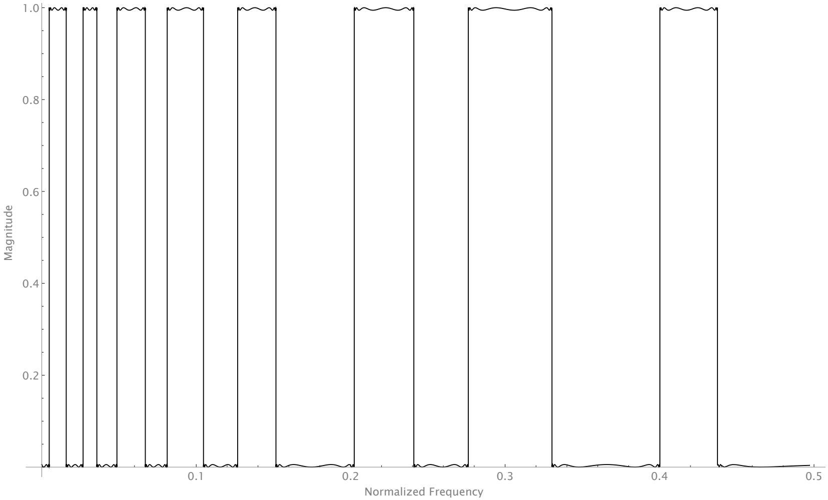



Figure: Optimal gain-frequency characteristic of multiband electrical filter, degree $n = 326$, $g = 16$ (computed by S.Lyamaev)

# Motivation: continued

Oscillatory behaviuor of solutions a.k.a. equiripple property or Chebyshev's alternation principle.

Chebyshev representation of polynomials

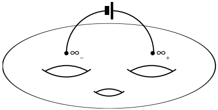
$$P(x) = \pm \cos(ni \int_{(e,0)}^{(x,w)} d\eta_M)$$

in terms of associated hyperelliptic curve $M(\mathrm{E})$

$$w^2 = \prod_{s=1}^{2g+2} (x - e_s), \qquad (x, w) \in \mathbb{C}^2, \qquad \mathrm{E} = \{e_s\},$$

and distinguished differential $d\eta_M$ on it:
simple poles at infinity with residues $\pm 1$ and purely imaginary periods:

$$d\eta_M = (x^g + \dots)\frac{dx}{w}$$



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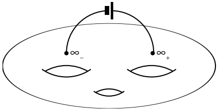


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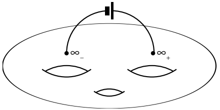

# Motivation: continued

Example:

Sphere $g = 0$ gives Chebyshëv polynomials (1853);
Tori $g = 1$ give a family of Zolotarëv polynomials (1868).

Abelian equations

$$\int_C d\eta_M \in \frac{2\pi i}{n}\mathbb{Z}, \quad \forall C \in H_1(M, \mathbb{Z}).$$

Q: How to concerve Abelian Eqs on a variable curve $M$?
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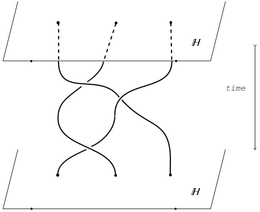
# Setting the problem: Moduli spaces of real curves

Consider moduli space $\mathcal{H}$ of real hyperelliptic curves with a marked point $\infty$ ($\neq$ branchpoint) on an oriented real oval. Curves with fixed topological invariants: the number $k$ of real ovals and the genus $g$ make up a component $\mathcal{H}_g^k$. Half of the symmetric branching divisor $\mathrm{E} = \bar{\mathrm{E}}$: $2k$ real points and $g-k+1$ points of the upper half plane (modulo translations and dilatations).




$$\mathcal{H}_g^k := \mathbb{H}^{g-k+1} \backslash \{diagonals\}/permutations \times \triangle_{2k-2}$$

$\dim_{\mathbb{R}} \mathcal{H}_g^k = 2g$;
$\pi_1(\mathcal{H}_g^k) = Br_{g-k+1}$ (braids on $g-k+1$ strands).

# Setting the problem: Periods mapping

Locally we can define the period map $\mathcal{H}_g^k \to \mathbb{R}^g$ as follows: Given a basis $C_j$ in $H_1(M, \mathbb{Z})$,

$$\Pi_j(\mathsf{E}) = -i \int_{C_j} d\eta_{M(E)}, \qquad j = 1, \ldots, 2g.$$

Globally the map is not correctly defined because of the monodromy of Gauss-Manin connection: braids entangle the basic cycles $C_j$ (Burau representation).
However, the period map $\Pi$ is well defined on the universal cover of the moduli space: $\Pi : \quad \tilde{\mathcal{H}_g^k} \to \mathbb{R}^g$.

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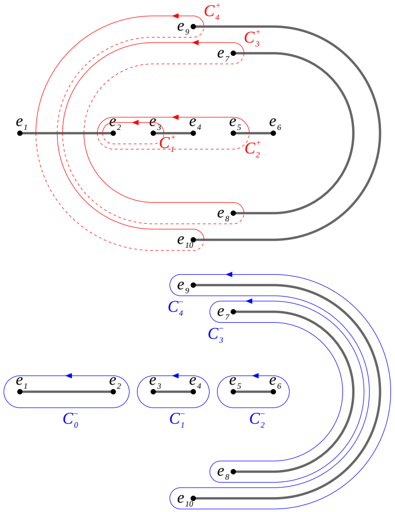
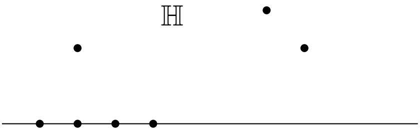
# Even and odd cycles

Simplification due to mirror symetry: cycles are split into even/odd. Real differential $d\eta_M$ has trivial periods along all even cycles due to its normalization.



# Labyrinth model of the moduli space universal cover




A point $E \in \mathcal{H}_4^2$ is lifted to the universal cover by choosing the labyrinth that accompanies it.

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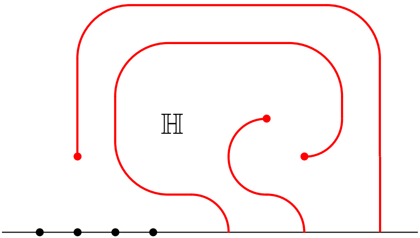


Labyrinth of a point $E \in \mathcal{H}$ gives a natural basis in odd 1-cycles of the curve $M(E)$. Fundamental group of the base (braids) acts on labyrinths as MCG of punctured half plane.

# Period mapping F.A.Q.

Natural questions about period mapping arise:

- Are fibers of $\Pi$ smooth?
- How $\Pi$ interacts with braids? Are there fixed fibers?
- How many rational fibers are there? (they parametrize solutions to optimization problems)
- What is the range of $\Pi$?
- What is the global topology of a fiber? Connected?

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**CONJECTURE(2001): Component of a fiber = $g$-cell.**

# Main theorem

**THEOREM** *Any fiber of the period mapping in* $\mathcal{H}_2^k$, $k = 1, 2, 3$, *is a cell.*

Same theorem is true for the (components of) 3D fibers.

# Machinery: Pictorial representation of curves

Let us fix a curve $M \in \mathcal{H}$. Due to normalization of distinguished differential, the function

$$W(x) := |\mathrm{Re} \int_{(e,0)}^{(x,w)} d\eta_M|$$

- is single valued on the plane,
- harmonic outside its zero set (containing all branchpoints)
- has logarithmic pole at infinity.
- it's level lines are the leaves of the foliation ${d\eta_M}^2 < 0$ on the sphere.

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Construction of a graph $\Gamma(M) := \Gamma_{\mid} \cup \Gamma_{-}$.

- $\Gamma_{\mid}$ is zero set of $W(x)$, not oriented;
- $\Gamma_{-}$ are all segments of the horizontal foliation $d\eta_M^2 > 0$ oriented with respect to the growth of $W(x)$ and connecting the finite critical points of the foliation to other such points or to zeroset of $W$.
- Each edge is equipped with its length in the metric $|d\eta_M|$.
- The vertices of the graph are the finite points of the divisor $(d\eta_M{}^2)$ and their projections to the vertical component along the horizontal foliation.

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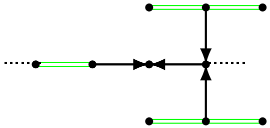
# Example of associated graph



Remarks:

- The multiplicity of $V$ in divisor of $(d\eta_M)^2$ equals to $ord(V) := d$[redacted]$(V) + 2d_{in}(V) - 2$. Hence, combinatorics of the graph $\Gamma(M)$ gives topological invariants $g, k$ of the curve $M$.
- The periods of the differential $d\eta_M$ are integer linear combinations of the lengths of the vertical edges.
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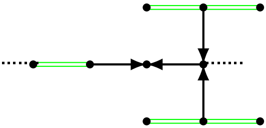

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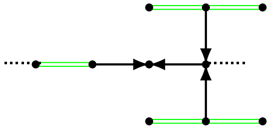

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# Axiomatic description of graphs

## Properties of associated graph:

1. $\Gamma$ is a tree with horizontal symmety axis (Topology)
2. Outcoming horizontal edges are separated (Topology)
3. $W(V)=0$ if $V$ lies on the vertical part of the graph (Width normalization)
4. The lengths of all vertical edges is $\pi$. (Height normalization)
5. If $\mathrm{ord}(V)=0$ then $V\in\Gamma_{-}\cap\Gamma_{|}$ (Minimal vertices)

THEOREM

Each weighted graph satisfying the above properties 1-5 stems from a unique curve $M\in\mathcal{H}_g^k$.

**Proof hint:** The Riemann surface may be glued from a finite number of stripes in a way determined by combinatorics and weights of graph.

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# Coordinate space of a graph

The weights of the admissible graph have obvious linear restrictions. They fill out a convex polyhedron $\mathcal{A}[\Gamma]$: simplex $\{H_s\}\times$ cone $\{W_j\}$ of dimension at most $2g$.

1. $\sum_s H_s = \pi$ simplex
2. *if* $V_1 \longrightarrow V_2$ *then* $W_1 < W_2$, $V_* \in \Gamma_{\_}$ cone




Example

$g = 2$;
$k = 1$;
$dim\mathcal{A}[\Gamma] = 2g = 4$;
$\mathcal{A}[\Gamma] = \{2(H_1 + H_2) + H_3 = \pi\}\times$
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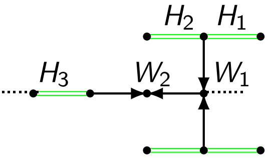


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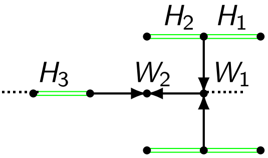
2. *if* $V_1 \longrightarrow V_2$ *then* $W_1 < W_2$, $V_* \in \Gamma_{\_}$ cone




Example

$g = 2$;
$k = 1$;
$dim\mathcal{A}[\Gamma] = 2g = 4$;
$\mathcal{A}[\Gamma] = \{2(H_1 + H_2) + H_3 = \pi\}\times$
$\{0 < W_1 < W_2\}$

Theorem

Space $\mathcal{A}[\Gamma]$ is real analytically embedded to the moduli space $\mathcal{H}$.

# Moduli space cellular decomposition

For given values of $g, k$ there are only finitely many admissible graphs.

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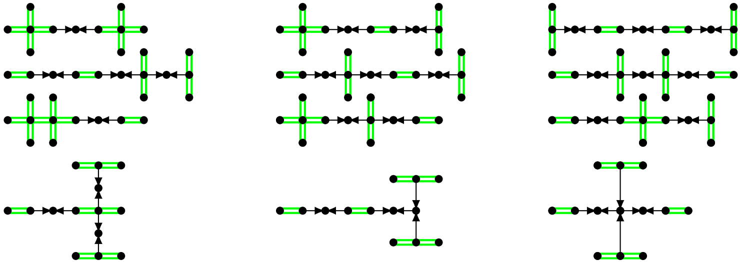
EXAMPLE: 20 codimension zero cells in the space $\mathcal{H}_3^2$ (up to symmetry)



# Moduli space cellular decomposition

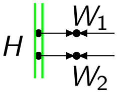
For given values of $g, k$ there are only finitely many admissible graphs.

ALGORITHM listing all admissible graphs with given topological invariants is available.

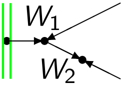
# Polyhedral model of Moduli space

We've built a cellular decomposition of the moduli space, cells are encoded by admissible types of trees. Full dimensional polyhedra are attached one to the other with the Neighboring relations

1. Contract edges of zero weight.
2. Zip neighbouring outcoming edges (to keep property 2 of $\Gamma$)




when $H \to 0$ goes to




if $W_1 < W_2$.

# Periods map

Periods map is a linear function in local coordinates (heights) of the cell $\mathcal{A}[\Gamma]$. It is easy to compute it for the labyrinth not intersecting the graph $\Gamma$.

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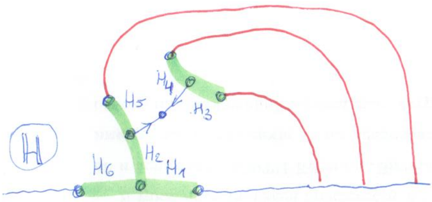
**EXAMPLE:**




$$\Pi(\Lambda) = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} =$$

$$= \begin{pmatrix} H_1 + H_2 + H_3 \\ H_1 + H_2 + H_3 + H_4 \\ H_1 + H_2 + H_3 + 2H_4 + H_5 \end{pmatrix}$$

Note that the value of $\Pi(\Lambda)$ lies in a simplex $\Delta_3$

$0 < h_1 < h_2 < h_3 < \pi$

# Periods map, continued

The (canonical) choice of a labyrinth means certain lifting of the cell $\mathcal{A}[\Gamma]$ to the universal covering. The value of the Periods mapping for other liftings of the polyhedron is related to the computed one via the Burau representation of braids

# Periods map, continued

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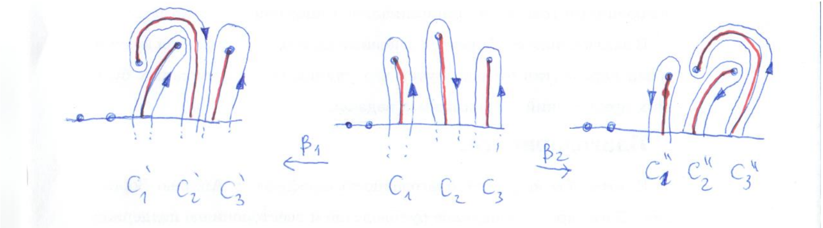
**EXAMPLE**




$$\beta_1 \cdot \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} C_2 \\ 2C_2 - C_1 \\ C_3 \end{pmatrix} \qquad \beta_2 \cdot \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} C_1 \\ C_3 \\ 2C_3 - C_1 \end{pmatrix}$$

# Fibers of Periods map

Locally, the fiber of the Periods map is an intersection of the cell $\mathcal{A}[\Gamma]$ and a $g$-dimensional plane. What remains is to assemble the arising $g$- polyhedra with the help of Neighbouring relations.

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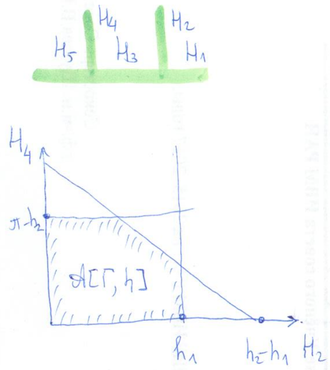
**EXAMPLE** A fiber inside the cell $\mathcal{A}[\Gamma] \subset \mathcal{H}_2^1$ for $\Gamma = \Gamma_{\mathsf{I}}$ (ambient space contains 9 full dimension cells)




Fix the periods:

$H_1 + H_2 = h_1$

$H_1 + 2H_2 + H_3 + H_4 = h_2$

$H_1 + 2H_2 + H_3 + 2H_4 + H_5 = \pi$

(normalization)

Positive coordinates $H_2, H_4$ in the polygon satisfy

$H_2 < h_1$

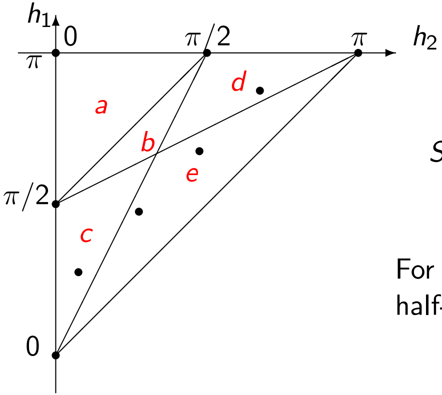
$H_4 < \pi - h_2$

$H_2 + H_4 < h_2 - h_1$

# Fibers of Periods map

In the above example the local section depends on the value $h \in \Delta_2$ of the Periods map.

**Phase diagram** for the space $\mathcal{H}_2^1$

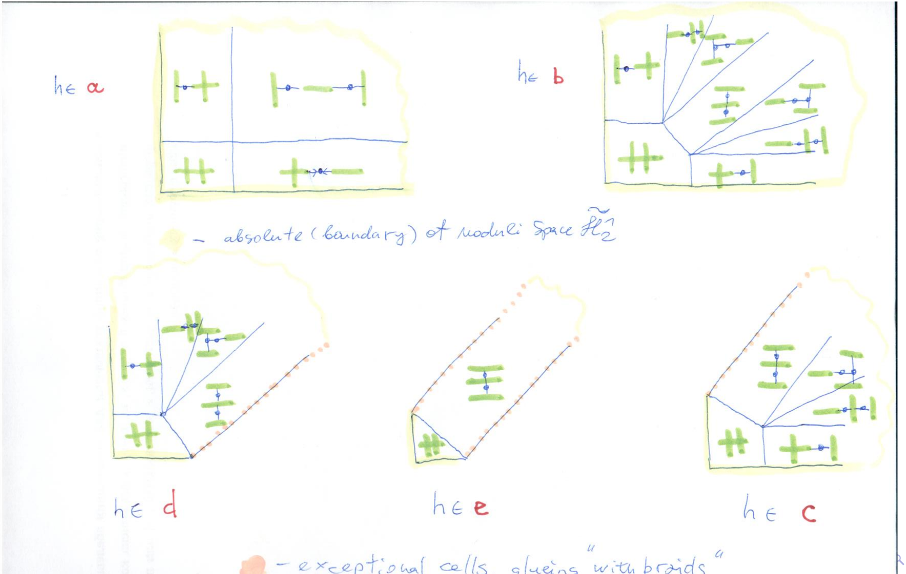



$$\mathit{Section} = \begin{cases} \mathit{rectangle} & h \in a \\ \mathit{pentagon} & h \in b \\ \mathit{trapezoid} & h \in c, d \\ \mathit{triangle} & h \in e \end{cases}$$

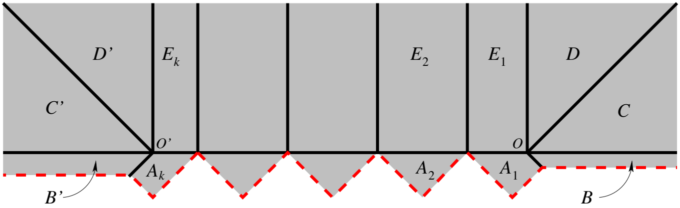
For other graphs $\Gamma$ the polygons are half-stripes or quadrants or empty.

# Assembling $\tilde{\mathcal{H}}_2^1$ fibers from cells

Fiber $\Pi^{-1}(h)$ with $h$ from the above phase diagram:



# The above fiber in 'e' phase assembled




This is Sasha Zvonkin's picture of a fiber.

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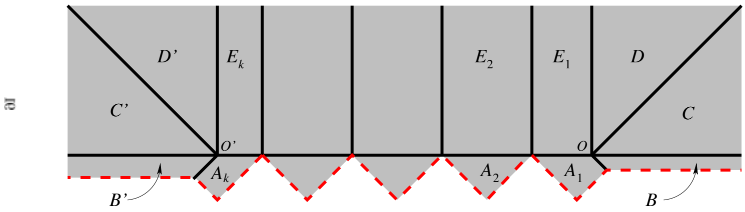


This is Sasha Zvonkin's picture of a fiber.

# I thank everyone for the patience!